

$$\alpha: \mathbb{Z}_2^t \xrightarrow{\cong} K(H) = \{ \mathbf{x} \in A \mid t_*^{\mathbf{x}} \mathcal{L} \cong \mathcal{L} \text{ if } c_1(\mathcal{L}) = H \}.$$

If  $A$  is an abelian surface with a polarisation  $H$  of type  $(1, t)$ ,  $t > 1$ , then a *canonical level structure*, or simply *level structure*, is a symplectic isomorphism

## I Background

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It follows from the theorem of L. Borisov [2] that  $\mathcal{A}_{\text{bil}}^t$  is of general type for  $t$  sufficiently large. If  $t = p$  is prime, then it follows from [7] and [12] that  $\mathcal{A}_{\text{bil}}^p$  is of general type for  $p \geq 37$ . Our result provides an effective bound in the general case and a better bound in the case  $t = p$ . As far as we know, all previous explicit general type results (for instance [7, 12, 15, 8, 16]) have been for the cases  $t = p$  or  $t = p^2$  only.

It is for brevity that we assume  $t$  is odd. If  $t$  is even the combinatorial details are more complicated, especially when  $t \equiv 2 \pmod 4$ , but the method is still applicable. In fact the method is essentially that of [12], with some modifications.

**Theorem**  $\mathcal{A}_{\text{bil}}^t$  is of general type for  $t$  odd and  $t \geq 17$ .

Abelian surfaces with weak bilinear structure were introduced by S. Mukai in [14]. There is a coarse moduli space, denoted  $\mathcal{A}_{\text{bil}}^t$ , for abelian surfaces of type  $(1, t)$  with weak bilinear structure.  $\mathcal{A}_{\text{bil}}^t$  is a Siegel modular threefold, and can be compactified in a standard way by Mumford's toroidal method [1]. We denote the toroidal compactification (in this situation also known as the Igusa compactification) by  $\mathcal{A}_{\text{bil}}^{t*}$ . It is a projective variety over  $\mathbb{C}$ , and it is shown in [14] that  $\mathcal{A}_{\text{bil}}^{t*}$  is rational for  $t \leq 5$ . In this paper we examine the Kodaira dimension  $\kappa(\mathcal{A}_{\text{bil}}^{t*})$  for larger  $t$ . Our main result is the following (Theorem VIII.1).

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Abelian surfaces with odd bilinear structure

The moduli space  $\mathcal{A}_{\text{lev}}^t$  of abelian surfaces with a canonical level structure has been studied in detail in [11], chiefly in the case  $t = p$ . A *collevel structure* on  $A$  is a level structure on the dual abelian surface  $\hat{A}$ : note that  $H$  induces a polarisation  $\hat{H}$  on  $\hat{A}$ , also of type  $(1, t)$ . Alternatively, a collevel structure may be thought of as a symplectic isomorphism

$$\beta : \mathbb{Z}_t^2 \rightarrow A[t]/K(H)$$

where  $A[t]$  is the group of all  $t$ -torsion points of  $A$ . Obviously the moduli space  $\mathcal{A}_{\text{col}}^t$  of abelian surfaces of type  $(1, t)$  with a collevel structure is isomorphic to  $\mathcal{A}_{\text{lev}}^t$ , and each of them has a forgetful morphism  $\psi_{\text{lev}}, \psi_{\text{col}}$  to the moduli space  $\mathcal{A}_t$  of abelian surfaces of type  $(1, t)$ . We define

$$\mathcal{A}_{\text{bil}}^t = \mathcal{A}_{\text{lev}}^t \times_{\mathcal{A}_t} \mathcal{A}_{\text{col}}^t.$$

