Algebraic Geometry

Exercises on graded rings

A. Elliptic curves

The first questions calculate the graded rings $R(E,kP)$, where E is an elliptic curve (that is, a curve of genus 1 with a marked point $P\in E$). You should use the RR formula

$$h^0(E,\mathcal{O}_E(D)) = \deg D \quad \text{for } D \text{ a divisor of } \deg > 0.$$

1. Let E be an elliptic curve and $P\in E$ a point. Consider the graded ring $R(E,P)=\bigoplus_{n\ge0}H^0(nP)$. Prove that $R(E,P)=k[x,y,z]/f$, where $\operatorname{wt}x,y,z=1,2,3$, and the relation f of degree 6 is of the form $z^2=y^3+ax^4y+bx^6$. [Hint: Choose x,y,z so that

$$H^0(E,\mathcal{O}_E(P))=\langle x\rangle\,,\quad H^0(E,\mathcal{O}_E(2P))=\left\langle x^2,y\right\rangle$$
$$\text{and}\quad H^0(E,\mathcal{O}_E(3P))=\left\langle x^3,xy,z\right\rangle.$$

Prove that x^2,y defines a 2-to-1 map $E\to\mathbb{P}^1$, so that x,y are algebraically independent. Prove that y has a pole at P of order 2, and z a pole of order 3. Write out the monomials in x,y,z of weighted degree 4 and prove that they form a basis of $H^0(E,\mathcal{O}_E(4P))$, and ditto for degree 5. Prove that there must be a relation in degree 6, and that it must involve y^3 and z^2. Finally, adjust $z\mapsto z+\alpha_3(x,y)$ and $y\mapsto y+\beta_2(x)$ to remove unwanted terms.]

2. The subring $R(E,2P)\subset R(E,P)$ is generated by $u_1=x^2,u_2=y,t=xz$ with the relation $t^2=u_1(u_2^3+au_2u_1^2+bu_1^3)$. This is the model of E as a double cover of $\mathbb{P}^1$ with 4 branch points, one at infinity ($u_1=0$).

3. $R(E,3P)$ is generated by $v_1=x^3,v_2=xy,v_3=z$ with the single relation $v_1v_3^2=v_2^3+av_1^2v_2+bv_1^3$. This is the model of E as a plane cubic with an inflexion at infinity (the line $v_1=0$ has inflexional tangent at $(0,0,1)$).

4. $R(E,4P)$ is generated by $w_1=x^4,w_2=x^2y,w_3=y^2,w_4=xz$ with two relations $w_1w_3=w_2^2$ and $w_4^2=w_2w_3+aw_1w_2+bw_1^2$. This is the model of E as an intersection of two quadrics in $\mathbb{P}^3$ with the plane $w_1=0$ having 4-fold intersection at $(0,0,1,0)$.

5. $R(E,5P)$ is generated by $\xi_1 = x^5, \xi_2 = x^3y, \xi_3 = xy^2, \xi_4 = x^2z, \xi_5 = yz$ with 5 quadratic relations

$$\operatorname{rank}\begin{pmatrix}\xi_1 & \xi_2 & \xi_4\\ \xi_2 & \xi_3 & \xi_5\end{pmatrix} \le 1, \quad \text{and} \quad \begin{pmatrix}\xi_4^2 = \xi_2\xi_3 + a\xi_1\xi_2 + b\xi_1^2\\ \xi_4\xi_5 = \xi_3^2 + a\xi_2^2 + b\xi_1\xi_2\end{pmatrix}.$$

Show how to view these equations as the 5 Pfaffians of a 5×5 skewsymmetric matrix. This is the model of E as a curve of degree 5 in $\mathbb{P}^4$ obtained as a $\mathbb{P}^4$ section of the Grassmann variety $\operatorname{Grass}(2,5) \subset \mathbb{P}^9$.

6. Let C be a hyperelliptic curve of genus g; we say that P is a *Weierstrass point* if P is a branch point of the double cover $C \to \mathbb{P}^1$, that is, $2P \in g^1_2$.

Show that $R(C,P) = k[x,y,z]/(F_{4g+2})$ with $\deg x = 1$, $\deg y = 2$ and $\deg z = 2g+1$. Show that if z is chosen so that $F_{4g+2} = z^2 - f_{2g+1}(x^2,y)$ (by completing the square) then $\operatorname{div} z$ is the sum of the remaining $2g+1$ Weierstrass points.

Calculate $R(C,2P)$ by eliminating x from $R(C,P) = k[x,y,z]/f_{4g+2}$. [Hint: Compare Q. 1, Q. 2 above.]

7. If C is a hyperelliptic curve of genus $g=4$ and P a Weierstrass point, compute $R(C,3P)$.

B. Subcanonical curves with $K_C = (2g-2)P$

These examples are entertaining and have many applications to surfaces of general type. Consider a nonsingular curve C marked with a point P such that $K_C = (2g-2)P$. The Riemann–Roch theorem for curves

$$h^0(C,nP) = 1 - g + n + h^0(C, K_C - nP)$$

and Clifford's theorem $h^0(nP) \le \left[\frac{n}{2}\right] + 1$ determines the Hilbert function $P_n = h^0(nP)$ of $R(C,nP)$ up to a finite number of choices. Write $P(t) = \sum_{n=0}^{\infty} P_n t^n$.

