Kronecker double series and the dilogarithm

Andrey Levin

Abstract

In this article we give an explicit expression for the value of a certain Kronecker double series at any point of complex multiplication as a sum of dilogarithms whose arguments are values of some modular unit of higher level at the corresponding points. This result can be interpreted in the spirit of the Zagier conjecture. The special value of the Kronecker double series is equal to the value of the partial ζ-function of an ideal class for an order in an imaginary quadratic field. The values of the above mentioned modular unit belong to ray class field corresponding to this order. Thus we get an explicit formula for the value of a partial ζ-function at $s = 2$ as a combination of dilogarithms of algebraic numbers.

1 Introduction

1.1 Modular part

1.1.1 We start by fixing notation and recalling some standard facts about modular curves. We set $\mathcal{H} = \{\tau \in \mathbb{C} \mid \Im(\tau) > 0\}$. Then a matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z})$ acts on $\mathcal{H}$ by $M(\tau) = \frac{a\tau+b}{c\tau+d}$. Write $L = L_\tau$ for the lattice in $\mathbb{C}$ generated by τ and 1, and $E = E_\tau$ for the corresponding elliptic curve $\mathbb{C}/L_\tau$. The matrix M defines a map of lattices $M^\tau \colon L_{M(\tau)} \to L_\tau$ given by $w \to (c\tau+d)w$ and an isogeny $M_\tau \colon E_\tau \to E_{M(\tau)}$ given by $\xi \to (ad-bc)(c\tau+d)^{-1}\xi$. If τ is a fixed point of M, then M_τ is a map of the curve E_τ onto itself. In this case we omit the subscript τ in our notation.

A point $\xi \in E_\tau$ defines a character χ_ξ of the lattice L_τ given by $\chi_\xi(w) = \exp\left(\frac{2\pi i(w\overline{\xi}-\overline{w}\xi)}{\tau-\overline{\tau}}\right)$. This pairing is $\mathrm{GL}_2(\mathbb{Z})$-invariant: $\chi_{M(\xi)}(w) = \chi_\xi(M(w))$.

Definition 1.1.2 The *second Kronecker double series* $\mathcal{K}_2(\xi;\tau)$ is the C^∞-function on $\mathbb{C} \times \mathcal{H}$ defined by the convergent series

$$\mathcal{K}_2(\xi;\tau) = \left(\frac{\tau-\overline{\tau}}{2\pi i}\right)^2 {\sum_{w\in L}}' \frac{\chi_\xi(w)}{|w|^4}\,, \tag{1}$$

where, as usual, $\sum'$ denotes the sum over $L \setminus \{0\}$.

One checks that this function is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{C} \times \mathcal{H}$ defined above.

1.1.3 The Weierstrass $\wp$-function is the elliptic function defined by the convergent series

$$\wp(\xi;\tau) = \frac{1}{\xi^2} + \sum_{w\in L_\tau}{}' \Big(\frac{1}{(w+\xi)^2} - \frac{1}{w^2}\Big).$$

1.2 Dilogarithms

Definition 1.2.1 The *Euler dilogarithm* $Li_2(z)$ is the multivalued analytic function on $\mathbb{P}^1 \setminus \{0,1,\infty\}$ defined as the analytic continuation of the series $\sum_{j\geq 1} \frac{z^j}{j^2}$ (which converges for $|z| < 1$).

Definition 1.2.2 The formula

$$D_2(z) = \Im(Li_2(z)) + \arg(1-z)\cdot \log|z| \quad \text{for } z \notin \{0,1,\infty\},$$
$$D_2(0) = D_2(1) = D(\infty) = 0.$$

defines a single-valued real function on $\mathbb{P}^1$ that we call the *Bloch–Wigner dilogarithm.* It is continuous on $\mathbb{P}^1$ and smooth on $\mathbb{P}^1 \setminus \{0,1,\infty\}$

We can extend the function D_2 byt linearity to a function on the $\mathbb{Q}$-vector space $\mathbb{Q}[\mathbb{C}]$.

1.2.3 Define a map

$$\delta\colon \mathbb{Q}[\mathbb{C}\setminus\{0,1\}] \to \textstyle\bigwedge^2 \mathbb{C}^* \quad \text{by the formula} \quad [x] \to x \wedge (1-x).$$

1.3 Results

Main Theorem 1.3.1 *Let τ be a fixed point of $M = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \neq 0, 1$. Set $m = \det M$ and $n = \det(M-1)$. Then*

$$\begin{aligned} &-(m+1)(n+1)\frac{c(\tau-\overline{\tau})}{i}\mathcal{K}_2(0;\tau) \\ &\quad = 4mnD_2\Big(\frac{ad-bc}{c\tau+d}\Big) + \sum_{\substack{\alpha\in\mathrm{Ker}(M)\setminus 0\\ \beta\in\mathrm{Ker}(M-1)\setminus 0}} \sum_{k=1}^{m}\sum_{l=1}^{n} D_2\Big(\frac{\wp(k\alpha+l\beta)-\wp(\alpha)}{\wp(\beta)-\wp(\alpha)}\Big); \end{aligned}$$

the arguments of the dilogarithms are in the kernel of δ (see 1.2.3):

$$\delta\Big(4mn\times[c\tau+d] + \sum_{\alpha,\beta,k,l}\Big(\frac{\wp(k\alpha+l\beta)-\wp(\alpha)}{\wp(\beta)-\wp(\alpha)}\Big)\Big) = 0 \in \wedge^2(\mathbb{C}^*\otimes_{\mathbb{Z}}\mathbb{Q}).$$

1.3.2 Our proof is based on introducing a new function $\mathcal{L}_{1,1}$, which we call the *elliptic* $(1,1)$*-logarithm* and define in Section 2. The Main Theorem follows from Theorems A and B below.

Theorem A 1.3.3 *Let τ, M, m and n be as in the Theorem 1.3.1. Then*

$$\frac{c(\tau-\overline{\tau})}{4i}\frac{(m+1)(n+1)}{mn}\mathcal{K}_2(0;\tau) = -D_2\left(\frac{ad-bc}{c\tau+d}\right) + \sum_{\substack{\alpha\in\operatorname{Ker} M\backslash 0\\ \beta\in\operatorname{Ker}(M-1)\backslash 0}} \mathcal{L}_{1,1}(\alpha,\beta,0). \quad (2)$$

Theorem B 1.3.4 *Let α and β be two distinct nontrivial torsion points on an elliptic curve E_τ, say $m\alpha = n\beta = 0$ for some $m, n \in \mathbb{N}$. Then*

$$\sum_{k=1}^{m}\sum_{l=1}^{n} D_2\Big(\frac{\wp(k\alpha+l\beta)-\wp(\alpha)}{\wp(\beta)-\wp(\alpha)}\Big) = -2mn\big(\mathcal{L}_{1,1}(\alpha,\beta,0)+\mathcal{L}_{1,1}(-\alpha,\beta,0)\big). \quad (3)$$

1.4 Generalities and structure of the article

1.4.1 The fact that the value of the Kronecker double series $\mathcal{K}_2$ at a CM point can be expressed as a combination of dilogarithms is not new. It can be derived from a result of Deninger as follows. A CM point τ defines a ring R of endomorphisms of the correspondent lattice L_τ; this ring is an order in some imaginary quadratic extension $F = R \otimes_{\mathbb{Z}} \mathbb{Q}$ of $\mathbb{Q}$. Extend the field F by the value of the j-invariant at the point τ. Deninger [D1] constructed an element in the third algebraic K-group of the field $F(j(\tau))$ that the regulator maps to the value of $\mathcal{K}_2$ at τ. We know by Suslin and Bloch that the regulator on K_3 of a number field is given by the Bloch–Wigner dilogarithm. Hence we can conclude that the value of the $\mathcal{K}_2$ equals a combination of values of the dilogarithm at numbers in $F(j(\tau))$. The arguments of the dilogarithm in our formula belong to some extension of the field $F(j(\tau))$, but the set of them is Galois invariant.

Our proof is in some sense parallel to Deninger's construction.

1.4.2 We can also interpret the Main Theorem independently of algebraic K-theory. The value of $\mathcal{K}_2(0;\tau)$ for a CM point τ is (a rational multiple of) the value at $s = 2$ of the partial zeta function $\zeta_{F,\mathcal{A}}(s) = \sum_{\mathfrak{a}\in\mathcal{A}} N(\mathfrak{a})^{-s}$ for some ideal class $\mathcal{A}$. The fact that this value can be written as a combination of dilogarithms with arguments belonging to the associated class field over F is a special case of the refined version of Zagier's conjecture that the values of all partial zeta functions at arbitrary integer argument $s = m$ can be expressed in terms of m-logarithms.

For the case at hand, the same result, of course, also follows from the theorems of Deninger (existence of elements in K_2 with required values of regulator) and Bloch–Suslin (structure of K_2 and description of the regulator from K_2). Our proof, as well as giving an explicit formula, also has the advantage of avoiding algebraic K-theory. It is possible in principle that this method could be applied for higher values of m where it is not known that the regulator may be expressed in terms of polylogarithms.

1.4.3 Theorem A reflects a general phenomenon. Another reflection of this phenomenon is the following fact. For any elliptic modular curve over $\mathbb{Q}$ the value of its L-function at $s = 2$ can be expressed as a combination of the values of a special function (Goncharov's elliptic $(1, 2)$-logarithm [G2]), a "relative" of our elliptic $(1, 1)$-logarithm. On the other hand, for a CM-curve the value of the L-function is equal to a combination of the values of a certain Kronecker double series [D1]. Therefore this Kronecker double series must be equal to a combination of values of the elliptic $(1, 2)$-logarithm.

1.4.4 The paper is organized as follows. Section 2 defines the elliptic $(1, 1)$-logarithm for an arbitrary elliptic curve and studies its elementary properties. In Section 3 we realize the Kronecker series as an integral over the square of the elliptic curve, and, for a curve with complex multiplication, reduce this integral to an integral over the elliptic curve itself. In Section 4 we compare the elliptic $(1, 1)$-logarithm and the dilogarithm. In Section 5 we check that δ vanishes on the arguments of the dilogarithms on the right-hand side of the Main Theorem, thus completing its proof. In the final Section 6 we prove a more general formula, relating values of $\mathcal{K}_2$ at torsion points to the dilogarithm.