The forgetful map  $\psi_{\text{lev}}^t : \mathcal{A}_{\text{lev}}^t \rightarrow \mathcal{A}_t$  is the quotient map under the action of  $\text{SL}(2, \mathbb{Z}_t)$  given by

$$\gamma : [(A, H, \alpha)] \mapsto [(A, H, \alpha\gamma)]$$

where  $\gamma \in \text{SL}(2, \mathbb{Z}_t)$  is viewed as a symplectic automorphism of  $\mathbb{Z}_t^2$ . The action is not effective, because  $(A, H, \alpha)$  is isomorphic to  $(A, H, -\alpha)$  via the isomorphism  $\mathbf{x} \mapsto -\mathbf{x}$ : so  $-\mathbf{1}_2 \in \text{SL}(2, \mathbb{Z}_t)$  acts trivially. Thus  $\psi_{\text{lev}}^t$  is a Galois morphism with Galois group  $\text{PSL}(2, \mathbb{Z}_t) = \text{SL}(2, \mathbb{Z}_t) / \{\pm \mathbf{1}_2\}$ .

A point of  $\mathcal{A}_{\text{bil}}^t$  thus corresponds to an equivalence class  $[(A, H, \alpha, \beta)]$ , where  $(A, H)$  is a polarised abelian surface of type  $(1, t)$ ,  $\alpha$  and  $\beta$  are level and collevel structures, and  $(A, H, \alpha, \beta)$  is equivalent to  $(A', H', \alpha', \beta')$  if there is an isomorphism  $p : A \rightarrow A'$  such that  $p^*H' = H$ ,  $p\alpha = \alpha'$  and  $p^{-1}\beta = \beta'$ . In particular, for general  $A$ , we have  $(A, H, \alpha, \beta) \cong (A, H, -\alpha, -\beta)$  but  $(A, H, \alpha, \beta) \not\cong (A, H, -\alpha, \beta)$ . Another way to express this is to say that the wreath product  $\mathbb{Z}_2 \wr \text{PSL}(2, \mathbb{Z}_t)$ , acts on  $\mathcal{A}_{\text{bil}}^t$  with quotient  $\mathcal{A}_t$ .

**Theorem 1.1** (Mukai [14])  $\mathcal{A}_{\text{bil}}^t$  is the quotient of the Siegel upper half-plane  $\mathbb{H}_2$  by the group

$$\Gamma_{\text{bil}}^t = \Gamma_{\text{bil}}^t \cup \Gamma_{\text{bil}}^t$$

where

$$\Gamma_{\text{bil}}^t = \left\{ \gamma \in \text{Sp}(4, \mathbb{Z}) \mid \gamma - \mathbf{1}_4 \in \begin{pmatrix} * & * & * & * \\ t\mathbb{Z} & t\mathbb{Z} & * & * \\ t\mathbb{Z} & t\mathbb{Z} & t\mathbb{Z} & * \\ t\mathbb{Z} & t\mathbb{Z} & * & * \end{pmatrix} \right\}$$

and  $\zeta = \text{diag}(1, -1, 1, -1)$ , acting by fractional linear transformations.

that is,  $\mathbf{a} \in \Phi_m(r)$  if and only if  $\mathbf{a} = z\mathbf{a}'$  implies  $z \in \mathbb{Z}_*^r$ ; and put  $\phi_m(r) = \# \Phi_m(r)$ . We also put  $\underline{\Phi}_m(r) = \Phi_m(r)/\{\pm 1\}$ .

$$\Phi_m(r) = \{ \mathbf{a} \in \mathbb{Z}_m^r \mid \mathbf{a} \text{ is not a multiple of a zero-divisor in } \mathbb{Z}_m^r \},$$

For  $m, r \in \mathbb{N}$ , define

$X_\circ(r)$ , and the compactification obtained by adding the cusps by  $X(r)$ . We first collect some facts about congruence subgroups in  $\mathrm{SL}(2, \mathbb{Z})$  and some related combinatorial information. For  $r \in \mathbb{N}$  we denote by  $\Gamma(r)$  the principal congruence subgroup of  $\mathrm{SL}(2, \mathbb{Z})$ . We denote the modular curve  $\Gamma(r) \backslash \mathbb{H}$  by  $X(r)$ , and the compactification obtained by adding the cusps by  $X_\circ(r)$ .