1. If $g=2$ then C is always hyperelliptic, and $K_C = 2P$ for any branch point P. Check that

$$R(C,P) = k[x,y,z]/(z^2 = f_{10}(x,y)), \quad \text{where } \deg x,y,z = 1,2,5$$

and $f_{10}(x,y) = F_5(x^2,y)$ is the equation of the 5 other branch points (see Ex. A.6 above).

2. If $g = 3$, there are exactly two cases for $K_C = 4P$, giving $C_{14} \subset \mathbb{P}(1,2,7)$ if $2P = g^1_2$ and $C_{12} \subset \mathbb{P}(1,3,4)$ otherwise. In the latter case, show how to deduce the form of $R(C,P)$ from the model of C as a plane quartic touching a line to order 4 at P.

3. If $g = 4$, there are exactly 3 cases for $K_C = 6P$, of which two are hypersurfaces, and one is the codimension 2 complete intersection $C_{10,12} \subset \mathbb{P}(1,4,5,6)$. Prove that every C,P with $K_C = 6P$ has ring given by these constructions. Count moduli.

4. If $K_C = (2g-2)P$ then $(g-1)P$ is a theta characteristic. It is known that the parity of $h^0((g-1)P)$ (whether it is even or odd) is constant under deformation of C and P. Use this to determine the possible specialisations between the three families of curves in Ex. 3. [Hint: The parity argument does not exclude that trigonal curves $C_{15} \subset \mathbb{P}(1,3,5)$ could specialise to a hyperelliptic curve $C_{18} \subset \mathbb{P}(1,2,9)$. If this specialisation happens, you can see it by equations.]

5. If $g = 5$ and $K_C = 8P$ (and C is not hyperelliptic), prove that a priori, the possible Hilbert functions for $R(C,P)$ are the following 3 cases:

$$1 + t + t^2 + t^3 + t^4 + 2t^5 + 3t^6 + 4t^7 + 5t^8 + \sum_{n\ge 9}(n-4)t^n =$$
$$= \frac{1 - t^{12} - t^{13} - t^{14} - t^{15} - t^{16} + t^{19} + t^{20} + t^{21} + t^{22} + t^{23} - t^{35}}{(1-t)(1-t^5)(1-t^6)(1-t^7)(1-t^8)}$$

or

$$1 + t + t^2 + t^3 + 2t^4 + 2t^5 + 3t^6 + 4t^7 + 5t^8 + \sum_{n\ge 9}(n-4)t^n =$$
$$= \frac{1 - t^{12} - t^{14} + t^{26}}{(1-t)(1-t^4)(1-t^6)(1-t^7)}$$

or

$$1 + t + t^2 + 2t^3 + 2t^4 + 3t^5 + 3t^6 + 4t^7 + 5t^8 + \sum_{n\ge 9}(n-4)t^n =$$
$$= \frac{1 - t^6 - t^{10} - t^{14} - t^{18} - t^{22} + t^{13} + t^{17} + t^{21} + t^{25} + t^{29} - t^{35}}{(1-t)(1-t^5)(1-t^6)(1-t^7)(1-t^8)}$$

6. In question 5, show that the second case gives the codimension 2 complete intersection $C_{12,14} \subset \mathbb{P}(1,4,5,7)$ and the first gives the Pfaffian variety in $\mathbb{P}(1,5,6,7,8)$ corresponding to a skewsymmetric matrix of weighted degrees

$$\begin{pmatrix} 3 & 4 & 5 & 6 & 7 \\ & 5 & 6 & 7 & 8 \\ & & 7 & 8 & 9 \\ & -\mathrm{sym} & & 9 & 10 \\ & & & & 11 \end{pmatrix}$$

7. In question 5, show that the third case is not possible. [Hint: Let x, y, z be the generators of $R(C, P)$ in degrees $1, 3, 5$. Determine the behaviour of x, y, z at P. There is a relation in degree 6, that we can write with unknown coefficients. Depending on which is zero derive a contradiction to the assumptions that x is a nonzero divisor or y has pole of order 3 at P.]