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2 The elliptic $(1, 1)$-logarithm

In this section we define and study properties of the elliptic $(1, 1)$-logarithm. Some motivation for considering this function is Goncharov's integral representation of the Bloch–Wigner dilogarithm. This representation uses a special case of a very general differential operator, that we call A_*.

2.1 The operations A_n

Definition 2.1.1 Let $\varphi_1, \ldots, \varphi_n$ be smooth functions on a complex variety X. We set

$$A_n(\varphi_1, \ldots, \varphi_n) = \mathrm{Alt}_n \left(\sum_{j=0}^{n-1} (-1)^j \varphi_1 \, \partial\varphi_2 \wedge \cdots \wedge \partial\varphi_{n-j} \wedge \overline{\partial}\varphi_{n-j+1} \wedge \cdots \wedge \overline{\partial}\varphi_n \right), \quad (4)$$

where Alt_n denotes alternation under the symmetric group S_n, with a factor of $1/(n!)$:

$$\mathrm{Alt}_n(F(x_1, \ldots, x_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \mathrm{sign}(\sigma) F(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

Remark 2.1.2 If $\varphi_j = \log|f_j|^2$ for *analytic* functions f_j, then A_n is the so-called Beilinson–Deligne product of the f_j, up to a factor 2 for odd n and $2i$ for even n.

Remark 2.1.3 The (p,q)-component of A_n equals $(-1)^q \frac{p!q!}{(p+q)!}$ multiplied by the (p,q)-component of $\mathrm{Alt}_n(\varphi_1 \, \mathrm{d}\varphi_2 \wedge \cdots \wedge \mathrm{d}\varphi_n)$.

An important property of the operations A_n is the following:

Proposition 2.1.4 *For $n > 1$*

$$\mathrm{d}A_n(\varphi_1, \ldots, \varphi_n) = \partial\varphi_1 \wedge \partial\varphi_2 \wedge \cdots \wedge \partial\varphi_n + (-1)^{n-1}\overline{\partial}\varphi_1 \wedge \overline{\partial}\varphi_2 \wedge \cdots \wedge \overline{\partial}\varphi_n + \sum_{j=1}^{n} (-1)^j \overline{\partial}\partial \, \varphi_j \wedge A_{n-1}(\varphi_1, \ldots, \hat{\varphi}_j, \ldots, \varphi_n). \quad (5)$$

The proof is a straightforward computation.

2.2 Goncharov's integral representation of the Bloch–Wigner dilogarithm

Lemma 2.2.1 (Goncharov [G1]) *The value of the Bloch–Wigner dilogarithm at a point $a \neq 0, 1, \infty$ is equal to the following convergent integral:*

$$\frac{1}{4\pi} \int_{\mathbb{P}^1} A_3(\log|z|^2, \log|1-z|^2, \log|a-z|^2).$$

Proof Since we are integrating over a curve, only the $(1,1)$-component contributes. Thus we can replace A_3 by

$$-\frac{1}{6}\Big(\log|z|^2 \mathrm{d}\log|1-z|^2 \wedge \mathrm{d}\log|a-z|^2 \\ - \log|1-z|^2 \mathrm{d}\log|z|^2 \wedge \mathrm{d}\log|a-z|^2 \\ + \log|a-z|^2 \mathrm{d}\log|z|^2 \wedge \mathrm{d}\log|1-z|^2\Big).$$

We first prove that the integral converges. The integrand is smooth outside 0, 1, a and ∞. Let (r,φ) be a polar coordinate system near one of the first three points; any term of the integrand is asymptotic to one of $\log|r|rdr \wedge \mathrm{d}\varphi$ or $r^{-1}rdr \wedge \mathrm{d}\varphi$, and is integrable. As A_3 is trilinear and totally antisymmetric, we can replace the three arguments $(\log|z|^2, \log|1-z|^2, \log|a-z|^2)$ by

$$\big(\log|z|^2,\ \log|1-z|^2 - \log|z|^2,\ \log|a-z|^2 - \log|z|^2\big) = \\ \big(\log|z|^2,\ \log|1-z^{-1}|^2,\ \log|az^{-1}-1|^2\big);$$

the convergence at ∞ can be checked for these by the same considerations.

By Stokes's formula we reduce the integral to

$$-\frac{1}{16\pi}\int\Big(\log|z|^2 \mathrm{d}\log|1-z|^2 \wedge \mathrm{d}\log|a-z|^2 \\ - \log|1-z|^2 \mathrm{d}\log|z|^2 \wedge \mathrm{d}\log|a-z|^2\Big).$$

The $(1,1)$-component of $\mathrm{d}\varphi_1 \wedge \mathrm{d}\varphi_2$ is the negative of the $(1,1)$-component of $(\partial-\overline{\partial})\varphi_1 \wedge (\partial-\overline{\partial})\varphi_2$. Hence the integral is equal to

$$\frac{1}{16\pi}\int\Big(\log|z|^2(\partial-\overline{\partial})\log|1-z|^2 - \log|1-z|^2(\partial-\overline{\partial})\log|z|^2\Big) \\ \wedge (\partial-\overline{\partial})\log|a-z|^2.$$

An easy calculation shows that

$$dD_2(z) = -i\Big(\log|z|(\partial-\overline{\partial})\log|1-z| - \log|1-z|(\partial-\overline{\partial})\log|z|\Big).$$

Therefore the integral is equal to

$$\frac{i}{4\pi}\int \mathrm{d}D_2(z) \wedge (\partial-\overline{\partial})\log|a-z|^2 = \frac{1}{4\pi i}\int D_2(z)\mathrm{d}(\partial-\overline{\partial})\log|a-z|^2.$$

As $\mathrm{d}(\partial-\overline{\partial})\log|a-z|^2 = 2\overline{\partial}\partial\log|a-z|^2 = 4\pi i(\delta_a(z)-\delta_\infty(z))$ and $D_2(\infty)=0$, this completes the proof. $\square$

Remark 2.2.2 It follows from the proof that the integral representation

$$\sum_{p\in X}\operatorname{ord}_p(g)D_2(f(p)) = \frac{1}{4\pi}\int_X A_3(\log|f|^2, \log|1-f|^2, \log|g|^2) \tag{6}$$

holds for any two meromorphic functions f and g on a compact curve X.

Remark 2.2.3 For any curve C

$$\int_C A_3(\varphi_1,\varphi_2,\varphi_3) = \int_C \frac{1}{2}\varphi_1\, \mathrm{d}\varphi_2 \wedge \mathrm{d}\varphi_3,$$

so that this integral is zero if $\varphi_j = \text{const}$ for some j.

Lemma 2.2.4 *For any three distinct points $a, b, c \in \mathbb{C}$, we have*

$$\int_{\mathbb{P}^1} A_3\big(\log|z-a|^2, \log|z-b|^2, \log|z-c|^2\big) = 4\pi D_2\Big(\frac{c-a}{b-a}\Big). \tag{7}$$

Proof We write r for the ratio $\frac{c-a}{b-a}$ and make the change of variable $x = \frac{z-a}{b-a}$. Then

$$\log|z-a|^2 = \log|x|^2 + \log|b-a|^2, \quad \log|z-b|^2 = \log|x-1|^2 + \log|b-a|^2$$
$$\text{and} \quad \log|z-c|^2 = \log|x-r|^2 + \log|b-a|^2.$$

By the preceding remark, any terms that include the constant $\log|b-a|^2$ vanish. This reduces the integral to that of Lemma 2.2.1. □

Remark 2.2.5 The integrand in (7) vanishes formally by antisymmetry if any of a, b and c are equal (we say formally because we are not really allowed to multiply 1-forms if their singularities coincide). If only two of a, b, c are equal, then the r.-h.s. of (7) is also zero, since $D_2(0) = D_2(1) = D_2(\infty) = 0$. Along the triple diagonal the expression in (7) has a discontinuity. However, it is continuous on the blowup of $\mathbb{C}^3$ along the triple diagonal.

2.3 The elliptic $(1,1)$-logarithm

In this section we define a real-valued function $\mathcal{L}_{1,1}(\alpha,\beta,\gamma;\tau)$, called the *elliptic* $(1,1)$*-logarithm*, on the third power of an elliptic curve (more precisely, on the fibered third power of the universal elliptic curve over the modular curve), which is invariant under the diagonal action of the elliptic curve by translations and antisymmetric under permutations of the variables.

2.3.1 A natural generalization of the function $\log|x-a|^2$ on $\mathbb{P}^1$ to an elliptic curve E_τ is the *Kronecker double series*

$$\mathcal{L}_1(\xi;\tau) := \log\left|\widetilde{\theta}(\xi,\tau)\right|^2 + \frac{1}{2}\frac{2\pi i}{\tau-\overline{\tau}}(\xi-\overline{\xi})^2 = -\frac{\tau-\overline{\tau}}{2\pi i}\sum_{e\,w\in L}' \frac{\chi_\xi(w)}{|w|^2}$$

where $\sum_e$ denotes Eisenstein summation (see Weil [W]), and

$$\widetilde{\theta}(\xi,\tau) = \frac{\theta(\xi,\tau)}{\eta(\tau)} = q^{1/12}(z^{\frac{1}{2}} - z^{-\frac{1}{2}})\prod_{j=1}^{\infty}(1-q^j z)(1-q^j z^{-1}).$$

$q = \exp(2\pi i\tau)$ (or, more precisely, $q^{1/12} = \exp(\frac{1}{6}\pi i\tau)$), and $z = \exp(2\pi i\xi)$ (or, more precisely, $z^{\pm\frac{1}{2}} = \exp(\pm\pi i\xi)$). The notation $\mathcal{L}_1$ for this function is not standard. It is meant to emphasize that this function is the elliptic 1-logarithm.