## II Modular groups and modular forms

ate reference.

where the proofs of intermediate results are very similar to those of corresponding results in [11] we omit the details and simply indicate the appropriate reference. The geometry of  $\mathcal{A}_{\mathrm{bil}}^t$  shows many similarities with that of  $\mathcal{A}_{\mathrm{lev}}^t$ , which was studied (in the case of  $t$  an odd prime) in the book [11]. In many cases the geometry of  $\mathcal{A}_{\mathrm{bil}}^t$  shows many similarities with that of  $\mathcal{A}_{\mathrm{lev}}^t$ , which was studied (in the case of  $t$  an odd prime) in the book [11]. In many cases where the proofs of intermediate results are very similar to those of corresponding results in [11] we omit the details and simply indicate the appropriate reference.

importance in this paper. In the moduli space arises from the fixed locus of  $\zeta$  in  $\mathbb{H}_2$ , and will be of great importance in this paper. The automorphism  $(\mathbf{x}, \mathbf{y}) \mapsto (\mathbf{x}, -\mathbf{y})$  of  $A = E_1 \times E_2$  induces an isomorphism  $(A, H, \alpha, \beta) \rightarrow (A, H, -\alpha, \beta)$  in this case, so a product surface with a weak bilinear structure still has an extra automorphism. The corresponding locus in the moduli space arises from the fixed locus of  $\zeta$  in  $\mathbb{H}_2$ , and will be of great importance in this paper.

linear transformations.

For some purposes it is more convenient to work with the conjugate bilinear transformations. For some purposes it is more convenient to work with the conjugate bilinear transformations. For some purposes it is more convenient to work with the conjugate bilinear transformations. For some purposes it is more convenient to work with the conjugate bilinear transformations.

$$\Gamma_t = \left\{ \gamma \in \mathrm{Sp}(4, \mathbb{Q}) \mid \gamma - \mathbf{1}_4 \in \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix} \right\}.$$

Thus  $\Gamma_{\mathrm{bil}}^t$  should be thought of as a subgroup of the paramodular group

$$\phi_m(r) = \sum_n (-1)^i \sum_{i=0}^{d_{j_1} \dots d_{j_k}} \left( r \prod_{i=1}^k d_{j_i}^i \right)_m = r_m \prod_{i=1}^d (1 - d^{-i}).$$

**Lemma II.1** *If the primes dividing  $r$  are  $p_1 < p_2 < \dots < p_n$  then*

**Proof** We first prove that  $\phi_m(r)$  is a multiplicative function. Suppose first that  $r = pq$ , with  $\gcd(p, q) = 1$ . It is easy to see that  $\mathbf{a} \in \Phi_m(r)$  if and only if  $\mathbf{a}_p \in \Phi_m(p)$  and  $\mathbf{a}_q \in \Phi_m(q)$ , where  $\mathbf{a}_p$  denotes the reduction of  $\mathbf{a}$  mod  $p$ .

We divide  $\mathbb{Z}_m^r$  into residue classes mod  $p$ : that is, we write  $\mathbb{Z}_m^r$  as the disjoint union of subsets  $S_{\mathbf{c}}$  for  $\mathbf{c} \in \mathbb{Z}_m^p$ , where  $S_{\mathbf{c}} = \{\mathbf{a} \mid \mathbf{a}_p = \mathbf{c}\}$ . There are  $\phi_m(p)$  subsets  $S_{\mathbf{c}}$  such that  $\mathbf{r} \in \Phi_m(p)$ .