C. Regular surface of general type

The following questions treat the canonical ring $R(X, K_X)$ of a regular surface of general type. You need to use the RR formula

$$P_n = \begin{cases} 1 & \text{if } n = 0, \\ p_g & \text{if } n = 1, \\ 1 + p_g + \binom{n}{2} & \text{if } n \geq 2. \end{cases}$$

1. Show that $R(X, K_X)$ can be a hypersurface or a complete intersection in projective space or weighted projective space in the following cases:

1. $p_g = 1$, $K^2 = 1$;
2. $p_g = 2$, $K^2 = 1$;
3. $p_g = 2$, $K^2 = 2$;
4. $p_g = 3$, $K^2 = 2$;
5. $p_g = 3$, $K^2 = 3$;
6. $p_g = 3$, $K^2 = 4$;
7. $p_g = 4$, $K^2 = 5$;

8. $p_g = 4$, $K^2 = 6$;
9. $p_g = 5$, $K^2 = 8$;
10. $p_g = 5$, $K^2 = 9$;
11. $p_g = 6$, $K^2 = 12$;
12. $p_g = 7$, $K^2 = 16$;

2. Show that $R(X, K_X)$ can be Pfaffian in the following cases:

1. $p_g = 3$, $K^2 = 5$;
2. $p_g = 4$, $K^2 = 7$;
3. $p_g = 5$, $K^2 = 10$;
4. $p_g = 6$, $K^2 = 11$;
5. $p_g = 6$, $K^2 = 13$;
6. $p_g = 6$, $K^2 = 14$ (note: 7×7);

D. Rings and geometry

1. Suppose that $f_{2g+2}(x)$ has degree equal to $2g + 2$ and distinct roots, and let $C_0 : y^2 = f_{2g+2}$. Show how to resolve the singularity at $(0, 1, 0)$ of the $z^{2g}y^2 = f_{2g+2}(x, z)$ by successive birational changes of $\mathbb{P}^2$, leading to a nonsingular model $C \subset \mathbb{F}(0, g + 1)$ in the scroll $\mathbb{F}_{g+1}$.

2. Let C be a nonsingular hyperelliptic curve of genus g, with $D = g^1_2$. We know that $R(C, D) = k[x_1, x_2, y]/(y^2 = f_{2g+2}(x,x_2)$, and we have an interpretation of the image of the projective embedding of C into $\mathbb{P}^{g+2}$ by $\varphi_{(g+1)D}$ (as the intersection of the cone $\overline{\mathbb{F}}_{g+1}$ with a quadric).

Give an analogous interpretation of the image of C by φ_{aD} for $a \geq g+1$.

3. Consider surfaces of general type with $p_g = 4, K^2 = 6$ for which the canonical ring can be a complete intersections $X_{3,4} : (f_3 = g_4 = 0) \subset \mathbb{P}(1^4, 2)$. Show that this model can be nonsingular even if f_3 does not involve y; in this case the map $X \to \mathbb{P}^3$ is a double cover of a cubic surface.

Show how to construct a 1-parameter family X_λ so that $\lambda = 0$ gives a double cover of a cubic and $\lambda \neq 0$ a sextic with a singular curve.

4. Let C be a curve of genus 4 and D a divisor such that $2D = K_C$ (so that in particular, $\deg D = 3$) and $h^0(D) = 2$. Show that one of the following holds:

1. $|D|$ is a g^1_3 with no base point;
2. C is hyperelliptic and $|D| = P + g^1_2$ where the fixed point P is a Weierstrass point.

In case (1), prove that $R(C, D) = k[x_1, x_2, y]/f_6$ where $\deg x_i = 1$, $\deg y = 2$, and $f = y^3 + a_2y^2 + b_4y + c_6$. Show that the canonical ring $R(C, K_C) = R(C, 2D)$ is a complete intersection of type $(2, 3)$ where the quadric is a cone.

In case (2), prove that $R(C, D) = k[x_1, x_2, y, z]/(g_3, f_6)$ where $\deg z = 3$ and $g_3 = x_1y + d_3$. [Hint: use Worksheet 3, Q. 7.]

5. Use the previous question to construct a 1-parameter family C_λ, D_λ of curves of genus 4 with a marked g^1_3 such that $\lambda = 0$ corresponds to a hyperelliptic curve and $\lambda \neq 0$ a nonhyperelliptic curve.

6. Consider surfaces of general type with $p_g = 3, q = 0$ and $K^2 = 3$. It is known that the general curve $C \in K_C$ is a nonsingular curve, and that $|K_X|_{|C}$ is a divisor with $2D = K_C$ and $h^0(D) = 2$. Prove that $R(X, K_X)$ is either (1) $k[x_0, x_1, x_2, y]/f_6$ or (2) $k[x_0, x_1, x_2, y, z]/(g_3, f_6)$. [Hint: use Q. 4 and the hyperplane section principle.]

7. Consider surfaces of general type with $p_g = 3, q = 0$ and $K^2 = 3$. Using the form $R(X, K_X) = k[x_0, x_1, x_2, y, z]/(g_3, f_6)$, prove the following:

1. there exist surfaces X for which $|K_X|$ has a base point;
2. these surfaces deform to surfaces for which $|K_X|$ has no base point.
3. (harder) the 1-canonical map $\varphi_X \colon X \dashrightarrow \mathbb{P}^2$ of such a a surface is a double cover with branch locus consisting of a line L and a curve C_9 of degree 9 having a triple tacnode tangent to L. [Hint: eliminate y from $x_1y = d_3$, $z^2 = f_6(x_{,i}, y)$.]

   (This question is based on work of Horikawa and Iliev around 1980.)