

A counterexample to a conjecture of Selmer

Tom Fisher

Abstract

We present a counterexample to a conjecture cited by Cassels [Cal] and attributed to Selmer. The issues raised have been given new significance by the recent work of Heath-Brown [HB] and Swinnerton-Dyer [SWD] on the arithmetic of diagonal cubic surfaces.

1 Introduction

Let E be an elliptic curve over a number field k , with complex multiplication by $\mathbf{Z}[\omega]$ where ω is a primitive cube root of unity. Let $K = k(\omega)$, so that $[K : k] = 1$ or 2 according as $\omega \in k$ or $\omega \notin k$. In his work on cubic surfaces, Heath-Brown [HB] makes implicit use of the following statement.

Theorem 1.1 *If $[K : k] = 2$ and the Tate-Shafarevich group $\text{III}(E/k)$ is finite, then the order of $\text{III}(E/K)[\sqrt{-3}]$ is a perfect square.*

We explain how this result follows from the work of Cassels [CalV], and give an example to show that the condition $[K : k] = 2$ is necessary.

For the application to cubic surfaces, we only need a special case of the theorem, namely that $\text{III}(E/K)[\sqrt{-3}]$ cannot have order 3. This result, still conditional on the finiteness of the Tate-Shafarevich group, has already appeared in [BF] and [SWD]. In fact Swinnerton-Dyer [SWD] vastly generalises Heath-Brown's results. In the case $[K : k] = 2$ he proves the Hasse principle for diagonal cubic 3-folds over k , conditional only on the finiteness of the Tate-Shafarevich group for elliptic curves over k . The condition $[K : k] = 2$ is unnatural, and conjecturally should not appear. However, the counterexample presented in this article suggests that, if we are to follow the methods of Heath-Brown and Swinnerton-Dyer, then this condition on k is unavoidable.

In §2 we recall how it is possible to pass between the fields k and K . Then in §3 we give a modern treatment of the descent by 3-isogeny studied by Selmer [S1] and Cassels [Cal]. In §§4–5 we recall how the conjectures of

Selmer may be deduced from properties of the Cassels–Tate pairing. This culminates in a proof of Theorem 1.1. Finally in §6 we present our new example.

2 Decomposition into Galois eigenspaces

Let E be an elliptic curve over k with complex multiplication by $\mathbf{Z}[\omega]$. The isogeny $[\sqrt{-3}]: E \rightarrow E$ is defined over $K = k(\omega)$. But the kernel $E[\sqrt{-3}]$ is defined over k . It follows that there is a 3-isogeny $\phi: E \rightarrow \tilde{E}$ defined over k with $E[\sqrt{-3}] = E[\phi]$. Here \tilde{E} is a second elliptic curve defined over k , which we immediately recognise as the -3 -twist of E . The dual isogeny $\hat{\phi}: \tilde{E} \rightarrow E$ satisfies $\hat{\phi} \circ \phi = [3]$ and $\phi \circ \hat{\phi} = [3]$. Our notation for the Selmer groups and Tate–Shafarevich groups follows Silverman [Sil, Chapter X].

Lemma 2.1 *If $[K : k] = 2$ then the exact sequence*

$$(1) \quad 0 \rightarrow E(K)/\sqrt{-3}E(K) \rightarrow S^{\sqrt{-3}}(E/K) \rightarrow \text{III}(E/K)[\sqrt{-3}] \rightarrow 0$$

is the direct sum of the exact sequences

$$(2) \quad 0 \rightarrow E(k)/\phi E(k) \rightarrow S^{(\phi)}(E/k) \rightarrow \text{III}(E/k)[\phi] \rightarrow 0$$

and

$$(3) \quad 0 \rightarrow E(k)/\hat{\phi}E(k) \rightarrow S^{(\hat{\phi})}(\tilde{E}/k) \rightarrow \text{III}(\tilde{E}/k)[\hat{\phi}] \rightarrow 0.$$

Proof Since arguments of this type have already appeared in [BF], [N], [SWD] and presumably countless other places in the literature, we will not dwell on the proof. Suffice it to say that we decompose (1) into eigenspaces for the action of $\text{Gal}(K/k)$, and then use the inflation–restriction exact sequence to identify these eigenspaces as (2) and (3). The observation that $[K : k] = 2$ is prime to $\deg \phi = 3$ is crucial throughout the proof. \square

Remark 2.2 Each term of the exact sequence (1) is a $\mathbf{Z}/3\mathbf{Z}$ -vector space with an action of $\text{Gal}(K/k)$. Thus each term is a direct sum of the Galois modules of $\mathbf{Z}/3\mathbf{Z}$ and μ_3 . If we replace E by \tilde{E} in (1) we obtain the same exact sequence of abelian groups, but as Galois modules the summands $\mathbf{Z}/3\mathbf{Z}$ and μ_3 are interchanged.