2.3.2 The function $\mathcal{L}_1$ is the Green function for the operator $\overline{\partial}\partial$ on an elliptic curve:

$$\overline{\partial}\partial(\mathcal{L}_1(\xi;\tau)) = 2\pi i\delta_0 + \frac{2\pi i}{\tau-\overline{\tau}}\mathrm{d}\xi\wedge\mathrm{d}\overline{\xi}.$$

Here δ_0 denotes the delta function at zero.

2.3.3 The function $\mathcal{L}_1$ satisfies the distribution relation

$$\sum_{\alpha:M_\tau(\alpha)=\beta}\mathcal{L}_1(\alpha;\tau) = \mathcal{L}_1(\beta;M(\tau)).$$

2.3.4 Any elliptic function f with $\mathrm{ord}_\beta(f)=0$ has a "theta decomposition"

$$\log|f(\xi)|^2 - \log|f(\beta)|^2 = \sum \mathrm{ord}_\alpha(f)\Big(\mathcal{L}_1(\xi-\alpha) - \mathcal{L}_1(\beta-\alpha)\Big). \qquad (8)$$

Thus the natural elliptic generalization of the function $D_2\left(\frac{c-a}{b-a}\right)$ is the following

Definition 2.3.5 The *elliptic* $(1,1)$*-logarithm* $\mathcal{L}_{1,1}(\alpha,\beta,\gamma;\tau)$ is the *convergent* integral:

$$\frac{1}{4\pi}\int_{E_\tau} A_3\Big(\mathcal{L}_1(\xi-\alpha),\mathcal{L}_1(\xi-\beta),\mathcal{L}_1(\xi-\gamma)\Big).$$

The convergence of the integral can be checked by the same consideration as for $D_2(a)$.

Remark 2.3.6 As above, only the $(1,1)$-component of the integrand gives a nontrivial contribution, so

$$\mathcal{L}_{1,1}\big(\alpha,\beta,\gamma;\tau\big) = -\frac{1}{8\pi}\int_{E_\tau} \mathcal{L}_1(\xi-\alpha)\,\mathrm{d}\mathcal{L}_1(\xi-\beta)\wedge \mathrm{d}\mathcal{L}_1(\xi-\gamma).$$

We give another definition of this function based on Fourier expansions:

Lemma 2.3.7 *Considered as a distribution, the function* $\mathcal{L}_{1,1}(\alpha,\beta,\gamma;\tau)$ *is equal to the following series*

$$\frac{(\tau-\overline{\tau})^2}{16\pi^2 i}\times \sum_{\substack{w_1+w_2+w_3=0\\ w_i\neq 0}} \frac{\chi_\alpha(w_1)\chi_\beta(w_2)\chi_\gamma(w_3)(w_2\overline{w}_3-\overline{w}_2 w_3)}{|w_1|^2|w_2|^2|w_3|^2}.$$

A series of this kind was introduced by Deninger [D2]. In contrast to his case, however, our series is not absolutely convergent.

Proof By the preceding remark, we can compute the following integral

$$\begin{aligned}
&-\frac{1}{8\pi}\int -\frac{\tau-\overline{\tau}}{2\pi i}\times\sum_{w_1\neq 0}\frac{\chi_{\xi-\alpha}(w_1)}{|w_1|^2}\times\\
&\quad\Big(\sum_{w_2\neq 0}\frac{\chi_{\xi-\beta}(w_2)(\overline{w}_2\mathrm{d}\xi-w_2\mathrm{d}\overline{\xi})}{|w_2|^2}\Big)\wedge\Big(\sum_{w_3\neq 0}\frac{\chi_{\xi-\gamma}(w_3)(\overline{w}_3\mathrm{d}\xi-w_3\mathrm{d}\overline{\xi})}{|w_3|^2}\Big)\\
&=-\frac{1}{8\pi}\frac{\tau-\overline{\tau}}{2\pi i}\int\sum_{w_i\neq 0}\frac{\chi_{\xi-\alpha}(w_1)\chi_{\xi-\beta}(w_2)\chi_{\xi-\gamma}(w_3)(w_2\overline{w}_3-\overline{w}_2w_3)}{|w_1|^2|w_2|^2|w_3|^2}\mathrm{d}\xi\wedge\mathrm{d}\overline{\xi}.
\end{aligned}$$

The integrals of the terms with $w_1+w_2+w_3\neq 0$ vanish, as integrals of nontrivial harmonics over a torus, so we get

$$\begin{aligned}
&-\frac{1}{8\pi}\frac{\tau-\overline{\tau}}{2\pi i}\sum_{\substack{w_1+w_2+w_3=0\\ w_i\neq 0}}\frac{\chi_\alpha(w_1)\chi_\beta(w_2)\chi_\gamma(w_3)(w_2\overline{w}_3-\overline{w}_2w_3)}{|w_1|^2|w_2|^2|w_3|^2}\int\mathrm{d}\xi\wedge\mathrm{d}\overline{\xi}\\
&=-\frac{1}{8\pi}\frac{\tau-\overline{\tau}}{2\pi i}\sum_{\substack{w_1+w_2+w_3=0\\ w_i\neq 0}}\frac{\chi_\alpha(w_1)\chi_\beta(w_2)\chi_\gamma(w_3)(w_2\overline{w}_3-\overline{w}_2w_3)}{|w_1|^2|w_2|^2|w_3|^2}\times(-(\tau-\overline{\tau})).
\end{aligned}$$

□

Remark 2.3.8 This Fourier series is antisymmetric under permutations of α, β and γ, since

$$w_2\overline{w}_3-\overline{w}_2w_3=\frac{1}{3}\begin{vmatrix}1&1&1\\ w_1&w_2&w_3\\ \overline{w}_1&\overline{w}_2&\overline{w}_3\end{vmatrix}\quad\text{for } w_1+w_2+w_3=0.$$

2.3.9 The function $\mathcal{L}_{1,1}$ is smooth outside the diagonals; this follows from the general formula for differentiation of an integral with respect to a parameter. $\mathcal{L}_{1,1}$ is zero on the diagonals, because A_3 is antisymmetric.

Lemma 2.3.10 $\mathcal{L}_{1,1}$ *is continuous on the complement of the triple diagonal* $\alpha = \beta = \gamma$.

Proof We prove that

$$\lim_{t\to 0} \mathcal{L}_{1,1}(t\alpha, t\beta, \gamma; \tau) = 0 \quad \text{for } \gamma \neq 0.$$

We choose some rather small ε and represent the integral as the sum of the integral over the disk of radius ε around 0 and the integral over the complement of this disk inside the elliptic curve. The second integral tends to zero, as the integral of any term of A_3 converges and A_3 is antisymmetric. Inside the disk, $\mathcal{L}_1(\xi - t\alpha)$ equals $\log|\xi - t\alpha|^2 + \varphi(\xi, t\alpha)$, where $\varphi(\xi, t\alpha)$, is a smooth function, the same is true for $\mathcal{L}_1(\xi - t\beta)$. Substitute this decomposition into A_3; we get several types of summands: 1) all the arguments of A_3 are smooth, 2) one argument is singular and 3) two arguments are singular. In the first two cases the integral tends to zero for the same reason as above. To estimate the last summand, perform the change of variable $z = t^{-1}\xi$. We get

$$\begin{aligned}
&\int_{|\xi|<\varepsilon} A_3\Big(\log|\xi - t\alpha|^2, \log|\xi - t\beta|^2, \mathcal{L}_1(\xi - \gamma)\Big) \\
&\quad = \int_{|\xi|<t^{-1}\varepsilon} A_3\Big(\log|z - \alpha|^2, \log|z - t\beta|^2, \mathcal{L}_1(tz - \gamma) - \mathcal{L}_1(-\gamma)\Big) \\
&\qquad\qquad + \int_{|\xi|<t^{-1}\varepsilon} A_3\Big(\log|z - \alpha|^2, \log|z - t\beta|^2, \mathcal{L}_1(-\gamma)\Big),
\end{aligned}$$

and the second integral tends to zero because $\mathcal{L}_1(-\gamma)$ is a constant.

For small t, the first integral can be represented as a sum of the integral over the disk of radius $\sqrt{t^{-1}\varepsilon}$ and that over the annulus $\sqrt{t^{-1}\varepsilon} < |z| < t^{-1}\varepsilon$. In the second integral, replace $\log|z - \beta|^2$ by $\log|z - \beta|^2 - \log|z - \alpha|^2$. The integral over the disk is small since $\mathcal{L}_1(tz - \gamma)$ and also its derivatives are small; the integral over the annulus is small because $\log|z - \beta|^2 - \log|z - \alpha|^2|$ and its derivatives are small. □

Lemma 2.3.11 *Let* α, β *and* γ *be points on an elliptic curve, not all three coincident. Then the limit of* $\mathcal{L}_{1,1}(t\alpha, t\beta, t\gamma; \tau)$ *as* $t \to 0$ *equals* $D_2\left(\frac{\gamma-\alpha}{\beta-\alpha}\right)$.

Proof We fix some rather small ε and represent the integral as the sum of the integral over the disk of radius ε around 0 and the integral over the complement of this disk inside the elliptic curve. The second integral tends to zero, as A_3 is antisymmetric. Inside the disk, $\mathcal{L}_1(\xi - t\alpha)$ equals $\log|\xi - t\alpha|^2 + \varphi(\xi, t\alpha)$, where $\varphi(\xi, t\alpha)$ is a smooth function,and the same is true for $\mathcal{L}_1(\xi - t\beta)$ and for $\mathcal{L}_1(\xi - t\gamma)$. As in the proof of the preceding lemma, only the summand $A_3(\log|\xi - t\alpha|^2, \log|\xi - t\beta|^2, \log|\xi - t\gamma|^2)$ gives a nontrivial contribution in the limit. Therefore

$$\begin{aligned}
&\lim_{t\to 0}\mathcal{L}_{1,1}(t\alpha, t\beta, t\gamma; \tau)\\
&\quad= \frac{1}{4\pi}\lim_{t\to 0}\int_{|\xi|<\varepsilon} A_3\Big(\log|\xi - t\alpha|^2, \log|\xi - t\beta|^2, \log|\xi - t\gamma|^2\Big)\\
&\quad= \frac{1}{4\pi}\lim_{t\to 0}\int_{|\xi|<t^{-1}\varepsilon} A_3\Big(\log|t^{-1}\xi - \alpha|^2, \log|t^{-1}\xi - \beta|^2, \log|t^{-1}\xi - \gamma|^2\Big)\\
&\quad= \frac{1}{4\pi}\int_{\mathbb{C}} A_3\Big(\log|z - \alpha|^2, \log|z - \beta|^2, \log|z - \gamma|^2\Big)\\
&\quad= D_2\Big(\frac{\gamma - \alpha}{\beta - \alpha}\Big). \quad \square
\end{aligned}$$

Remark 2.3.12 $\mathcal{L}_{1,1}(\alpha, \beta, \gamma; \tau)$ is clearly invariant under "diagonal" translation of the arguments and changing the sign of the arguments, so that the elliptic $(1,1)$-logarithm is a function on the moduli space $\overline{\mathcal{M}}_{1,3}$ of curves of genus 1 with three marked points.