The reduction mod  $q$  map  $S_{\mathbf{c}} \rightarrow \mathbb{Z}_m^q$  is bijective, since it is the inverse of the injective map  $\mathbf{b} \mapsto \mathbf{c} + p\mathbf{b} \in \mathbb{Z}_m^r$ . Hence in each of the  $\phi_m(p)$  subsets  $S_{\mathbf{c}}$ ,  $\mathbf{c} \in \Phi_m(p)$  there are  $\phi_m(q)$  elements whose reduction mod  $q$  belongs to  $\Phi_m(q)$ . It follows that  $\phi_m(r) = \phi_m(p)\phi_m(q)$ .

Finally, we check that if  $r = p^k$ ,  $p$  prime, then  $\phi_m(r) = r_m(1 - p^{-m})$ . If  $\mathbf{a} \notin \Phi_m(r)$ , then  $\mathbf{a} = p\mathbf{a}'$  for a unique  $\mathbf{a}' \in \mathbb{Z}_m^{r/d}$ , so there are  $(p^{k-1})_m$  such elements  $\mathbf{a}$ .  $\square$

Note that  $\phi_1$  is the Euler  $\phi$  function, and  $\Phi_1(r)$  the set of non-zero-divisors of  $\mathbb{Z}_r$ .

**Corollary II.2** *The order of  $\text{SL}(2, \mathbb{Z}_t)$  is given by*

$$|\text{SL}(2, \mathbb{Z}_t)| = t^3 \prod_{t^3 | t} (1 - d^{-2}).$$

**Proof** (See also [18, §1.6].) If  $A \in \text{SL}(2, \mathbb{Z}_t)$ , then  $A_1 = (a_{11}, a_{12}) \in \Phi_2^2(t)$ . So by Euclid's algorithm we can find  $A_2 = (a_{21}, a_{22})$  such that  $\det \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \gcd(a_{11}, a_{12}) = r$ . Replacing  $A_2$  by  $A_2 = r^{-1}A_2$ , we get a matrix  $A$  with  $\det A = 1$ . Furthermore, if  $B_j = \begin{pmatrix} A_1 \\ A_2 + jA_1 \end{pmatrix}$ ,  $j = 0, \dots, t-1$ , then  $\det B_j = \det A = 1$ , and  $B_j \neq B_{j'}$  if  $j \neq j'$ . So  $|\text{SL}(2, \mathbb{Z}_t)| = t\phi_2(t)$ .  $\square$

For  $r > 2$ , put  $h(r) = [\text{PSL}(2, \mathbb{Z}) : \Gamma_1(r)]$ . By Corollary II.2 we have

$$h(r) = r^3 \prod_{r^3 | r} (1 - d^{-2}).$$

We need the following well-known lemma.

**Lemma II.3** *If  $r > 2$  then  $X(r)$  has*

$$v(r) = h(r)/r = \prod_{r^2 | r} (1 - d^{-2})^{p|r}$$

*cusps and is a smooth complete curve of genus  $g = 1 + \frac{12}{r} - \frac{2}{r}$ .*

**Proof** See [18, pp. 23–24].  $\square$

We denote  $\mu(t)$  by  $\mu$  and  $\nu(t)$  by  $\nu$ . Note that  $\phi_2(1) = \nu(1) = 1$  and  $\phi_2(r) = 2\nu(r)$  for  $r < 2$ .

Now we turn to subgroups of  $\mathrm{Sp}(4, \mathbb{Q})$  and modular forms. Denote by  $\mathfrak{G}_n^*(\Gamma)$  the space of weight  $n$  cusp forms for  $\Gamma \subseteq \mathrm{Sp}(4, \mathbb{Q})$ . We need the groups  $\Gamma(1) = \mathrm{P}\mathrm{Sp}(4, \mathbb{Z})$  and, for  $\ell \in \mathbb{N}$ ,

$$\Gamma(\ell) = \{ \gamma \in \mathrm{Sp}(4, \mathbb{Z}) \mid \gamma = \mathbf{1}_4 \in \mathrm{Sp}(4, \mathbb{Z}_\ell) \}.$$

If  $\ell^2 \mid \ell$  then  $\Gamma(\ell) \triangleright \Gamma^t_{\mathrm{bil}}$ , because  $\Gamma(\ell) \subseteq \Gamma^t_{\mathrm{bil}}$  and  $\Gamma(\ell)$  is normal in  $\Gamma(1) = \mathrm{Sp}(4, \mathbb{Z})$ .

By a previous calculation [19] we know that

$$\dim \mathfrak{G}_n^*(\Gamma(\ell)) = \frac{n^3}{8640} [\Gamma(1) : \Gamma(\ell)] + O(n^2)$$

(as long as  $\ell > 2$  we can consider  $\Gamma(\ell)$  as a subgroup of  $\mathrm{P}\mathrm{Sp}(4, \mathbb{Z})$  rather than  $\mathrm{Sp}(4, \mathbb{Z})$ ). A standard application of the Atiyah–Bott fixed-point theorem (see [9], or in this context [12]) gives

$$\dim \mathfrak{G}_n^*(\Gamma^t_{\mathrm{bil}}) = \frac{a}{a} \dim \mathfrak{G}_n^*(\Gamma(\ell)) + O(n^2)$$

where  $a$  is the number of elements  $\gamma \in \Gamma^t_{\mathrm{bil}}$  whose fixed locus in  $\mathbb{H}_2$  has dimension 3. Thus  $a$  is the number of elements of  $\Gamma^t_{\mathrm{bil}}$  that act trivially on  $\mathbb{H}_2$ . In  $\mathrm{Sp}(4, \mathbb{Z})$  there are two such elements,  $\pm \mathbf{1}_4$ , but if  $t > 2$  then  $-\mathbf{1}_4 \notin \Gamma^t_{\mathrm{bil}}$ . So  $a = 1$ , and hence

$$\begin{aligned} \dim \mathfrak{G}_n^*(\Gamma^t_{\mathrm{bil}}) &= \frac{1}{1} \dim \mathfrak{G}_n^*(\Gamma(\ell)) + O(n^2) \\ &= \frac{n^3}{8640} [\Gamma(\ell) : \Gamma(\ell)] + O(n^2) \\ &= \frac{n^3}{8640} [\Gamma(1) : \Gamma^t_{\mathrm{bil}}] + O(n^2). \end{aligned} \tag{1}$$

The number  $[\Gamma(1) : \Gamma^t_{\mathrm{bil}}]$  is equal to the degree of the map  $\mathcal{A}^t_{\mathrm{bil}} \rightarrow \mathcal{A}_1$  (actually there are two such maps of the same degree), where  $\mathcal{A}_1$  is the moduli space of principally polarized abelian surfaces. Now

$$\begin{aligned} [\Gamma(1) : \Gamma^t_{\mathrm{bil}}] &= \frac{1}{2} [\Gamma(1) : \Gamma^t_{\mathrm{bil}}] \\ &= \frac{1}{2} [\Gamma(1) : \Gamma^t_{\mathrm{lev}}] \cdot \end{aligned}$$

We can see directly that  $\Gamma_{\text{lev}}^t \subset \Gamma_{\text{h}}^t$  since

$$\Gamma_{\text{lev}}^t = \left\{ \gamma \in \text{Sp}(4, \mathbb{Z}) \mid \gamma - \mathbf{1}_4 \in \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ t\mathbb{Z} & t\mathbb{Z} & t\mathbb{Z} & * \\ t\mathbb{Z} & t\mathbb{Z} & * & * \end{pmatrix} \right\}.$$

**Lemma II.4** *The map*

$$\phi : \Gamma_{\text{lev}}^t \rightarrow \text{SL}(2, \mathbb{Z}_t) \quad \text{given by} \quad A \mapsto \begin{pmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{pmatrix}$$

*is a surjective group homomorphism, and the kernel is  $\Gamma_{\text{h}}^t$ .*

**Proof** The surjectivity follows from the well-known fact that the reduction mod  $t$  map  $\text{red}_t : \text{SL}(2, \mathbb{Z}) \rightarrow \text{SL}(2, \mathbb{Z}_t)$  is surjective; the rest is obvious.  $\square$

**Lemma II.5** *For  $t < 2$ , the index  $[\Gamma(1) : \Gamma_{\text{lev}}^t]$  is equal to  $t\phi_4(t)/2$ .*

**Proof** The proof is almost the same as that of [13, Lemma 0.5]. In place of the chain of groups  $\Gamma_{1^p} > \Gamma_{1^{2p}} > \Gamma' = \Gamma(1)$ , we use the chain  $\Gamma_{\text{lev}}^t > \Gamma_{1^t} > \Gamma(1)$ . Furthermore, we use the set  $\Phi_4(t)$  where  $\text{SL}(4, \mathbb{Z}_t)$  acts. Note that  $\text{SL}(4, \mathbb{Z})$  still acts transitively on  $\Phi_4(t)$ , via

$$\begin{pmatrix} b_{11} & 0 & b_{12} & 0 \\ 0 & 1 & 0 & 0 \\ b_{21} & 0 & b_{22} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & B \\ 0 & {}_t B^{-1} \end{pmatrix},$$

for  $B \in \text{SL}(2, \mathbb{Z})$ .

Following the same steps as in [13], and substituting  $\phi_m(t)$  for  $p_m - 1 = \phi_m(p)$ , we then find that  $[\Gamma_{1^t}^t : \Gamma_{\text{lev}}^t] = t\phi_1(t)$  and  $[\Gamma(1) : \Gamma_{1^t}^t] = t\phi_1(t)/2$ , so  $[\Gamma(1) : \Gamma_{\text{lev}}^t] = t\phi_4(t)/2$ .  $\square$

**Theorem II.6** *The number of cusp forms of weight  $n$  for  $\Gamma_{\text{bil}}^t$  (for  $t > 2$ ) is given by*

$$\dim \mathcal{G}_*^n(\Gamma_{\text{bil}}^t) = \frac{n^3}{34560} t^2 \phi_2(t) \phi_4(t) \quad = \quad \prod_{p|t} (1 - p^{-2}) (1 - p^{-4}).$$

**Proof** Immediate from equation (1), Corollary II.2 and Lemma II.5.  $\square$

### III Torsion in the modular group

We know that  $\Gamma_{\text{bill}}^t \subset \text{Sp}(4, \mathbb{Z})$ , and the conjugacy classes of torsion elements in  $\text{Sp}(4, \mathbb{Z})$  are known ([6, 20]). See [10] for a summary of the relevant information.

If  $\gamma \in \Gamma_{\text{bill}}^t$  then the reduction mod  $t$  of  $\gamma$  is

$$\bar{\gamma} = \begin{pmatrix} 1 & * & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & * & 1 & 0 \\ * & * & * & 1 \end{pmatrix} \in \text{Sp}(4, \mathbb{Z}_t),$$

so the characteristic polynomial  $\chi(\bar{\gamma})$  is  $(1-x)^4 \in \mathbb{Z}_t[x]$ . On the other hand, if  $\gamma \in \zeta \Gamma_{\text{bill}}^t$  then

$$\bar{\gamma} = \zeta \begin{pmatrix} 1 & * & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ * & * & * & 1 \end{pmatrix} = \begin{pmatrix} * & * & * & 1 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & * & * & -1 \end{pmatrix} \in \text{Sp}(4, \mathbb{Z}_t),$$

so  $\chi(\bar{\gamma}) = (1-x)^2(1+x)^2 \in \mathbb{Z}_t[x]$ .

The only classes in the list in [20], up to conjugacy, where the characteristic polynomials have this reduction mod  $t$  ( $t > 2$ ) are I(1), where  $\chi(\gamma) = (1-x)^4$ , II(1)a and II(1)b. Class I(1) consists of the identity; class II(1)a includes  $\zeta$  so this just gives us the conjugacy class of  $\zeta$ . Class II(2)b is the  $\text{Sp}(4, \mathbb{Z})$ -conjugacy class of  $\xi$ , where

$$\xi = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \in \Gamma_{\text{bill}}^t.$$

**Proposition III.1** Every nontrivial element of finite order in  $\Gamma_{\text{bill}}^t$  (for  $t > 2$ ) has order 2, and is conjugate to  $\zeta$  or to  $\xi$  in  $\Gamma_{\text{bill}}^t$  if  $t$  is odd.

**Proof** It follows from the list in [20] that the only torsion for  $t > 2$  is 2-torsion (this is still true if  $t$  is even). The 2-torsion of the group  $\Gamma_{\text{lev}}^t$  was studied by Brascch [3]. There are five types but only two of them occur for odd  $t$ . The representatives for these conjugacy classes given in [3] are (up to sign)  $\zeta$  and  $\xi$ ; so the assertion of the theorem is that the  $\Gamma_{\text{bill}}^t$ -conjugacy

classes of  $\zeta$  and  $\xi$  coincide with the intersections of their  $\Gamma_{\text{lev}}^t$ -conjugacy classes with  $\Gamma_{\text{bil}}^t$ . This is checked in [17, Proposition 3.2] for the case  $t = 6$  (the relevant cases are called  $\zeta_0$  and  $\zeta_3$  there), but the proof works for all  $t > 2$ .  $\square$

We put

$$(2) \quad \mathcal{H}_1 = \left\{ \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_3 \end{pmatrix} \mid \text{Im } \tau_1 > 0, \text{Im } \tau_3 > 0 \right\} \subset \mathbb{H}_2$$

and

$$(3) \quad \mathcal{H}_2 = \left\{ \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} \mid 2\tau_2 + \tau_3 = 0 \right\} \subset \mathbb{H}_2.$$

These are the fixed loci of  $\zeta$  and  $\xi$  respectively. We denote by  $H_1^\circ$  and  $H_2^\circ$  the images of  $\mathcal{H}_1$  and  $\mathcal{H}_2$  in  $\mathcal{A}_{\text{bil}}^t$ , and by  $H_1$  and  $H_2$  their respective closures in  $\mathcal{A}_{\text{bil}}^t$ .

**Lemma III.2**  $H_i^\circ$  is irreducible for  $i = 1, 2$ .

**Proof** This follows at once from Proposition III.1 together with equations (2) and (3).  $\square$

The abelian surfaces corresponding to points in  $H_1^\circ$  and  $H_2^\circ$  are, respectively, product surfaces and bielliptic abelian surfaces, as described in [13] for the case  $t$  prime.

We define the subgroup  $\Gamma(2t, 2t)$  of  $\Gamma(t) \times \Gamma(t)$  by

$$\Gamma(2t, 2t) = \{ (M, N) \in \Gamma(t) \times \Gamma(t) \mid M \equiv \mathbb{1}_{N^{-1}} \pmod{2} \}$$

**Lemma III.3**  $H_1^\circ$  is isomorphic to  $X^\circ(t) \times X^\circ(t)$ , and  $H_2^\circ$  is isomorphic to  $\Gamma(2t, 2t) \backslash \mathbb{H} \times \mathbb{H}$ .