$$(x_0 : x_1 : x_2) \mapsto (x_0 x_1 x_2 : a_1 x_1^2 : a_2 x_2^2 : a_0 x_0^3).$$

then the 3-covering map $T[a_0, a_1, a_2] \rightarrow \widetilde{E}_A$ is given by
 it is clear that $A \in \text{im } \delta$. If \widetilde{E}_A has Weierstrass equation $Y^2 Z = -4AX^3 + Z^3$ where the products are over $\nu \in \mathbf{Z}/3\mathbf{Z}$. Since $T[a_0, a_1, a_2] \simeq T[a_1, a_2, a_0]$

$$(6) \quad T[a_0, a_1, a_2] \simeq C_{A, \alpha} \quad \text{for } A = \prod a_\nu \text{ and } \alpha = \prod a_\nu^r$$

that
 the torsor under E_A described by $\alpha \in k^*/k^{*3}$. The proof of Lemma 3.1 shows
 The group $H^1(k, E_A)$ parametrises the torsors under E_A . We write $C_{A, \alpha}$ for

$$(5) \quad 0 \rightarrow E^A(k)/\phi E^A(k) \xrightarrow{\delta} k^*/k^{*3} \rightarrow H^1(k, E^A)[\phi] \rightarrow 0.$$

exact sequence
 where \widetilde{E}_A is the -3 -twist of E_A . Taking Galois cohomology we obtain an

$$0 \rightarrow \mu_3 \rightarrow E_A \xrightarrow{\phi} \widetilde{E}_A \rightarrow 0$$

map ϕ which gives an exact sequence of Galois modules
 $E^A[1 - \omega] = E^A[\sqrt{-3}]$ is generated by $(0 : \omega : -\omega^2)$. So as in §2 there is a
 multiplication by $\mathbf{Z}[\omega]$ where $\omega : (x_0 : x_1 : x_2) \mapsto (\omega x_0 : x_1 : x_2)$ and that
 Temporarily working over $K = k(\omega)$ we note that E_A has complex mul-

is a torsor under E_A and that E_A is the jacobian of $T[a_0, a_1, a_2]$.
 belongs to the translation subgroup of $\text{Aut}(E_A)$. It follows that $T[a_0, a_1, a_2]$
 by $x_i \mapsto \omega^i x_i$. But since μ_3 acts on E_A without fixed points, this action
 The cocycle $\sigma(\psi)\psi^{-1}$ takes values in the subgroup $\mu_3 \subset \text{Aut}(E_A)$ generated

$$\psi : (x_0 : x_1 : x_2) \mapsto (a_2 x_0 : \alpha^{2/3} x_1 : \alpha^{1/3} a_2 x_2).$$

$\alpha = a_1 a_2^2$, given by
Proof There is an isomorphism $T[a_0, a_1, a_2] \simeq E_A$ defined over $k(\sqrt[3]{\alpha})$ where

1 with jacobian E_A where $A = a_0 a_1 a_2$.
Lemma 3.1 *The diagonal plane cubic $T[a_0, a_1, a_2]$ is a smooth curve of genus*

be found in [CaL, §18].
 equation $y^2 = x^3 - 432A^2$. An alternative proof of the following lemma may
 element $0 = (0 : 1 : -1)$. It is well known [St] that E_A has Weierstrass
 where $a_0, a_1, a_2 \in k^*/k^{*3}$. Let E_A be the elliptic curve $T[A, 1, 1]$ with identity

$$(4) \quad a_0 x_0^3 + a_1 x_1^3 + a_2 x_2^3 = 0$$

Let k be a number field. Let $T[a_0, a_1, a_2]$ be the diagonal plane cubic

3 Computation of Selmer groups

The Selmer group attached to ϕ is

$$S^{(\phi)}(E_A/k) = \{ \alpha \in k^*/k^{*3} \mid C_{A,\alpha}(k_{\mathfrak{p}}) \neq \emptyset \text{ for all primes } \mathfrak{p} \}.$$

Since $\deg \phi = 3$ is odd we have ignored the infinite places. We write $\delta_{\mathfrak{p}}$ for the local connecting map obtained when we apply (5) to the local field $k_{\mathfrak{p}}$. Then the condition $C_{A,\alpha}(k_{\mathfrak{p}}) \neq \emptyset$ may also be written $\alpha \in \text{im } \delta_{\mathfrak{p}}$. Using (6) to give equations for $C_{A,\alpha}$ it is easy to prove

Lemma 3.2 *Let k be a number field, and \mathfrak{p} a prime not dividing 3. Let $\mathfrak{o}_{\mathfrak{p}}$ denote the ring of integers of $k_{\mathfrak{p}}$. Then*

$$\text{im } \delta_{\mathfrak{p}} = \begin{cases} \mathfrak{o}_{\mathfrak{p}}^{\mathfrak{d}}/\mathfrak{o}_{\mathfrak{p}}^{\mathfrak{d}3} & \text{if } \text{ord}_{\mathfrak{p}}(A) \equiv 0 \pmod{3} \\ \langle A \rangle & \text{if } \text{ord}_{\mathfrak{p}}(A) \not\equiv 0 \pmod{3}. \end{cases}$$

If \mathfrak{p} divides 3 the situation is more complicated, although we still have

$$(7) \quad \text{im } \delta_{\mathfrak{p}} \subset \mathfrak{o}_{\mathfrak{p}}^{\mathfrak{d}}/\mathfrak{o}_{\mathfrak{p}}^{\mathfrak{d}3} \quad \text{if } \text{ord}_{\mathfrak{p}}(A) \equiv 0 \pmod{3}.$$

If $\omega \in k_{\mathfrak{p}}$ Tate local duality tells us that $\text{im } \delta_{\mathfrak{p}}$ is a maximal isotropic subspace with respect to the Hilbert norm residue symbol

$$(8) \quad k_{\mathfrak{p}}^{\mathfrak{d}}/k_{\mathfrak{p}}^{\mathfrak{d}3} \times k_{\mathfrak{p}}^{\mathfrak{d}}/k_{\mathfrak{p}}^{\mathfrak{d}3} \rightarrow \mu_3.$$

The next lemma treats the case $k = \mathbb{Q}(\omega)$. This field has ring of integers $\mathbb{Z}[\omega]$ and class number 1. The unique prime above 3 is $\pi = \omega - \omega^2$.

Lemma 3.3 *Let $A \in \mathbb{Z}[\omega]$ be nonzero and cube-free. Then*

$$\text{im } \delta_{\pi} = \begin{cases} \langle A, (1+A)/(1-A) \rangle & \text{if } \text{ord}_{\pi}(A) \neq 0 \\ \langle A, 1 - \pi^3 \rangle & \text{if } \text{ord}_{\pi}(A) = 0 \text{ and } A \not\equiv \pm 1 \pmod{\pi^3} \\ \langle \omega(1+3a), 1 - \pi^3 \rangle & \text{if } A = \pm(1+a\pi^3) \text{ for some } a \in \mathbb{Z}[\omega]. \end{cases}$$

Proof We recall [CF, Exercise 2.13] that $k_{\pi}^{\mathfrak{d}}/k_{\pi}^{\mathfrak{d}3}$ has basis $\pi, \omega, 1 - \pi^2, 1 - \pi^3$ and that these elements define a filtration compatible with the pairing (8). By Tate local duality it follows that $\text{im } \delta_{\pi}$ has order 9. So to prove the lemma it suffices to prove the inclusions \supset . As always $A \in \text{im } \delta_{\pi}$, whereas (7) and Tate local duality tell us that $1 - \pi^3 \in \text{im } \delta_{\pi}$. There is at most one more element to find.

(i) Suppose $\text{ord}_{\pi}(A) \neq 0$. If α satisfies $\alpha - \alpha^{-1} = A$ then $T[A, \alpha, \alpha^{-1}]$ is soluble. Splitting into the cases $\text{ord}_{\pi}(A) = 1$ or 2 we find

$$4A/(1 - A^2) \equiv A \pmod{\pi^4}.$$

Thus $\alpha = (1 + A)/(1 - A)$ provides a solution mod π^4 .

Remark 4.4 According to the formulae and tables of Stephens [St], the above examples have $L(E_A, 1) \neq 0$. So the claims $\text{rank } E_A(\mathbf{Q}) = 0$ could equally be deduced from the work of Coates and Wiles [CW].

$$\text{III}(E_{473}/\mathbf{Q})[\phi] \simeq \mathbf{Z}/3\mathbf{Z} \quad \text{and} \quad \text{III}(E_{473}/\mathbf{Q})[\phi] \simeq \mathbf{Z}/3\mathbf{Z}.$$

Then (9) gives $S(\phi)(E_{473}/\mathbf{Q}) \simeq (\mathbf{Z}/3\mathbf{Z})^2$ and $S(\phi)(E_{473}/\mathbf{Q}) \simeq \mathbf{Z}/3\mathbf{Z}$. But a 2-descent [S2], [Cr] shows that $E_{473}(\mathbf{Q})$ has rank 0. We deduce

$$S(\sqrt{-3})(E_{473}/K) \simeq \langle 11, 1 - 6\omega, 1 - 6\omega^2 \rangle \subset K^*/K^{*3}.$$

Example 4.3 Let $A = 473$. Lemmas 3.2 and 3.3 tell us that

$$\text{III}(E_{60}/\mathbf{Q})[3] \simeq (\mathbf{Z}/3\mathbf{Z})^2.$$

Then (9) gives $S(\phi)(E_{60}/\mathbf{Q}) \simeq (\mathbf{Z}/3\mathbf{Z})^3$ and $S(\phi)(E_{60}/\mathbf{Q}) = 0$. But a 2-descent [Gal, §15], [Cr] shows that $E_{60}(\mathbf{Q})$ has rank 0. We deduce

$$S(\sqrt{-3})(E_{60}/K) \simeq \langle 2, 3, 5 \rangle \subset K^*/K^{*3}.$$

Example 4.2 Let $A = 60$. Lemmas 3.2 and 3.3 tell us that

The following examples were found by Selmer [S1], [S2].

$$(9) \quad S(\sqrt{-3})(E_A/K) \simeq S(\phi)(E_A/\mathbf{Q}) \oplus S(\phi)(E_A/\mathbf{Q}).$$

Lemma 2.1 gives a decomposition into $\text{Gal}(K/\mathbf{Q})$ -eigenspaces

Proof See [St, §6] or [K, Chapter 1, Problem 7]. \square

$$E_A(\mathbf{Q})^{\text{tors}} = 0 \quad \text{and} \quad \widetilde{E}_A(\mathbf{Q})^{\text{tors}} \simeq \mathbf{Z}/3\mathbf{Z}.$$

Lemma 4.1 If $A \geq 3$ then the torsion subgroups are

In this section we take $k = \mathbf{Q}$, so that $K = \mathbf{Q}(\omega)$. We consider the elliptic curves E_A and \widetilde{E}_A over \mathbf{Q} where $A \geq 2$ is a cube-free integer.

4 Selmer's conjectures

we see that $\alpha = \omega(1 + 3a)$ provides a solution mod π^4 . \square

$$(1 + \pi^3 a) + \omega(1 + 3a) + \omega^2(1 - 3a) = 0$$

(ii) Suppose $A = 1 + a\pi^3$ for some $a \in \mathbf{Z}[\omega]$. If α satisfies $A + \alpha + \alpha^{-1} = 0$ then $T[A, \alpha, \alpha^{-1}]$ is soluble. In view of the identity

Example 4.2 tells us that each of the curves

$$(10) \quad \begin{aligned} T[3, 4, 5] : & 3x_3^0 + 4x_3^1 + 5x_3^2 = 0 \\ T[1, 3, 20] : & x_3^0 + 3x_3^1 + 20x_3^2 = 0 \\ T[1, 4, 15] : & x_3^0 + 4x_3^1 + 15x_3^2 = 0 \\ T[1, 5, 12] : & x_3^0 + 5x_3^1 + 12x_3^2 = 0 \end{aligned}$$

is a counterexample to the Hasse Principle for smooth curves of genus 1 defined over \mathbf{Q} . Selmer proves this without the need for a 2-descent. Instead he shows that the equations (10) are insoluble over \mathbf{Q} by writing them as norm equations. As Cassels explains [Cal, §11] this is equivalent to performing a second descent, *i.e.* computing the middle group in

$$(11) \quad \widetilde{E}_A(\mathbf{Q})/\phi E_A(\mathbf{Q}) \subset \widehat{\phi} S^{(3)}(E_A/\mathbf{Q}) \subset S^{(\phi)}(E_A/\mathbf{Q}).$$

In fact Selmer's calculations suffice to show that $\text{III}(E_{60}/\mathbf{Q})(3) \simeq (\mathbf{Z}/3\mathbf{Z})^2$. In other words $\text{III}(E_{60}/\mathbf{Q})$ does not contain an element of order 9. More recent work of Rubin [M] improves this to $\text{III}(E_{60}/\mathbf{Q}) \simeq (\mathbf{Z}/3\mathbf{Z})^2$.

Selmer also gave practical methods for computing the two right hand groups in

$$(12) \quad E_A(\mathbf{Q})/\widehat{\phi} E_A(\mathbf{Q}) \subset \phi S^{(3)}(E_A/\mathbf{Q}) \subset S^{(\phi)}(E_A/\mathbf{Q}).$$

Following Stephens [St] we write $g_1 + 1, \chi_1^1 + 1, \lambda_1 + 1$ for the dimensions of the $\mathbf{Z}/3\mathbf{Z}$ -vector spaces (11) and g_2, χ_2^2, λ_2 for the dimensions of the $\mathbf{Z}/3\mathbf{Z}$ -vector spaces (12). Trivially we have $0 \leq g_1 \leq \chi_1^1 \leq \lambda_1, 0 \leq g_2 \leq \chi_2^2 \leq \lambda_2$ and $\text{rank } E_A(\mathbf{Q}) = g_1 + g_2$. Based on a large amount of numerical evidence, Selmer [S3] made the following

Conjecture 4.5 Let $A \geq 2$ be a cube-free integer. Let E_A be the elliptic curve $x^3 + y^3 = Az^3$ defined over \mathbf{Q} . Then

Weak form The second descent excludes an even number of generators, *i.e.* $\lambda_1 \equiv \chi_1^1 \pmod{2}$ and $\lambda_2 \equiv \chi_2^2 \pmod{2}$.

Strong form The number of generators of infinite order for $E_A(\mathbf{Q})$ is an even number less than what is indicated by the first descent, *i.e.*

$$\lambda_1 + \lambda_2 \equiv g_1 + g_2 \pmod{2}.$$

For $A = 473$, Selmer found $\lambda_1 = \chi_1^1 = \lambda_2 = \chi_2^2 = 1$ yet $g_1 = g_2 = 0$. He was thus aware of the need to combine the contributions from ϕ and $\widehat{\phi}$ in the strong form of his conjecture.

The pairing was later generalised to abelian varieties by Tate, and so is known as the Cassels–Tate pairing. The most striking applications in the case of elliptic curves come from the following easy lemma.

Proof This was proved by Cassels [CaIV] in the case $\phi = [m]$ for m a rational integer. The general case follows by his methods and is explained in [E]. \square

Theorem 5.1 Let $\phi: E \rightarrow E'$ be an isogeny of elliptic curves over k . Then $x \in \text{III}(E/k)$ belongs to the image of $\hat{\phi}: \text{III}(E'/k) \rightarrow \text{III}(E/k)$ if and only if $\langle x, y \rangle = 0$ for all $y \in \text{III}(E/k)[\phi]$.

with the following nondegeneracy property.

$$(13) \quad \langle \cdot, \cdot \rangle: \text{III}(E/k) \times \text{III}(E/k) \rightarrow \mathbf{Q}/\mathbf{Z}$$

[CaIV] defines an alternating bilinear pairing on elliptic curves over k we shall write $\hat{\phi}: E' \rightarrow E$ for the dual isogeny. Cassels Let E be an elliptic curve over a number field k . For $\phi: E \rightarrow E'$ an isogeny of

5 The Cassels–Tate pairing

In the next section we recall how Conjecture 4.7 follows from the work of Cassels, the strong form being conditional on the finiteness of $\text{III}(E_A/k)$.

Strong form The order of $\text{III}(E_A/k)[\phi]$ times that of $\text{III}(E_A/k)[\hat{\phi}]$ is a perfect square.

Weak form The index of the subgroup $\hat{\phi}(\text{III}(E_A/k)[3]) \subset \text{III}(E_A/k)[\phi]$ is a perfect square. The same is true for $\phi(\text{III}(E_A/k)[3]) \subset \text{III}(E_A/k)[\hat{\phi}]$.

Conjecture 4.7 Suppose that $A \in k^*$ is not a perfect cube and let E_A be the elliptic curve $x^3 + y^3 = Az^3$ defined over k . Then

Now let k be any number field. Conjecture 4.5 is equivalent to the case $k = \mathbf{Q}$ of the following

of Selmer’s conjecture as $r(A) \equiv s(A) \pmod{2}$. that Heath-Brown uses to define $s(A)$. Naturally he writes the strong form By (9) the order of $S^{(\sqrt{-3})}(E_A/K)$ is $3^{s(A)+1}$ and in fact it is this relation

$$r(A) = \text{rank } E_A(\mathbf{Q}) = g_1 + g_2 \quad \text{and} \quad s(A) = \lambda_1 + \lambda_2.$$

Remark 4.6 In Heath-Brown’s notation [HB] we have

Lemma 5.2 *If a finite abelian group admits a nondegenerate alternating bilinear pairing, then its order must be a perfect square.*

The weak form of Conjecture 4.7 is a special case of

Corollary 5.3 *Let $\phi: E \rightarrow E'$ be an m -isogeny of elliptic curves over k . Then the subgroup $\widehat{\phi}(\text{III}(E'/k)[m]) \subset \text{III}(E/k)[\phi]$ has index a perfect square.*

Proof According to Theorem 5.1 the pairing (13) restricted to $\text{III}(E/k)[\phi]$ has kernel $\widehat{\phi}(\text{III}(E'/k)[m])$. We are done by Lemma 5.2. \square

Let us assume that $\text{III}(E/k)$ is finite. So by Theorem 5.1 and Lemma 5.2 the order of $\text{III}(E/k)$ is a perfect square. If $\phi: E \rightarrow E'$ is an isogeny of elliptic curves over k then the same conclusions will hold for E' . We define

$$(14) \quad \langle \cdot, \cdot \rangle_{\phi}: \text{III}(E/k) \times \text{III}(E'/k) \rightarrow \mathbf{Q}/\mathbf{Z}; \quad (x, y) \mapsto \langle \phi x, y \rangle = \langle x, \widehat{\phi} y \rangle$$

where the equality on the right is [CaVIII, Theorem 1.2]. The strong form of Conjecture 4.7 is a special case of

Corollary 5.4 *Let $\phi: E \rightarrow E'$ be an isogeny of elliptic curves over k . If $\text{III}(E/k)$ is finite then the order of $\text{III}(E/k)[\phi]$ times that of $\text{III}(E'/k)[\widehat{\phi}]$ is a perfect square.*

Proof According to Theorem 5.1 the left and right kernels of $\langle \cdot, \cdot \rangle_{\phi}$ are $\text{III}(E/k)[\phi]$ and $\text{III}(E'/k)[\widehat{\phi}]$. We obtain a nondegenerate pairing

$$\text{III}(E/k)[\phi]/\text{III}(E/k)[\phi] \times \text{III}(E'/k)[\widehat{\phi}]/\text{III}(E'/k)[\widehat{\phi}] \rightarrow \mathbf{Q}/\mathbf{Z}.$$

We deduce that these quotients have the same order and are done since $\text{III}(E/k)$ and $\text{III}(E'/k)$ each have order a perfect square. \square

Another well known consequence is

Corollary 5.5 *Let E be an elliptic curve over k whose Tate–Shafarevich group is finite, and let m be a rational integer. Then the order of $\text{III}(E/k)[m]$ is a perfect square.*

Proof According to Theorem 5.1 the kernel of $\langle \cdot, \cdot \rangle_m$ is $\text{III}(E/k)[m]$. We obtain a nondegenerate alternating pairing

$$\text{III}(E/k)/\text{III}(E/k)[m] \times \text{III}(E/k)/\text{III}(E/k)[m] \rightarrow \mathbf{Q}/\mathbf{Z}.$$

We apply Lemma 5.2 to this pairing and are done since $\text{III}(E/k)$ has order a perfect square. \square

Remark 5.6 We could equally deduce Corollary 5.4 from Corollaries 5.3 and 5.5.

Proof of Theorem 1.1 Let E be an elliptic curve over k with complex multiplication by $\mathbf{Z}[\omega]$ and suppose that $[K : k] = 2$. Lemma 2.1 tells us that

$$\text{III}(E/K)[\sqrt{-3}] \simeq \text{III}(E/k)[\phi] \oplus \text{III}(E/k)[\phi].$$

Assuming $\text{III}(E/k)$ is finite, Corollary 5.4 shows that the group on the right has order a perfect square. So the group on the left has order a perfect square, and this is precisely the statement of Theorem 1.1. \square

In the first of his celebrated series of papers, Cassels [Cas] defines a pairing $S^{(\sqrt{-3})}(E^A/K) \times S^{(\sqrt{-3})}(E^A/K) \rightarrow \mu_3$. It is of course a special case of the pairing (13). He uses it to prove the weak form of Conjecture 4.7 in the case $[K : k] = 1$. However in the introduction to the same paper he misquotes the strong form of Selmer's conjecture. The statement he gives is equivalent to

- If $[K : k] = 1$ then the order of $\text{III}(E^A/K)[\sqrt{-3}]$ is a perfect square.

It is this statement to which we have found a counterexample. It is possible that Cassels was misled by earlier work of Selmer at a time when he did not appreciate the need to combine the contributions from ϕ and $\tilde{\phi}$ in the strong form of his conjecture.

Remark 5.7 It is tempting to try and prove Theorem 1.1 also in the case $[K : k] = 1$ by imitating the proof of Corollary 5.5. However the isogeny $[\sqrt{-3}]$ has dual $[-\sqrt{-3}]$ and this extra sign means that the pairing $\langle \cdot, \cdot \rangle_{\sqrt{-3}}$ is symmetric rather than alternating. Lemma 5.2 does not apply.

6 A new example

In this section we take $K = \mathbf{Q}(\omega)$. Let E_A be the elliptic curve $x^3 + y^3 = Az^3$. We aim to find $A \in K$ such that the order of $\text{III}(E_A/K)[\sqrt{-3}]$ is not a perfect square. As in Example 4.3 our method is to compare a 3-descent with a 2-descent. The form of the curves E_A makes the 3-descent easy. We use the results of §3 to compute the Selmer group $S^{(\sqrt{-3})}(E_A/K)$. But `mwrnk` is written specifically for elliptic curves over \mathbf{Q} , whereas Theorem 1.1 tells us that there are no examples of the required form with $A^2 \in \mathbf{Q}$. Fortunately we were able to use a program of Denis Simon [Si1], [Si2], written using the computer algebra package pari [BBBCO], that extends Cremona's work on 2-descents to general number fields (in practice of degrees 1 up to 5). We consider all cube-free $A \in \mathbf{Z}[\omega]$ with $A^2 \notin \mathbf{Q}$ and $\text{Norm}(A) \leq 150$. We ignore repeats of the form $\pm\sigma(A)$ for $\sigma \in \text{Gal}(K/\mathbf{Q})$. In all 123 cases a calculation based on Lemmas 3.2 and 3.3 shows that $S^{(\sqrt{-3})}(E_A/K)$ is isomorphic

to either $\mathbf{Z}/3\mathbf{Z}$ or $(\mathbf{Z}/3\mathbf{Z})^2$. In the 98 cases where $S^{(\sqrt{-3})}(E_A/K) \simeq \mathbf{Z}/3\mathbf{Z}$ it follows immediately that $\text{rank } E_A(K) = 0$. In the remaining 25 cases we run Simon's program. For 20 of these curves the program exhibits a point of infinite order. Since $E_A(K)$ has the structure of $\mathbf{Z}[\omega]$ -module, we are able to deduce that $\text{rank } E_A(K) = 2$. The remaining 5 cases are

$$A = \pm(3 + 7\omega), \pm(9 + \omega), \pm(12 + 5\omega), \pm(6 + 13\omega), \pm(13 + 7\omega)$$

and their Galois conjugates. In each of these cases Simon's program reports that $\text{rank } E_A(K) = 0$. Reducing modulo some small primes we find $E_A(K) \simeq \mathbf{Z}/3\mathbf{Z}$. Thus

$$\text{III}(E_A/K)[\sqrt{-3}] \simeq \mathbf{Z}/3\mathbf{Z}.$$

For the remainder of this article we restrict attention to the first of these examples, namely $A = 3 + 7\omega$, and give further details of the descent calculations involved. In particular we establish the counterexample of the title in a way that is independent of Simon's program.

We begin by checking the above computation of $S^{(\sqrt{-3})}(E_A/K)$ for $A = 3 + 7\omega$. Since (A) is prime, Lemma 3.2 tells us that

$$(15) \quad S^{(\sqrt{-3})}(E_A/K) \subset \langle \omega, 3 + 7\omega \rangle.$$

We check the local conditions at the primes (π) and (A) above 3 and 37 respectively.

- Since $37 \equiv 1 \pmod{9}$ we know that ω is a cube locally at (A) .
- Lemma 3.3 gives $\text{im } \delta_\pi = \langle A, 1 - \pi^3 \rangle \subset K_\pi^*/K_\pi^{*3}$. Since $A = \omega - \pi^3$ it is clear that ω belongs to this subgroup.

It follows that equality holds in (15) as required. Given the provisional nature of Simon's program we have taken the liberty of writing out the 2-descent for $A = 3 + 7\omega$ in the style of Cassels [CaL, p. 72-73]. The curve E_A has Weierstrass form

$$(16) \quad Y^2 = X^3 - 2^4 3^3 (3 + 7\omega)^2.$$

The 2-descent takes place over the field $L = K(\delta)$ where $\delta^3 = 4(3 + 7\omega)$. According to part [BBCO]₁, L has class number $h = 3$, and fundamental units

$$\eta_1 = (-7 - 3\omega) + (-3 - 2\omega)\delta + (-2 + \omega)\delta^2/2$$

$$\eta_2 = (-7 - 3\omega) + (2 - \omega)\delta + (3 + 2\omega)\delta^2/2.$$

[†]These calculations were performed using Version 2.0.20 (beta)

Furthermore part 1 is able to certify these results, independent of any conjecture. We have chosen η_1 and η_2 to be K -conjugates. They have minimal polynomial

$$x^3 + (21 + 9\omega)x^2 + (102 - 165\omega)x - 1.$$

If $(X, Y) = (r/t^2, s/t^3)$ is a solution of (16), with fractions in lowest terms, then a common prime divisor of any two of

$$r - 3\delta^2 t^2, \quad r - 3\omega\delta^2 t^2, \quad r - 3\omega^2\delta^2 t^2$$

must divide $2(1 - \omega)(3 + 7\omega)$. Since 2, $(1 - \omega)$, $(3 + 7\omega)$ ramify completely, $r - 3\delta^2 t^2$ must be a perfect ideal square. Since h is odd it follows that $S_{(2)}(E/K)$ is a subgroup of $\langle -1, \eta_1, \eta_2 \rangle \subset L^*/L^{*2}$. We claim that $S_{(2)}(E/K)$ is trivial. By considering norms from L to K , it suffices to show that the

$$r - 3\delta^2 t^2 = \eta\alpha^2 \quad \text{with } \eta = \eta_1, \eta_2 \text{ or } 1/(\eta_1\eta_2)$$

is insoluble for $r, t \in K$ and $\alpha \in L$. The action of $\text{Gal}(L/K)$ shows that we need only consider the case $\eta = \eta_1$. Put $\alpha = n + v\delta + w\delta^2$. Equating coefficients of powers of δ we obtain

$$\begin{aligned} 0 &= (-3 - 2\omega)n^2 + (-14 - 6\omega)nv + (-26 - 36\omega)v^2 \\ &\quad + (-52 - 72\omega)nw + (40 - 104\omega)vw + (-148\omega)w^2 \\ -3t^2 &= ((-2 + \omega)/2)n^2 + (-6 - 4\omega)nv + (-7 - 3\omega)v^2 \\ &\quad + (-14 - 6\omega)nw + (-52 - 72\omega)vw + (20 - 52\omega)w^2. \end{aligned}$$

On putting

$$\begin{aligned} n &= (-8 + 6\omega)e + (-6 - 34\omega)f + (-20 + 15\omega)g \\ v &= (-4 - 4\omega)e + (12 + 4\omega)f + (-10 - 11\omega)g \\ w &= (1 - \omega)e + (1 + 4\omega)f + (2 - 2\omega)g \end{aligned}$$

in the first equation, it becomes

$$0 = (3 + 7\omega)g^2 - 16ef.$$

Hence there are m, n such that

$$e : f : g = m^2 : (3 + 7\omega)n^2 : 4mn.$$

On substituting into the second equation, we get

$$\begin{aligned} -3t^2 &= 2(-1 - 4\omega)m^4 + 8(-4 + 3\omega)m^3n + 4(21 + 12\omega)m^2n^2 \\ &\quad + 8(4 - 3\omega)mn^3 + 2(-33 - 40\omega)n^4. \end{aligned}$$

But this is impossible over the 2-adic completion of K . Hence $S_{(2)}(E^A/K)$ is trivial and $\text{rank } E^A(K) = 0$ as claimed.

Acknowledgments

The author would like to thank Laura Basile, Alexei Skorobogatov and Sir Peter Swinnerton-Dyer for useful conversations.

References

- [BF] C.L. Basile and T.A. Fisher, Diagonal cubic equations in four variables with prime coefficients, *Rational points on algebraic varieties*, 1–12, Progr. Math., **199**, Birkhäuser, Basel, 2001
- [BBBCO] C. Batut, K. Belabas, D. Bernardi, H. Cohen and M. Olivier, pari/gp, a computer algebra package, <http://www.pari/gp-home.de>
- [CaI] J.W.S. Cassels, Arithmetic on curves of genus 1. I, On a conjecture of Selmer, *J. Reine Angew. Math.* **202** (1959) 52–99
- [CaIV] J.W.S. Cassels, Arithmetic on curves of genus 1. IV, Proof of the Hauptvermutung, *J. reine angew. Math.* **211** (1962) 95–112
- [CaVIII] J.W.S. Cassels, Arithmetic on curves of genus 1. VIII, On conjectures of Birch and Swinnerton-Dyer, *J. Reine Angew. Math.* **217** (1965) 180–199
- [CaL] J.W.S. Cassels, *Lectures on elliptic curves*, LMSST **24**, Cambridge University Press, Cambridge, 1991
- [CF] J.W.S. Cassels and A. Fröhlich (Eds.), *Algebraic number theory*, Academic Press, London, 1967
- [CW] J. Coates and A. Wiles, On the conjecture of Birch and Swinnerton-Dyer, *Invent. Math.* **39** (1977) 223–251
- [Cr] J.E. Cremona, mwrank, a program for performing 2-descent on elliptic curves over \mathbb{Q} , <http://www.maths.nottingham.ac.uk/personal/jec/ftp/progs>
- [F] T.A. Fisher, The Cassels–Tate pairing and the Platonic solids, *J. Number Theory* **98** (2003) 105–155
- [HB] D.R. Heath-Brown, The solvability of diagonal cubic Diophantine equations, *Proc. London Math. Soc.* (3) **79** (1999) 241–259
- [K] N. Koblitz, *Introduction to elliptic curves and modular forms*, GTM **97**, Springer-Verlag, New York, 1993

[M] B. Mazur, On the passage from local to global in number theory, *Bull. Amer. Math. Soc.* **29** (1993) 14–50

[N] J. Nekoř, Class numbers of quadratic fields and Shimura’s correspondence, *Math. Ann.* **287** (1990) 577–594

[S1] E.S. Selmer, The diophantine equation $ax^3 + by^3 + cz^3 = 0$, *Acta Math.* **85** (1951) 203–362

[S2] E.S. Selmer, The diophantine equation $ax^3 + by^3 + cz^3 = 0$, completion of the tables, *Acta Math.* **92** (1954) 191–197

[S3] E.S. Selmer, A conjecture concerning rational points on cubic curves, *Math. Scand.* **2** (1954) 49–54

[S11] J.H. Silverman, *The arithmetic of elliptic curves*, GTM **106**, Springer-Verlag, New York, 1986

[S12] D. Simon, Computing the rank of elliptic curves over number fields, *LMS J. Comput. Math.* **5** (2002) 7–17

[S22] D. Simon, e11.gp, a program for calculating the rank of elliptic curves over number fields, <http://www.math.uicaen.fr/~simon>

[St] N.M. Stephens, The diophantine equation $X^3 + Y^3 = DZ^3$ and the conjectures of Birch and Swinnerton-Dyer, *J. Reine Angew. Math.* **231** (1968) 121–162

[SWD] H.P.F. Swinnerton-Dyer, The solubility of diagonal cubic surfaces, *Ann. Sci. École Norm. Sup.* (4) **34** (2001) 891–912

Tom Fisher,
 DPMMS, Centre for Mathematical Sciences,
 Wilberforce Road, Cambridge CB3 0WB, UK
 e-mail: T.A.Fisher@dpmms.cam.ac.uk
 web: <http://www.dpmms.cam.ac.uk/~taf1000>