3 From the Kronecker series to the elliptic $(1,1)$-logarithm

The reduction of the Kronecker series $\mathcal{K}_2$ to the elliptic $(1,1)$-logarithm for an elliptic curve with complex multiplication splits up into two steps. We first represent $\mathcal{K}_2(\xi;\tau)$ as an integral over the square of the elliptic curve. This representation is valid for any elliptic curve and for any point ξ on it. We then reduce the integral over the square of the elliptic curve to an integral over the elliptic curve itself; this is possible for a curve with complex multiplication, and uses the existence of an extra projection of the square of the curve onto itself.

3.1 An integral representation of the Kronecker series

We start from the simplest example which illustrates the main idea of this representation.

Proposition 3.1.1

$$\mathcal{K}_2(\alpha;\tau) = \frac{\tau-\overline{\tau}}{2\pi i}\int\limits_{E_\tau\times E_\tau} \mathcal{L}_1(\alpha-\eta_1-\eta_2)\frac{\partial\mathcal{L}_1(\eta_1)}{\partial\eta_1}\frac{\partial\mathcal{L}_1(\eta_2)}{\partial\overline{\eta}_2}\frac{\mathrm{d}\eta_1\wedge\mathrm{d}\overline{\eta}_1}{\tau-\overline{\tau}}\wedge\frac{\mathrm{d}\eta_2\wedge\mathrm{d}\overline{\eta}_2}{\tau-\overline{\tau}}. \quad (9)$$

Proof We use the Fourier expansions of $\mathcal{L}_1$ and its derivatives

$$\frac{\tau-\overline{\tau}}{2\pi i}\int\Big(-\frac{\tau-\overline{\tau}}{2\pi i}{\sum_{w_1\in L}}'_e \frac{\chi_{\alpha-\eta_1-\eta_2}(w_1)}{|w_1|^2}\Big)\times \Big({\sum_{w_2\in L}}'_e\frac{\chi_{\eta_1}(w_2)}{w_2}\Big)\Big(-{\sum_{w_3\in L}}'_e\frac{\chi_{\eta_2}(w_3)}{\overline{w}_3}\Big)\times\frac{d\eta_1\wedge d\overline{\eta}_1}{\tau-\overline{\tau}}\wedge\frac{d\eta_2\wedge d\overline{\eta}_2}{\tau-\overline{\tau}}.$$

The integral of the term with $w_1\neq w_2, w_1\neq w_3$ vanishes, as an integral of a nontrivial harmonic over a torus, so we get the integral

$$\frac{\tau-\overline{\tau}}{2\pi i}\int\Big(-\frac{\tau-\overline{\tau}}{2\pi i}{\sum_{w_1\in L}}'_e\frac{\chi_\alpha(w_1)}{|w_1|^2}\Big)\Big(\frac{1}{-w_1}\Big)\Big(-\frac{1}{-\overline{w}_1}\Big)\frac{d\eta_1\wedge d\overline{\eta}_1}{\tau-\overline{\tau}}\wedge\frac{d\eta_2\wedge d\overline{\eta}_2}{\tau-\overline{\tau}}$$
$$=\Big(\frac{\tau-\overline{\tau}}{2\pi i}\Big)^2\sum_{w_1\in L}\frac{\chi_\alpha(w_1)}{|w_1|^4}(-1)^2=\mathcal{K}_2(\xi;\tau). \quad \square$$

At the end of this section we discuss a more symmetric representation of $\mathcal{K}_2$ for any τ and ξ; this result will not be used further. We now state the main result of this section.

Lemma 3.1.2 *Let τ be a fixed point of $M=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\neq 0,1$; we set $m=\det M$ and $n=\det(M-1)$. Write η_3 for the expression $\eta_1+M_\tau(\eta_2)-\alpha$ and η_4 for $\eta_1+\eta_2-\beta$. Then*

$$\int\limits_{E_\tau^2} A_3\big(\mathcal{L}_1(\eta_1),\mathcal{L}_1(\eta_2),\mathcal{L}_1(\eta_3)\big)\Big(\frac{\mathrm{d}\eta_4\wedge\mathrm{d}\overline{\eta}_4}{\tau-\overline{\tau}}\Big)=\frac{\pi ic(\tau-\overline{\tau})}{m}\mathcal{K}_2(-\alpha;\tau);$$
$$\int\limits_{E_\tau^2} A_3\big(\mathcal{L}_1(\eta_1),\mathcal{L}_1(\eta_2),\mathcal{L}_1(\eta_4)\big)\Big(\frac{\mathrm{d}\eta_3\wedge\mathrm{d}\overline{\eta}_3}{\tau-\overline{\tau}}\Big)=-\pi ic(\tau-\overline{\tau})\mathcal{K}_2(-\beta;\tau);$$
$$\int\limits_{E_\tau^2} A_3\big(\mathcal{L}_1(\eta_1),\mathcal{L}_1(\eta_3),\mathcal{L}_1(\eta_4)\big)\Big(\frac{\mathrm{d}\eta_2\wedge\mathrm{d}\overline{\eta}_2}{\tau-\overline{\tau}}\Big)=\frac{\pi ic(\tau-\overline{\tau})}{mn}\mathcal{K}_2(M_\tau(\beta)-\alpha;\tau);$$
$$\int\limits_{E_\tau^2} A_3\big(\mathcal{L}_1(\eta_2),\mathcal{L}_1(\eta_3),\mathcal{L}_1(\eta_4)\big)\Big(\frac{\mathrm{d}\eta_1\wedge\mathrm{d}\overline{\eta}_1}{\tau-\overline{\tau}}\Big)=-\frac{\pi ic(\tau-\overline{\tau})}{n}\mathcal{K}_2(\beta-\alpha;\tau).$$
$$(10)$$

Proof We only prove the third equation; the others can be proved by the same considerations. The last factor of the integrand has type $(1,1)$ and is closed, so we can replace the expression $A_3\big(\mathcal{L}_1(\eta_1),\mathcal{L}_1(\eta_3),\mathcal{L}_1(\eta_4)\big)$ by $\frac{1}{2}\mathcal{L}_1(\eta_3)\mathrm{d}\mathcal{L}_1(\eta_1)\wedge\mathrm{d}\mathcal{L}_1(\eta_4)$. Thus we can calculate the integral

$$\frac{1}{2}\int\Big(-\frac{\tau-\overline{\tau}}{2\pi i}\sum_{w_3\in L}{}_e^{\prime}\frac{\chi_{\eta_1+M_\tau(\eta_2)-\alpha}(w_3)}{|w_3|^2}\Big)\times$$
$$\Bigg(\Big(\sum_{w_1\in L}{}_e^{\prime}\frac{\chi_{\eta_1}(w_1)\mathrm{d}\eta_1}{w_1}\Big)\wedge\Big(-\sum_{w_4\in L}{}_e^{\prime}\frac{\chi_{\eta_1+\eta_2-\beta}(w_4)(\mathrm{d}\overline{\eta}_1+\mathrm{d}\overline{\eta}_2)}{\overline{w}_4}\Big)$$
$$+\Big(\sum_{w_1\in L}{}_e^{\prime}\frac{\chi_{\eta_1}(w_1)\mathrm{d}\overline{\eta}_1}{\overline{w}_1}\Big)\wedge\Big(-\sum_{w_4\in L}{}_e^{\prime}\frac{\chi_{\eta_1+\eta_2-\beta}(w_4)(\mathrm{d}\eta_1+\mathrm{d}\eta_2)}{w_4}\Big)\Bigg)\wedge\frac{\mathrm{d}\eta_2\wedge\mathrm{d}\overline{\eta}_2}{\tau-\overline{\tau}}.$$

Since $\chi_{M_\tau}(\eta_2)(w)=\chi_{\eta_2}(M(w))$, only the terms with $w_3+w_1+w_4=0$ and $M(w_3)+w_4=0$ give nontrivial contributions in the integral. So we get

$$\frac{1}{2}\frac{(\tau-\overline{\tau})^2}{2\pi i}\sum_{w_3\in L}{}^{\prime}\frac{\chi_{-\alpha}(w_3)}{|w_3|^2}\times\Bigg(\frac{1}{(M-1)(w_3)}\times\frac{\chi_{-\beta}(-M(w_3))}{-\overline{M(w_3)}}$$
$$-\frac{1}{\overline{\overline{(M-1)(w_3)}}}\times\frac{\chi_{-\beta}(-M(w_3))}{-M(w_3)}\Bigg)\times\int\frac{\mathrm{d}\eta_1\wedge\mathrm{d}\overline{\eta}_1}{\tau-\overline{\tau}}\wedge\frac{\mathrm{d}\eta_2\wedge\mathrm{d}\overline{\eta}_2}{\tau-\overline{\tau}}$$
$$=-\frac{1}{2}\frac{(\tau-\overline{\tau})^2}{2\pi i}\sum_{w_3\in L}{}^{\prime}\frac{\chi_{M(\beta)-\alpha}(w_3)}{|w_3|^4}\times$$
$$\Big(\frac{1}{(c\tau+d-1)(c\overline{\tau}+d)}-\frac{1}{(c\overline{\tau}+d-1)(c\tau+d)}\Big)$$
$$=\frac{1}{2}\frac{(\tau-\overline{\tau})^2}{2\pi i}\frac{c(\tau-\overline{\tau})}{nm}\sum_{w_3\in L}{}^{\prime}\frac{\chi_{M(\beta)-\alpha}(w_3)}{|w_3|^4}$$
$$=\frac{\pi i c(\tau-\overline{\tau})}{mn}\mathcal{K}_2(M_\tau(\beta)-\alpha;\tau).$$

We used above the following simple formula: for any isogeny of E_τ to itself, $|c\tau+d|^2=ad-bc=\det M$. $\square$

Remark 3.1.3 For general values of τ any isogeny of E_τ to itself is multiplication by some integer, and the r.-h.s. vanishes. Hence the result is only interesting for an elliptic curve with complex multiplication.

