MA505 Algebraic geometry

Worksheet A

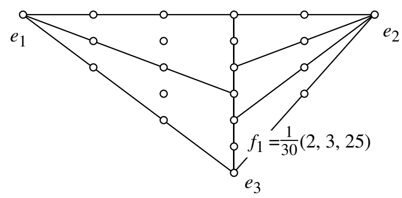
Questions on $\mathbb{C}^3/A$ and A-Hilb $\mathbb{C}^3$ for $A \subset \mathrm{SL}(3,\mathbb{C})$

1 (Too easy.) The quotient singularity $\frac{1}{3}(1,1,1)$ is the cone on the third Veronese embedding of $\mathbb{P}^2$. Show how to write down its "cylinder" resolution as a union of 3 affine pieces.

Draw the junior simplex and its barycentric subdivision at $\frac{1}{3}(1,1,1)$. Calculate the dual cone of each triangle in the lattice of invariant monomials (for example $\xi = x/y, \eta = y^3, \zeta = z/y$). Make them parametrise A-clusters in $\mathbb{C}^3$.

2 Draw the regular subdivision of $\frac{1}{r}(1,1,r-2)$, and calculate the monomials that base the dual cone. [Hint: consider the triangle with vertexes $(1,0,0)$, $\frac{1}{r}(i,i,r-2i)$, $\frac{1}{r}(i+1,i+1,r-2i-2)$ for $i \leq (r-2)/2$. The monomials dual to two of its sides are $\xi = x/y$ and $\eta = y^{r-2i}/z^i$.]

3 Calculate the regular subdivision of $\frac{1}{30}(2,3,25)$, whose junior simplex is given in the figure: this example should help clear up any lingering doubts about whether the constructions depend on things being coprime. It also



illustrates the notion of a *long side*.

4 Let $r = (ab-1)$; the Newton polygon of $1/r(a,1)$ has two nodes $(a,1)$ and $(1,b)$ spanning a basic cone. The dual basic cone is $\langle \xi, \eta \rangle$, and $\mathbb{C}^2_{\xi,\eta}$ parametrises systems of equations

$$x^a = \xi y, \quad y^b = \eta x, x^{a-1}y^{b-1} = \xi\eta.$$

Check that these define an A-cluster for any ξ, η. [Hint: write down a basis of $k[x,y]/I$ consisting of monomials not made redundant by the displayed equations. Each monomial belongs to an eigenspace of $\mathbb{Z}/r$, and you have to see that you get one in each.]

5 Draw the the junior simplex for $\frac{1}{17}(1,2,14)$ and $\frac{1}{17}(1,3,13)$ together with the lattice points. In each case, calculate the Newton polygons at each vertex e_i, and check that the vectors out of e_i satisfy the continued fraction rule. [Hint: for $\frac{1}{17}(1,3,13)$, draw a generous sized equilateral triangle with e_1 at the top, e_3 at the bottom left, and e_2 at the right. Draw the line through e_3 cutting the opposite side e_1e_2 1/4 of the way up, and distribute $(1,3,13),\dots,(4,12,1)$ along it, with the last close to the far edge. Since the vector from $(2,6,9)$ to e_1 is $(-15,6,9)$, you can trisect the line from e_1 to $(2,6,9)$, etc.]

6 For each of the two examples of Q. 5, write out the concatenated continued fraction and carry out the MMP and the knock-out competition to determine the regular subdivisions and their regular tesselations.

7 If you finished $\frac{1}{17}(1,3,13)$ in Q. 5–6, you found the meeting of champions given by the 3 lines $x^3 : y$, $z^4 : x$ and $y^3 : z^2$, enclosing a triangle of side 2. It has a single down triangle, with vertexes $(3,9,5)$, $(7,4,6)$, $(8,7,2)$. Check that its dual monomials are $\lambda = y^2z/x^2$, $\mu = z^3x/y^2$, $\nu = x^2y/z^3$. [Hint: there are two possible methods: either calculate $\langle(7,4,6),(8,7,2)\rangle^\perp$, etc. by writing down 2×2 minors, or parallel translate the sides by multiplying their equations by $(xyz)^i$, obtaining $x^3 : y \mapsto x^2 : y^2z$, etc.]

8 Write out the full set of equations of the corresponding cluster given by λ,μ,ν of Q. 7, and check that it is an A-cluster. [Hint: the λ equation is $y^2z = \lambda x^2$. Multiply the λ and μ equation together and cancel to get $x^3 = \mu\nu y$, etc. To show it's a cluster, as in Q. 4, you must show that $\mathcal{O}_Z$ has a basis made up of 17 monomials different in all the different character spaces: well, you don't need x^3 or y^2z, for a start, ...]

9 For r and s coprime consider the cyclic group $\mathbb{Z}/(rs) \subset \mathrm{SL}(3,\mathbb{C})$ generated by $\frac{1}{r}(1,-1,0)$ and $\frac{1}{s}(0,1,-1)$. Show that it has a long line, and calculate its regular subdivision. The case $r=5, s=12$ gives the figure: