MA505 Algebraic geometry, Miles Reid

Homework sheet 2

Resolution of quotient singularities $\frac{1}{r}(1,a)$

1 The quotient singularity $\frac{1}{2}(1,1)$ is the ordinary quadratic cone

$$X : (uw = v^2) \subset \mathbb{C}^3;$$

write $Y \to X$ for its "cylinder" resolution. Calculate the divisor of zeros on Y of each of $u = x^2$, $v = xy$, and $w = y^2$.

2 The quotient singularity $\frac{1}{r}(1,-1)$ is the hypersurface $X : (uw = v^r)$, where $u = x^r$, $v = xy$, and $w = y^r$. Resolve the singularity by successively blowing up the ambient $\mathbb{C}^3$. Check that this resolution is isomorphic to that given by Jung–Hirzebruch continued fractions.

3 Let $Y \to X$ be the resolution of Q. 2, with exceptional curves $E_1, E_2, \dots,$ E_{r-1}, and write E_0, E_r for the birational transform on Y of the u- and w-axes. Calculate the divisor of zeros on Y of each of u, v, w.

4 Recall the notation for the Jung–Hirzebruch resolution: we consider the lattice $L = \mathbb{Z}^2 + \mathbb{Z}\frac{1}{r}(1,a)$, and the Newton polygon of L intersect the positive quadrant; its successive vertexes along the boundary are $e_0, e_1, \dots, e_k, e_{k+1}$ with $e_i = (c_i, d_i)$ given by $e_0 = (0, r)$, $e_1 = (1, a)$ and

$$e_{i-1} + e_{i+1} = b_i e_i \quad \text{for } i = 1, \dots, k,$$

where $r/a = [b_1, \dots, b_k]$.

The resolution Y is covered by affine pieces $Y_i = \mathbb{C}^2$ with coordinates ξ_i, η_i with $\xi_i = x^{d_i}/y^{c_i}$ and $\eta_i = y^{c_{i+1}}/x^{d_{i+1}}$. Write $E_1, \dots, E_k$ for the exceptional curves of $Y \to X$ and E_0, E_{k+1} for the birational transforms of the coordinate axes.

Prove that the divisors of zeros on Y of the invariant monomials x^r and y^r are given by $\sum c_i E_i$ and $\sum d_i E_i$. [Hint: x^r and y^r are regular functions on each Y_i, so can be written in terms of ξ_i, η_i.]

5 For general $\frac{1}{r}(1,a)$, consider the resolution $Y \to X$ and the neighbourhood $S_i = Y_{i-1} \cup Y_i$ of the exceptional curve E_i. Prove that S_i is isomorphic to the cylinder resolution of the singularity $\frac{1}{b_i}(1,1)$, the cone over the rational normal curve of degree b. [Compare the glueing map $Y_{i-1} \dashrightarrow Y_i$ with that for the cone.]

6 Let $S \to T$ be the cylinder resolution of the singularity $\frac{1}{b}(1,1)$ and E its exceptional curve. Calculate the selfintersection $E^2 = -b$ of E in S. [Hint: show that the divisor of zeros on S of the invariant monomial $u = x^b$ is $bE_0 + E$, where E_0 is the coordinate axis. Then show $EE_0 = 1$, and use the fact that $E \operatorname{div}(f) = 0$ for any principal divisor $\operatorname{div}(f)$.]

7 If $Y \to X$ is a general Jung–Hirzebruch resolution, calculate the selfintersections of the exceptional curves E_i. [Hint: you can do this using Q. 6 + Q. 7, or slightly more directly using Q. 4.]

8 Prove the following: the expression

$$s = \frac{\mathrm{d}x}{x} \wedge \frac{\mathrm{d}y}{y}$$

is a rational differential 2-form on $\mathbb{C}^2$ having poles of order 1 along each of the coordinate axes. It is invariant under the group action $\frac{1}{r}(1,a)$. It defines a rational differential 2-form on the quotient $X = \mathbb{C}^2/(\mathbb{Z}/r)$ with poles of order 1 along the images of the coordinate axes. It also defines a rational differential 2-form on the resolution $Y \to X$ with poles of order 1 along $E_0, E_1, \dots, E_k, E_{k+1}$.

Therefore $K_Y \overset{\text{lin}}{\sim} \operatorname{div} s = -\sum_{i=0}^{k+1} E_i$.

9 Check the adjunction formula $K_Y E_i + E_i^2 = 2g(E_i) - 2 = 2$.

10 Prove that $K_Y \overset{\text{num}}{\sim} \sum_{i=1}^{k} (-1 + c_i + d_i) E_i$. In other words, the canonical class of Y can be written as a rational combination of the exceptional curves.

[Hint: there are two methods. (1) Check that this formula satisfies the adjunction formula, giving the right value of $K_Y E_i$ for each i. (2) Alternatively, Q. 8 gives an expression for K_Y involving the two coordinate axes E_0, E_{k+1}; you can get rid of these from the formula at the expense of passing to a multiple, by adding a principal divisor, considering $rK_Y + \operatorname{div}(x^r y^r)$. Then calculate $\operatorname{div}(x^r y^r)$ by Q. 4.]

11 The quantity $-1 + c_i + d_i$ calculated in Q. 10 is called the *discrepancy* of E_i in the resolution $Y \to X$. You can visualise it in the Newton polygon as the distance of the point $e_i = (c_i, d_i)$ under the diagonal of the unit square. Prove that if $a \neq r - 1$ then $-1 + c_i + d_i < 0$ for all i.