3.1.4 The function $\frac{2\pi i}{\tau-\overline{\tau}}\mathcal{K}_2$ can be treated as a component of the vector valued function

$$\mathcal{L}_3(\xi;\tau) = \frac{\tau-\overline{\tau}}{2\pi i}\begin{pmatrix} \sum'_{w\in L}\dfrac{\chi_\xi(w)}{w^3\overline{w}} \\ \sum'_{w\in L}\dfrac{\chi_\xi(w)}{w^2\overline{w}^2} \\ \sum'_{w\in L}\dfrac{\chi_\xi(w)}{w\overline{w}^3} \end{pmatrix},$$

(the elliptic trilogarithm), taking values in the symmetric square $S^2(\mathcal{H})$ of the homology group $\mathcal{H}$ of the elliptic curve E_τ with complex coefficients. This space is isomorphic to a direct summand of the second cohomology group of the square of the elliptic curve. A “natural” basis of this space is

$$f_1 = \frac{\mathrm{d}\overline{\eta}_1\wedge\mathrm{d}\overline{\eta}_2}{(\tau-\overline{\tau})^2}, \quad f_2 = \frac{\mathrm{d}\eta_1\wedge\mathrm{d}\overline{\eta}_2 - \mathrm{d}\eta_2\wedge\mathrm{d}\overline{\eta}_1}{(\tau-\overline{\tau})^2} \quad f_3 = \frac{\mathrm{d}\eta_1\wedge\mathrm{d}\eta_2}{(\tau-\overline{\tau})^2}.$$

Proposition 3.1.5

$$\mathcal{L}_3(\xi;\tau) = \int_{E_\tau^2} A_3\left(\mathcal{L}_1(\eta_1),\mathcal{L}_1(\eta_2),\mathcal{L}_1(\eta_1+\eta_2-\xi)\right)\wedge\begin{pmatrix} f_1\\ f_2\\ f_3\end{pmatrix}. \qquad (11)$$

The proof is a straightforward calculation.

3.2 Reduction to the elliptic $(1,1)$-logarithm

3.2.1 Consider the current $\Phi = \frac{1}{2\pi^2}A_4\big(\mathcal{L}_1(\eta_1),\mathcal{L}_1(\eta_2),\mathcal{L}_1(\eta_3),\mathcal{L}_1(\eta_4)\big)$ on E_τ^2, where, as above, $\eta_3 = \eta_1 + M_\tau(\eta_2) - \alpha$ and $\eta_4 = \eta_1+\eta_2-\beta$ for some isogeny $M = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \neq 0,1$. The wedge product of currents is not defined, but for general α and β the divisors of the singularities of $\mathcal{L}_1$'s are in general position, and the wedge product is well defined; the formula for the differential of the wedge product also holds.

3.2.2 The differential $\mathrm{d}\Phi$ of Φ equals

$$\begin{aligned}\frac{1}{2\pi^2}\times\Big(&-\overline{\partial}\partial\mathcal{L}_1(\eta_1)\wedge A_3\Big(\mathcal{L}_1(\eta_2),\mathcal{L}_1(\eta_3),\mathcal{L}_1(\eta_4)\Big)\\ &+\overline{\partial}\partial\mathcal{L}_1(\eta_2)\wedge A_3\Big(\mathcal{L}_1(\eta_1),\mathcal{L}_1(\eta_3),\mathcal{L}_1(\eta_4)\Big)\\ &-\overline{\partial}\partial\mathcal{L}_1(\eta_3)\wedge A_3\Big(\mathcal{L}_1(\eta_1),\mathcal{L}_1(\eta_2),\mathcal{L}_1(\eta_4)\Big)\\ &+\overline{\partial}\partial\mathcal{L}_1\Big(\eta_4)\wedge A_3(\mathcal{L}_1(\eta_1),\mathcal{L}_1(\eta_2),\mathcal{L}_1(\eta_3)\Big)\Big),\end{aligned}$$

because the $(4,0)$-part and the $(0,4)$-part vanish on a surface. We split the $\overline{\partial}\partial\mathcal{L}_1$ into a δ-function part and a smooth part. Integrals with δ-functions are integrals over the elliptic curve and are equal to sums of values of the elliptic $(1,1)$-logarithm. Integrals of smooth parts of the $\partial\overline{\partial}\mathcal{L}_1$ are calculated in the preceding lemma.

3.2.3 We write $B_1,\dots,B_4$ for the δ-parts of the four components of $\mathrm{d}\Phi$ and $B'_1,\dots,B'_4$ for the smooth ones. We first calculate the integrals using δ-functions:

$$\begin{aligned}\int B_1 &= -\frac{2\pi i}{2\pi^2}\int_{E_\tau} A_3\Big(\mathcal{L}_1(\eta_2),\mathcal{L}_1(\eta_3),\mathcal{L}_1(\eta_4)\Big)_{|\eta_1=0}\\ &= \frac{1}{\pi i}\int_{E_\tau} A_3\Big(\mathcal{L}_1(\eta_2),\mathcal{L}_1(M_\tau\eta_2-\alpha),\mathcal{L}_1(\eta_2-\beta)\Big)\\ &= \frac{1}{\pi i}\int_{E_\tau}\sum_{\alpha':M_\tau(\alpha')=\alpha} A_3\Big(\mathcal{L}_1(\eta_2),\mathcal{L}_1(\eta_2-\alpha'),\mathcal{L}_1(\eta_2-\beta)\Big)\\ &= -4i\sum_{\alpha':M_\tau(\alpha')=\alpha}\mathcal{L}_{1,1}(0,\alpha',\beta).\end{aligned}$$

The same consideration shows that

$$\begin{aligned}B_2 &= 4i\mathcal{L}_{1,1}(0,\alpha,\beta),\\ B_3 &= -4i\sum_{\substack{\alpha':M_\tau(\alpha')=\alpha\\ \gamma':(M_\tau-1)(\gamma')=\alpha-\beta}}\mathcal{L}_{1,1}(\alpha',0,\gamma'),\quad \text{and}\\ B_4 &= 4i\sum_{\gamma':(M_\tau-1)(\gamma')=\alpha-\beta}\mathcal{L}_{1,1}(0,\beta,\gamma').\end{aligned}$$

3.2.4 Since the smooth part of $\overline{\partial}\partial\mathcal{L}_1(\xi;\tau)$ equals $2\pi i\frac{\mathrm{d}\xi\wedge\mathrm{d}\overline{\xi}}{\tau-\overline{\tau}}$, the integrals of B'_j were already calculated in (10).

3.2.5 The integral of the differential of a current over a compact variety is zero, so that we get $\sum B_i = -\sum B'_i$, and we have proved the following result:

Proposition 3.2.6 *Let τ be a fixed point of $M \neq 0,1$, with $m=\det M$ and*

$n = \det(M-1)$. *Suppose that* $\alpha \neq 0$ *and* $\beta \neq 0, \alpha, M_\tau(\alpha)$. *Then*

$$\frac{c(\tau-\overline{\tau})}{4i}\Big(\frac{1}{m}\mathcal{K}_2(-\alpha;\tau)+\mathcal{K}_2(-\beta;\tau)+\frac{1}{mn}\mathcal{K}_2(M_\tau(\beta)-\alpha;\tau)+\frac{1}{n}\mathcal{K}_2(\beta-\alpha;\tau)\Big)$$
$$= -\sum_{\alpha'\in M^{-1}(\alpha)} \mathcal{L}_{1,1}(0,\alpha',\beta) + \mathcal{L}_{1,1}(0,\alpha,\beta)$$
$$- \sum_{\substack{\alpha'\in M^{-1}(\alpha)\\ \gamma'\in(M-1)^{-1}(\alpha-\beta)}} \mathcal{L}_{1,1}(\alpha',0,\gamma') + \sum_{\gamma'\in(M-1)^{-1}(\alpha-\beta)} \mathcal{L}_{1,1}(0,\beta,\gamma').$$

As the function $\mathcal{K}_2$ is continuous, we can "degenerate" this formula:

3.2.1 Theorem. *Let* τ *be a fixed point of* $M \neq 0, 1$ *and* m, n *as above. Then*

$$\frac{c(\tau-\overline{\tau})}{4i}\left(\frac{m+n+1}{mn}\mathcal{K}_2(-\alpha;\tau)+\mathcal{K}_2(0;\tau)\right)$$
$$= -\sum_{\substack{\alpha'\in M^{-1}(\alpha)\\ \gamma'\in(M-1)^{-1}(\alpha)}} \mathcal{L}_{1,1}(\alpha',0,\gamma'), \quad \text{for } \alpha \neq 0; \quad (12)$$

$$\frac{c(\tau-\overline{\tau})}{4i}\frac{(m+1)(n+1)}{mn}\mathcal{K}_2(0;\tau)$$
$$= -D_2\left(\frac{ad-bc}{c\tau+d}\right) - \sum_{\substack{\alpha'\in \mathrm{Ker}(M)\backslash 0\\ \gamma'\in \mathrm{Ker}(M-1)\backslash 0}} \mathcal{L}_{1,1}(\alpha',0,\gamma'). \quad (13)$$

Proof The first formula is the result of substituting $\beta = 0$; and the second one is the limit of the first as $\alpha \to 0$. This completes the proof of Theorem A. □

4 From the elliptic $(1,1)$-logarithm to the dilogarithm

In this section we relate the elliptic $(1,1)$-logarithm to the dilogarithm. Specifically, we express the combination $\mathcal{L}_{1,1}(0,\alpha,\beta;\tau)+\mathcal{L}_{1,1}(0,\alpha,-\beta;\tau)$ as a sum of dilogarithms for any torsion points α and β on *any* elliptic curve E_τ. The proof uses the representation of an elliptic curve as a covering of degree 2 of the projective line.

4.1 The dilogarithm as a combination of elliptic $(1,1)$-logarithms

We start by expressing the dilogarithm as a combination of elliptic $(1,1)$-logarithms.

4.1.1 The Weierstrass $\wp$-function maps the elliptic curve as a double cover of $\mathbb{P}^1$. Suppose that $\pm\alpha$ on the elliptic curve are the inverse images of a point a on $\mathbb{P}^1$, that is, $\wp(\pm\alpha) = a$; similarly, suppose that $\wp(\pm\beta) = b$, $\wp(\pm\gamma) = c$. Then by 2.2.4,

$$
\begin{aligned}
D_2\left(\frac{c-a}{b-a}\right) &= \frac{1}{4\pi}\int_{\mathbb{P}^1} A_3(\log|z-a|^2, \log|z-b|^2, \log|z-c|^2) \\
&= \frac{1}{2}\frac{1}{4\pi}\int_{E_\tau} A_3(\log|\wp(\xi)-\wp(\alpha)|^2, \log|\wp(\xi)-\wp(\beta)|^2, \log|\wp(\xi)-\wp(\gamma)|^2).
\end{aligned}
$$

The extra factor of $1/2$ reflects the number of branches.

4.1.2 The "theta decomposition" of 2.3.4 implies that

$$
\log|\wp(\xi)-\wp(\alpha)|^2 = \mathcal{L}_1(\xi+\alpha) + \mathcal{L}_1(\xi-\alpha) - 2\mathcal{L}_1(\xi) - 2\mathcal{L}_1(\alpha).
$$

We substitute this expression into A_3 and integrate. A straightforward computation gives

Lemma 4.1.3

$$
\begin{aligned}
D_2\left(\frac{\wp(\gamma)-\wp(\alpha)}{\wp(\beta)-\wp(\alpha)}\right) &= \mathcal{L}_{1,1}(\alpha,\beta,\gamma) + \mathcal{L}_{1,1}(-\alpha,\beta,\gamma) + \mathcal{L}_{1,1}(\alpha,-\beta,\gamma) \\
&+ \mathcal{L}_{1,1}(\alpha,\beta,-\gamma) - 2\Big(\mathcal{L}_{1,1}(0,\beta,\gamma) + \mathcal{L}_{1,1}(0,-\beta,\gamma) + \mathcal{L}_{1,1}(\alpha,0,\gamma) \\
&+ \mathcal{L}_{1,1}(-\alpha,0,\gamma) + \mathcal{L}_{1,1}(\alpha,\beta,0) + \mathcal{L}_{1,1}(-\alpha,\beta,0)\Big). \quad (14)
\end{aligned}
$$

4.2 The elliptic $(1,1)$-logarithms as a combination of dilogarithms

Now we combine the expressions of the preceding lemma to cancel almost all terms on the r.-h.s.

Theorem 4.2.1 *For any two nontrivial torsion points $\alpha \neq \pm\beta$ on an elliptic curve E_τ, say $m\alpha = n\beta = 0$ for some $m, n \in \mathbb{N}$. Then*

$$
\sum_{k=1}^{m}\sum_{l=1}^{n} D_2\Big(\frac{\wp(k\alpha+l\beta)-\wp(\alpha)}{\wp(\beta)-\wp(\alpha)}\Big) = -2mn(\mathcal{L}_{1,1}(\alpha,\beta,0) + \mathcal{L}_{1,1}(-\alpha,\beta,0)). \quad (15)
$$

Sketch proof We must check that all except the two last terms on the r.-h.s. of (14) cancel after summation. We show that the first one cancels:

$$\begin{aligned}\mathcal{L}_{1,1}(\alpha,\beta,k\alpha+l\beta) &= \mathcal{L}_{1,1}(\alpha-(\alpha+\beta),\beta-(\alpha+\beta),k\alpha+l\beta-(\alpha+\beta))\\ &= \mathcal{L}_{1,1}(-\beta,-\alpha,(k-1)\alpha+(l-1)\beta)\\ &= \mathcal{L}_{1,1}(\beta,\alpha,(1-k)\alpha+(1-k)\beta)\\ &= -\mathcal{L}_{1,1}(\alpha,\beta,(1-k)\alpha+(1-k)\beta);\end{aligned}$$

hence the first summands with arguments (k,l) and $(1-k,1-l)$ only differ by the sign and the first summands cancel on averaging. The arguments for the other terms are similar.

This completes the proof of Theorem B and hence of the formula for $\mathcal{K}_2(0,\tau)$ in the Main Theorem.

Remark 4.2.2 We have proved that the *combination*

$$\mathcal{L}_{1,1}(\alpha,\beta,0)+\mathcal{L}_{1,1}(-\alpha,\beta,0)$$

is equal to a sum of dilogarithms for torsion points α and β. It is even true that the single term $\mathcal{L}_{1,1}(\alpha,\beta,0)$ is equal to a combination of dilogarithms; but we do not need this in this expression for the Main Theorem, and it is rather complicated. We will derive it in Section 6.

5 Vanishing of the map δ

In this section we prove that the argument of the dilogarithm in the Main Theorem belongs to the kernel of the map δ of 1.2.3.

5.1 Values of the θ-function at torsion points

5.1.1 The theta function

$$\widetilde{\theta}(\xi,\tau)=\frac{\theta(\xi,\tau)}{\eta(\tau)}=q^{1/12}(z^{\frac{1}{2}}-z^{-\frac{1}{2}})\prod_{j=1}^{\infty}(1-q^jz)(1-q^jz^{-1})$$

is not elliptic. It is only *quasiperiodic*"

$$\widetilde{\theta}(\xi+1,\tau)=-\widetilde{\theta}(\xi,\tau)\quad\text{and}\quad\widetilde{\theta}(\xi+\tau,\tau)=-z^{-1}q^{-1/2}\times\widetilde{\theta}(\xi,\tau);$$

but for any torsion point $\xi=r\tau+s$ with $r,s\in\frac{1}{N}\mathbb{Z}$ of order N, we can redefine the *value* of the theta function at this point by the formula:

$$\widetilde{\theta}[\xi](\tau)=z^{1/2+r}q^{-r^2/2-r}\widetilde{\theta}(\xi,\tau).$$

Translating the argument by a point of the lattice multiplies this value by some root of unity. We write $\equiv$ for equality modulo multiplication by a root of unity.

Remark 5.1.2 If τ is imaginary quadratic over $\mathbb{Q}$, the numbers $\widetilde{\theta}[\xi](\tau)$ are *algebraic.*

5.1.3 We have the theta decomposition

$$\wp(\alpha;\tau) - \wp(\beta;\tau) \equiv \frac{\widetilde{\theta}'(0,\tau)^2 \times \widetilde{\theta}[\alpha-\beta](\tau) \times \widetilde{\theta}[\alpha+\beta](\tau)}{\widetilde{\theta}[\alpha](\tau)^2 \times \widetilde{\theta}[\beta](\tau)^2}.$$

for any two torsion α and β points.

5.1.4 Let α be a torsion point on the elliptic curve $E_{M(\tau)}$. Then

$$\prod_{\beta\in M^{-1}\alpha} \widetilde{\theta}[\beta](\tau) \equiv \widetilde{\theta}[\alpha](M(\tau)) \quad \text{if } \alpha \neq 0; \text{ and}$$

$$\widetilde{\theta}'(0,\tau) \times \prod_{\beta\in \operatorname{Ker} M\setminus 0} \widetilde{\theta}[\beta](\tau) \equiv \frac{ad-bc}{c\tau+d} \times \widetilde{\theta}'(0,M(\tau)).$$

5.2 Computations

5.2.1 We first calculate the value of the map δ on $\frac{\wp(\gamma)-\wp(\alpha)}{\wp(\beta)-\wp(\alpha)}$

$$\begin{aligned}\delta\left(\frac{\wp(\gamma)-\wp(\alpha)}{\wp(\beta)-\wp(\alpha)}\right) &= \left(\frac{\wp(\gamma)-\wp(\alpha)}{\wp(\beta)-\wp(\alpha)}\right) \wedge \left(\frac{\wp(\gamma)-\wp(\beta)}{\wp(\alpha)-\wp(\beta)}\right) \\ &\equiv \left(\frac{\widetilde{\theta}[\alpha-\gamma]\times\widetilde{\theta}[\alpha+\gamma]\times\widetilde{\theta}[\beta]^2}{\widetilde{\theta}[\alpha-\beta]\times\widetilde{\theta}[\alpha+\beta]\times\widetilde{\theta}[\gamma]^2}\right) \wedge \left(\frac{\widetilde{\theta}[\beta-\gamma]\times\widetilde{\theta}[\beta+\gamma]\times\widetilde{\theta}[\alpha]^2}{\widetilde{\theta}[\alpha-\beta]\times\widetilde{\theta}[\alpha+\beta]\times\widetilde{\theta}[\gamma]^2}\right).\end{aligned}$$

Thus the answer is the same as the result of the following procedure:

1. Define the map ν by the formula

$$\begin{aligned}\{\alpha,\beta,\gamma\} \to -\Big(\{\alpha-\beta\}\wedge\{\beta-\gamma\} + \{\beta-\gamma\}\wedge\{\gamma-\alpha\} \\ + \{\gamma-\alpha\}\wedge\{\alpha-\beta\}\Big).\end{aligned}$$

2. Apply ν to the arguments of the function $\mathcal{L}_{1,1}$ on the r.-h.s. of (14).

3. Apply the map $\bigwedge^2 \widetilde{\theta}\colon \xi_1\wedge\xi_2 \to \widetilde{\theta}[\xi_1]\wedge\widetilde{\theta}[\xi_2] \in \bigwedge^2 \mathbb{C}^*$ to the result of the previous step.

5.2.2 The map $\bigwedge^2 \widetilde{\theta} \circ \nu$ satisfies the same properties as $\mathcal{L}_{1,1}$ under translations or permutations of arguments. Hence after summation over $\gamma = j\alpha + k\beta$, we get

$$\delta\left(\sum_{k=1}^{m}\sum_{l=1}^{n}\left\{\frac{\wp(k\alpha+l\beta)-\wp(\alpha)}{\wp(\beta)-\wp(\alpha)}\right\}\right)$$
$$\equiv -2mn \wedge^2 \widetilde{\theta} \circ \nu(\{\alpha,\beta,0\} + \{-\alpha,\beta,0\})$$
$$\equiv 2mn(\beta)\Big(\widetilde{\theta}[\alpha-\beta]\wedge\widetilde{\theta}[\beta] + \widetilde{\theta}[\beta]\wedge\widetilde{\theta}[\alpha] + \widetilde{\theta}[\alpha]\wedge\widetilde{\theta}[\alpha-\beta]$$
$$+ \widetilde{\theta}[\alpha+\beta]\wedge\widetilde{\theta}[\beta] + \widetilde{\theta}[\beta]\wedge\widetilde{\theta}[\alpha] + \widetilde{\theta}[\alpha]\wedge\widetilde{\theta}[\alpha+\beta]\Big).$$

5.2.3 Finally, we perform the last summation:

$$\delta\Bigg(\sum_{\substack{\alpha\in\operatorname{Ker} M\setminus 0\\ \beta\in\operatorname{Ker}(M-1)\setminus 0}}\sum_{k=1}^{m}\sum_{l=1}^{n}\left\{\frac{\wp(k\alpha+l\beta)-\wp(\alpha)}{\wp(\beta)-\wp(\alpha)}\right\}\Bigg)$$
$$\equiv 4mn \sum_{\substack{\alpha\in\operatorname{Ker} M\setminus 0\\ \beta\in\operatorname{Ker}(M-1)\setminus 0}} \Big(\widetilde{\theta}[\alpha-\beta]\wedge\widetilde{\theta}[\beta] + (\widetilde{\theta}[\beta]\wedge\widetilde{\theta}[\alpha] + \widetilde{\theta}[\alpha]\wedge\widetilde{\theta}[\alpha-\beta]\Big).$$

now sum the first terms over the α, the second terms over the α and β and the third terms over the β. We get

$$4mn\Bigg(\sum_{\beta\in\operatorname{Ker}(M-1)\setminus 0}\Big(\widetilde{\theta}[M\beta]\wedge(\widetilde{\theta}[\beta] - \widetilde{\theta}[\beta]\wedge(\widetilde{\theta}[\beta]\Big)$$
$$+\left(\frac{(a-1)(d-1)-bc}{c\tau+d-1}\right)\wedge\left(\frac{ad-bc}{c\tau+d}\right)$$
$$+\sum_{\alpha\in\operatorname{Ker} M\setminus 0}\Big(\widetilde{\theta}[\alpha]\wedge\widetilde{\theta}[(M-1)\alpha] - \widetilde{\theta}[\alpha]\wedge\widetilde{\theta}[\alpha]\Big)\Bigg)$$
$$= 4mn \times \left(\frac{(a-1)(d-1)-bc}{c\tau+d-1}\right)\wedge\left(\frac{ad-bc}{c\tau+d}\right).$$

This expression is the negative of δ of the first term $4mn\left\{\frac{ad-bc}{c\tau+d}\right\}$ in the formula (14).

6 Values of the Kronecker double series at torsion points

In this section we express the value of the Kronecker double series $\mathcal{K}_2(\alpha;\tau)$ at a CM point τ and a torsion point α on the elliptic curve E_τ as a sum of dilogarithms. In (12) we proved that this value of the Kronecker double series can be reduced to a combination of $\mathcal{L}_{1,1}(\alpha', 0, \gamma')$ (with $\alpha' \in M^{-1}(\alpha)$ and $\gamma' \in (M-1)^{-1}(\alpha)$), and $\mathcal{K}_2(0,\tau)$. So we will prove that $\mathcal{L}_{1,1}(\alpha', \gamma', 0)$ equals a sum of dilogarithms.

6.1 Reduction to "standard" functions

6.1.1 Let α be a torsion point of exact order $N(\alpha) = 2^{\rho(\alpha)} \times N_0(\alpha)$, with odd $N_0(\alpha)$. Denote by f_α a function with divisor $N(\alpha) \times (\alpha - 0)$. Clearly, $\log|f_\alpha(\eta)|^2 = \text{const} + N\mathcal{L}_1(\eta - \alpha) - N\mathcal{L}_1(\eta)$. This yields the formula

Lemma 6.1.2 *For any three distinct torsion points α, β and γ*

$$\frac{1}{4\pi}\int_{E_\tau} A_3(f_\alpha, f_\beta, f_\gamma) = N(\alpha)N(\beta)N(\gamma) \times \Big(\mathcal{L}_{1,1}(\alpha,\beta,\gamma) - \mathcal{L}_{1,1}(0,\beta,\gamma) - \mathcal{L}_{1,1}(\alpha,0,\gamma) - \mathcal{L}_{1,1}(\alpha,\beta,0)\Big). \quad (16)$$

Lemma 6.1.3 *For any distinct nontrivial torsion points α and β on the elliptic curve E_τ*

$$\sum_{k=1}^{N(\alpha)}\sum_{l=1}^{N(\beta)} \frac{1}{N(\alpha)^2 N(\beta)^2 N(k\alpha + l\beta)} \int_{E_\tau} A_3(f_\alpha, f_\beta, f_{k\alpha+l\beta}) = -\mathcal{L}_{1,1}(\alpha,\beta,0) \quad (17)$$

The proof is parallel to that of (15). We now reduce the integrals on the l.-h.s. of the previous equation to "dilogarithmic" integrals. First, we have a decomposition of $f_\alpha(\eta)$ into a product of "standard" functions $g_\xi = \wp(\eta-\xi) - \wp(\xi)$. Let $\lambda = \lambda(\alpha)$ be a natural number such that $2^\lambda \equiv 1 \mod (N_0)$ and write α_0 for the point $2^\rho\alpha$.

Lemma 6.1.4

$$f_\alpha(\eta)^{-\frac{2^\lambda - 1}{N_0}} = \text{const} \prod_{i=0}^{\rho-1} \big(\wp(\eta - 2^i\alpha) - \wp(2^i\alpha)\big)^{2^{\rho-i-1}(2^\lambda-1)} \times \prod_{j=0}^{\lambda-1} \big(\wp(\eta - 2^j\alpha_0) - \wp(2^j\alpha_0)\big)^{2^{\lambda-j-1}} \quad (18)$$

Proof Compare the divisors of both sides. □

Remark 6.1.5 If F is a function on an elliptic curve and ξ a torsion point, we can define the operation of *averaging with a factor* 2 by the formula

$$\begin{aligned}\mathrm{Av}_2^{\xi}(F)(\xi) &= \sum_{j=0}^{\infty} 2^{-j} F(2^j\xi)\\ &= \sum_{j=0}^{\rho(\xi)-1} 2^{-j} F(2^j\xi) + \sum_{j=\rho(\xi)}^{\rho(\xi)+\lambda(\xi)-1} \frac{2^{-j}}{1-2^{-\lambda(\xi)}} F(2^j\xi). \qquad (19)\end{aligned}$$

The last equality is nothing more than the formula for the sum of a geometric progression. Hence we can rewrite the statement of the preceding lemma formally as

$$\log(f_\alpha(\eta)) = -\frac{N(\alpha)}{2}\,\mathrm{Av}_2^{\alpha}(\log(\wp(\eta-\alpha)-\wp(\alpha)))$$

the superscript α denotes the variable over which we average.

6.2 From standard functions to dilogarithms

Lemma 6.2.1 *Let $k \subset K$ be a quadratic field extension, and write σ for the involution of K over k. Then*

$$f \times \frac{g^\sigma - g}{fg^\sigma - gf^\sigma} + g \times \frac{f^\sigma - f}{gf^\sigma - fg^\sigma} = 1 \quad \textit{and} \quad \frac{g^\sigma - g}{fg^\sigma - gf^\sigma} \in k.$$

for any $f, g \in K \setminus k$.

The proof is obvious.

6.2.2 We apply this lemma to the extension $\mathbb{C}(\mathbb{P}^1) \subset \mathbb{C}(E)$ and "standard" functions. The involution σ is given by changing sign of the argument of a function.

Lemma 6.2.3 *The "standard" functions $g_\alpha(\eta) = \wp(\eta-\alpha)-\wp(\alpha)$ and $g_\beta(\eta) = \wp(\eta-\beta)-\wp(\beta)$ satisfy*

$$\frac{g_\beta^\sigma(\eta) - g_\beta(\eta)}{g_\alpha(\eta)g_\beta^\sigma(\eta) - g_\beta(\eta)g_\alpha^\sigma(\eta)} = K(\alpha,\beta) \times \frac{(\wp(\eta)-\wp(\alpha))^2}{\wp(\eta) - C(\alpha,\beta)},$$

where

$$C(\alpha,\beta) = \frac{\wp(\alpha)\wp'(\beta)\left(\wp(\alpha+\beta)-\wp(\alpha)\right) + \wp(\beta)\wp'(\alpha)\left(\wp(\alpha+\beta)-\wp(\beta)\right)}{\wp'(\beta)\left(\wp(\alpha+\beta)-\wp(\alpha)\right) + \wp'(\alpha)\left(\wp(\alpha+\beta)-\wp(\beta)\right)},$$

and

$$K(\alpha,\beta) = \frac{-\wp'(\beta)}{(\wp(\alpha)-\wp(\beta)) \times \left(\wp'(\beta)\left(\wp(\alpha+\beta)-\wp(\alpha)\right) + \wp'(\alpha)\left(\wp(\alpha+\beta)-\wp(\beta)\right)\right)}$$

Proof We consider separately the numerator and denominator on the l.-h.s. The numerator $g^\sigma_\beta - g_\beta$ is odd, so it vanishes at half-periods. On the other hand it has double poles at $\eta = \pm\beta$. Hence,

$$g^\sigma_\beta - g_\beta = c \times \frac{\wp'(\eta)}{(\wp(\eta) - \wp(\beta))^2}.$$

The constant c can be calculated by considering the leading term at $\eta = \beta$, and is equal to $-\wp'(\beta)$.

