MA505 Algebraic geometry, Miles Reid

Homework sheet 1

Jung–Hirzebruch continued fractions

1 Reminder: the usual notation $\frac{1}{7}(1,4)$ stands for the cyclic group $G = \mathbb{Z}/7$ acting on $\mathbb{C}^2$ by $x \mapsto \varepsilon x$, $y \mapsto \varepsilon^4 y$, where ε is a primitive 7th root of 1. Show that the invariant monomials of the action are

$$u = x^7, \quad v_1 = x^3y, \quad v_2 = x^2y^3, \quad v_3 = xy^5, \quad v_4 = y^7;$$

prove that these generate the ring $k[x,y]^G$ of all G-invariant polynomials.

Prove that these generators satisfy the relations:

$$\operatorname{rank}\begin{pmatrix} u & v_1 & v_2 & v_3 \\ v_1^2 & v_2 & v_3 & v_4 \end{pmatrix} \le 1; \tag{1}$$

that is, 6 polynomial relations 2×2 minor $= 0$.

How to prove that these are the only relations? Equivalently, if I is the ideal of $k[u, v_1, v_2, v_3, v_4]$ generated by (1), prove that $k[u, v_1, v_2, v_3, v_4]/I \hookrightarrow k[x,y]$.

2 Carry out the same analysis for

$$\frac{1}{9}(1,5), \quad \frac{1}{2a-1}(1,a) \text{ for all } a \ge 2, \quad \frac{1}{8}(1,3).$$

3 $G = \mathbb{Z}/3$ acts on $\mathbb{R}^3$ by $x \mapsto y \mapsto z \mapsto x$. Express the action as a rotation about a suitable axis through 120°.

Now let $G = \mathbb{Z}/3$ act on $\mathbb{C}^3$ by $x \mapsto y \mapsto z \mapsto x$. Find (if possible) a G-invariant subspace $V \subset \mathbb{C}^3$ and a G-invariant complement W such that $V \oplus W = \mathbb{C}^3$.

Decompose $\mathbb{C}^3$ into irreducible representations of $\mathbb{Z}/3$. Show that with respect to suitable coordinates the action can be expressed in terms of the notation $\frac{1}{r}(a,b,c)$.

[Hint: Find eigenvalues and eigenvectors of the permutation matrix $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$.] It is an interesting experiment to try this question out on your algebraist friends, especially those interested in representation theory; you can make it even more provocative by throwing in some mysterious hints about the regular representation or Maschke's theorem.

4 Reminder: a typical Hirzebruch continued fraction is

$$[4,2,7] = 4 - \frac{1}{2-\frac{1}{7}} = 4 - \frac{7}{13} = \frac{45}{13}.$$

Express the following fractions as Hirzebruch continued fractions:

$$\frac{19}{12}, \quad \frac{39}{22}, \quad \frac{19}{7}, \quad \frac{n}{n-1}, \quad \frac{ab-1}{ab-a-1}.$$

Compute the following Hirzebruch continued fractions:

$$[5,2,3], \quad [2,6,3], \quad [3,3,5,2].$$

5 Calculate the matrix product

$$\begin{pmatrix} 0 & 1 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 7 \end{pmatrix}.$$

More generally, relate the computation of any Hirzebruch continued fractions $[b_1, \dots, b_k]$ to the matrix product

$$\begin{pmatrix} 0 & 1 \\ -1 & b_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ -1 & b_k \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

6 Reminder: The *Newton polygon* of the lattice $L = \mathbb{Z}^2 + \mathbb{Z} \cdot \frac{1}{r}(1,a)$ is the convex hull in $\mathbb{R}^2$ of $L \cap$ (positive quadrant). To calculate it, just write out all the lattice points in the unit square you can think of and draw the convex hull. For example, $\frac{1}{7}(1,2)$ gives the polygon with vertexes $(7,0),(1,2),(4,1),(7,0)$ (the denominator 1/7 is omitted for brevity).

Compute from first principles the Newton polygons for

$$\frac{1}{19}(1,12), \quad \frac{1}{39}(1,22), \quad \frac{1}{n}(1,1), \quad \frac{1}{n}(1,n-1), \quad \frac{1}{ab-1}(1,ab-a-1).$$

7 If $a,b < r$ and $ab \equiv 1$ modulo r, determine the relation between the Hirzebruch continued fraction expressions of r/a and r/b. Start by considering 17/5 and 17/7.

8 The Hirzebruch continued fractions of $\frac{r}{a}$ and its *inverse* $\frac{r}{r-a}$ are determined from one another by the following rule: wherever $\frac{r}{a} = [b_1, \dots, b_k]$ has an entry $b_i > 2$, the inverse $\frac{r}{r-a}$ has $b_i - 2$ repeated 2s, and vice-versa: whenever $\frac{r}{a}$ has a number $c - 2$ of repeated 2s (with $c \geq 3$), its inverse fraction $\frac{r}{r-a}$ has an entry c. This is represented by the *Riemenschneider staircase*:

$$\begin{array}{lcccc} & \multicolumn{4}{l}{\text{read down for } [2,2,4,2]} \\ & \downarrow & & & \\ \text{read across} \to & \bullet & \bullet & \bullet & \\ \text{for } [4,2,3] & & & \bullet & \\ & & & \bullet & \bullet \end{array} \tag{2}$$

with successive rows of length $b_i - 1$. If we write $c_i - 1$ for the height of the columns, then $\frac{r}{r-a} = [c_1, \dots, c_l]$. For example, $\frac{17}{5} = [4, 2, 3]$ gives the staircase (2) with rows of length $3, 1, 2$. The vertical steps have height $1, 1, 3, 1$, and $\frac{17}{12} = [2, 2, 4, 2]$.

Draw the Riemenschneider staircases for all the fractions of Q. 4, and check that the recipe works.