The denominator $g_\alpha g^\sigma_\beta - g_\beta g^\sigma_\alpha$ is also odd. It has a zero of order ≥ 2 at 0, and hence (by oddness) of order ≥ 3. On the other hand, it has double poles at $\eta = \pm\alpha$ and $\pm\beta$. Therefore,

$$g_\alpha g^\sigma_\beta - g_\beta g^\sigma_\alpha = c' \times \frac{\wp'(\eta) \times (\wp(\eta) - C(\alpha, \beta))}{(\wp(\eta) - \wp(\alpha))^2 \times (\wp(\eta) - \wp(\beta))^2}$$

for some constants c' and $C(\alpha, \beta)$. These constants can be calculated by computing the leading terms at α and β;

$$c' = -\big(\wp(\alpha) - \wp(\beta)\big)\big(\wp'(\beta)\big(\wp(\alpha+\beta) - \wp(\alpha)\big) - \wp'(\alpha)\big(\wp(\alpha+\beta) - \wp(\beta)\big)\big).$$

□

6.2.4 We write $F_{\alpha\beta}$ for the function $\frac{g^\sigma_\beta - g_\beta}{g_\alpha g^\sigma_\beta - g_\beta g^\sigma_\alpha}$. Thus the "non-obvious" zeros $\pm\xi_{\alpha\beta}$ of the function $F_{\alpha\beta}$ are defined by the condition $\wp(\pm\xi_{\alpha\beta}) = C(\alpha, \beta)$ (and $\wp'(\pm\xi_{\alpha\beta}) = \pm\sqrt{4C(\alpha,\beta)^3 - c_4(\tau)C(\alpha,\beta) - c_6(\tau)}$, where c_4 and c_6 are the coefficients of the Weierstrass equation).

Lemma 6.2.5 *For any three points α, β and γ on an elliptic curve*

$$\begin{aligned}
&g_\alpha \wedge g_\beta \wedge g_\gamma + g^\sigma_\alpha \wedge g^\sigma_\beta \wedge g^\sigma_\gamma \\
&\quad = (g_\alpha F_{\alpha\beta}) \wedge (g_\beta F_{\beta\alpha}) \wedge g_\gamma + (g^\sigma_\alpha F_{\alpha\beta}) \wedge (g^\sigma_\beta F_{\beta\alpha}) \wedge g^\sigma_\gamma \\
&\quad - \Big((g_\alpha F_{\alpha\gamma}) \wedge F_{\beta\alpha} \wedge (g_\gamma F_{\gamma\alpha}) + (g^\sigma_\alpha F_{\alpha\gamma}) \wedge F_{\beta\alpha} \wedge (g^\sigma_\gamma F_{\gamma\alpha}) \\
&- F_{\alpha\beta} \wedge (g_\beta F_{\beta\gamma}) \wedge (g_\gamma F_{\gamma\beta}) + F_{\alpha\beta} \wedge (g^\sigma_\beta F_{\beta\gamma}) \wedge (g^\sigma_\gamma F_{\gamma\beta})\Big) - F_{\alpha\beta} \wedge F_{\beta\alpha} \wedge (g_\gamma g^\sigma_\gamma) \\
&\quad + \Big((g_\alpha g^\sigma_\alpha) \wedge F_{\beta\alpha} \wedge F_{\gamma\alpha} + F_{\alpha\gamma} \wedge F_{\beta\alpha} \wedge (g_\gamma g^\sigma_\gamma) + 2 \times (F_{\alpha\gamma} \wedge F_{\beta\alpha} \wedge F_{\gamma\alpha}) \\
&\quad + F_{\alpha\beta} \wedge (g_\beta g^\sigma_\beta) \wedge F_{\gamma\beta} + F_{\alpha\beta} \wedge F_{\beta\gamma} \wedge (g_\gamma g^\sigma_\gamma) + 2 \times (F_{\alpha\beta} \wedge (F_{\beta\gamma}) \wedge F_{\gamma\beta})\Big)
\end{aligned}$$

The proof is a straightforward computation.

6.2.6 The first six terms on the r.-h.s. of the previous equation are of the form $\varphi\wedge(1-\varphi)\wedge\psi$, so that by (6) the integral of A_3 of any such term equals a sum of dilogarithms. The last seven terms contain only σ-invariant functions, so the corresponding integrals can be reduced to integrals over $\mathbb{CP}^1$ and are also equal to combinations of dilogarithms. On the other hand, the integral of A_3 of the second term on the l.h.s is equal to the corresponding integral of the first one for obvious geometric reasons.

6.3 Results

If we combine all previous results, we get the following.

Theorem 6.3.1 *For any two distinct nontrivial torsion points α and β on an elliptic curve, we have*

$$\mathcal{L}_{1,1}(\alpha,\beta,0) = D_2(\Phi(\alpha,\beta)), \tag{20}$$

where

$$\Phi(\alpha,\beta) = \frac{1}{\operatorname{ord}(\alpha)\operatorname{ord}(\beta)} \sum_{\gamma\in\langle\alpha,\beta\rangle\setminus 0} \mathrm{Av}_2^{\alpha}\,\mathrm{Av}_2^{\beta}\,\mathrm{Av}_2^{\gamma}\,\Phi_1(\alpha,\beta,\gamma);$$

here $\langle\alpha,\beta\rangle$ denotes the subgroup generated by α and β, $\mathrm{Av}_2^{\xi}(F)(\xi)$ is defined by (19) for a torsion point ξ, and

$$\begin{aligned}
\Phi_1(\alpha,\beta,\gamma) &= \{G_{\alpha\beta}(\gamma)\}_2 + 8\Big\{\frac{\wp(\gamma)-\wp(\alpha)}{\wp(\beta)-\wp(\alpha)}\Big\}_2 \\
&+ \mathrm{Alt}_{\alpha,\beta}\Big(2\{G_{\alpha\gamma}(\beta)\}_2 + 2\{G_{\alpha\gamma}(-\beta)\}_2 - \{G_{\alpha\gamma}(\xi_{\beta\alpha})\}_2 - \{G_{\alpha\gamma}(-\xi_{\beta\alpha})\}_2\Big) \\
&+ 4\,\mathrm{Cyc}_{\alpha,\beta,\gamma}\Big(\Big\{\frac{\wp(2\gamma)-\wp(\alpha)}{\wp(\beta)-\wp(\alpha)}\Big\}_2\Big) - 4\,\mathrm{Cyc}_{\alpha,\beta,\gamma}\Big(\Big\{\frac{C(\alpha,\gamma)-\wp(\alpha)}{\wp(\beta)-\wp(\alpha)}\Big\}_2\Big) \\
&- 2\,\mathrm{Alt}_{\alpha,\beta,\gamma}\Big(\Big\{\frac{\wp(2\gamma)-\wp(\alpha)}{C(\beta,\gamma)-\wp(\alpha)}\Big\}_2\Big) + \mathrm{Alt}_{\alpha,\beta}\Big(2\Big\{\frac{C(\alpha,\gamma)-\wp(\alpha)}{C(\alpha,\beta)-\wp(\alpha)}\Big\}_2 \\
&+2\Big\{\frac{\wp(\gamma)-C(\alpha,\gamma)}{C(\alpha,\beta)-C(\alpha,\gamma)}\Big\}_2 + \Big\{\frac{\wp(2\gamma)-C(\alpha,\gamma)}{C(\alpha,\beta)-C(\alpha,\gamma)}\Big\}_2 + \Big\{\frac{C(\alpha,\gamma)-\wp(2\alpha)}{C(\alpha,\beta)-\wp(2\alpha)}\Big\}_2\Big),
\end{aligned}$$

where

$$G_{\alpha,\beta}(\eta) = \Big(\wp(\eta-\alpha)-\wp(\alpha)\Big)\times K(\alpha,\beta)\frac{(\wp(\eta)-\wp(\alpha))^2}{\wp(\eta)-C(\alpha,\beta)}.$$

Here $C(\alpha,\beta)$ and $K(\alpha,\beta)$ are as in Lemma 6.2.3., the points $\pm\xi_{\alpha\beta}$ are solutions of the equation $\wp(\pm\xi_{\alpha\beta}) = C(\alpha,\beta)$; $\mathrm{Alt}_{\alpha,\beta}$,$\mathrm{Alt}_{\alpha,\beta,\gamma}$, $\mathrm{Cyc}_{\alpha,\beta,\gamma}$ denote the (anti)symmetrization with respect to S_2, S_3, A_3 (with a factor 1).

Theorem 6.3.2 *Let* τ *be a fixed point of* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0, 1$; $m = \det M$, $n = \det(M-1)$. *Let* α *be a torsion point on the curve* E_τ. *Then*

$$\frac{c(\tau-\overline{\tau})}{i}\Big(\frac{m+n+1}{mn}\mathcal{K}_2(\alpha;\tau)+\mathcal{K}_2(0;\tau)\Big) = D_2\Big(\sum_{\substack{\alpha' \in M^{-1}(\alpha) \\ \gamma' \in (M-1)^{-1}(\alpha)}} \Phi(\alpha',\gamma')\Big), \quad (21)$$

where Φ *is defined in the preceding theorem. The argument of the dilogarithm belongs to the kernel of the map* $\delta\colon \{x\}_2 \to x \wedge (1-x)$.

We have proved all statements of this theorem except the last one. It can be proved by the same consideration as in Section 5.

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Andrey Levin,
L. D. Landau Institute of Theoretical Physics,
Russian Academy of Sciences,
117940 Moscow, Russia
e-mail: andrl@landau.ac.ru and alevin@mpim-bonn.mpg.de