**Proof** Identical to the proofs of the corresponding results [11, Lemma I.5.43] and [11, Lemma I.5.45]. The level  $t$  structure now occurs in both factors, whereas in [11] there is level 1 structure in the first factor and level  $p$  structure in the second. In [11] the level  $p$  is assumed to be an odd prime but this fact is not used at that stage:  $p$  odd suffices, so we may replace  $p$  by  $t$ . Thereafter one simply replaces all the groups with their intersection with  $\Gamma_{\text{bil}}^t$ , which imposes a level  $t$  structure in the first factor and causes it to behave exactly like the second factor.  $\square$

**Lemma III.4**  $H_1^\circ$  and  $H_2^\circ$  are disjoint.



The group  $\widetilde{\Gamma}_{\text{bil}}^t(t)$  acts on the set  $(\mathbb{Z}_4^t)^\times$  of nonzero elements of  $\mathbb{Z}_4^t$  by  $\underline{v}_2 \mapsto \underline{v}_2 + k\underline{v}_1 + l\underline{v}_3$  and  $\underline{v}_4 \mapsto \underline{v}_4 + k'\underline{v}_1 + l'\underline{v}_3$ : so  $\underline{\mathbf{V}}$  is equivalent to  $\underline{\mathbf{W}}$  if and only if  $(\underline{v}_1, \underline{v}_3) = (\underline{w}_1, \underline{w}_3)$ , so they have the same  $t$ -divisor, and  $\underline{v}_2 \in \mathbb{Z}_4^t \underline{v}_1 + \mathbb{Z}_4^t \underline{v}_4$  and  $\underline{w}_2 \in \mathbb{Z}_4^t \underline{w}_1 + \mathbb{Z}_4^t \underline{w}_4$ . These are therefore the conditions for primitive vectors  $\mathbf{v}$  and

at that point). We claim that two primitive vectors  $\mathbf{v}$  and  $\mathbf{w}$  are equivalent modulo  $\Gamma(t)$  if and only if  $\underline{v} = \underline{w}$ . It is obvious that  $\Gamma(t)$  preserves the residue classes mod  $t$ . Conversely, suppose that  $\underline{v} = \underline{w}$ . Then we can find  $\gamma \in \text{Sp}(4, \mathbb{Z})$  such that  $\gamma \mathbf{v} = (1, 0, 0, 0)$  (the corresponding geometric fact is that the moduli space  $\mathcal{A}_2$  of principally polarised abelian surfaces has only one rank 1 cusp). Since  $\Gamma(t) \triangleright \text{Sp}(4, \mathbb{Z})$  this means that in order to prove the claim we may assume  $\mathbf{v} = (1, 0, 0, 0)$ . Then we proceed exactly as in the proof of [5, Lemma 3.3], taking  $p = 1$  and  $q = t$  (the assumptions that  $p$  and  $q$  are prime are not used

$$\widetilde{\Gamma}_{\text{bil}}^t(t) = \left\{ \begin{pmatrix} 1 & k & 0 & k' \\ 0 & 1 & 0 & 0 \\ 0 & l & 1 & l' \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \text{Sp}(4, \mathbb{Z}_4^t) \right\} \cong \mathbb{Z}_4^t.$$

**Proof** Note that if  $\Gamma(t)$  is the principal congruence subgroup of level  $t$  in  $\text{Sp}(4, \mathbb{Z})$  then  $\Gamma(t) \triangleright \widetilde{\Gamma}_{\text{bil}}^t$  and the quotient is

**Proposition IV.1** *Positive primitive vectors  $\mathbf{v}, \mathbf{w} \in \mathbb{Z}_4^t$  span lines  $\mathbb{Q}\mathbf{v}$  and  $\mathbb{Q}\mathbf{w}$  in the same  $\widetilde{\Gamma}_{\text{bil}}^t$ -orbit if and only if  $(\underline{v}_1, \underline{v}_3) = (\underline{w}_1, \underline{w}_3)$  (in particular  $\mathbf{v}$  and  $\mathbf{w}$  have the same  $t$ -divisor,  $r$ ), and  $(v_2, v_4) \equiv \pm(w_2, w_4) \pmod r$ .*

If  $\mathbf{v} = (v_1, v_2, v_3, v_4) \in \mathbb{Z}_4^t$ , we define the  $t$ -divisor to be  $r = \text{hcf}(t, v_1, v_3)$ . each line has a unique positive primitive generator.  $\mathbb{Z}_4^t$  with  $\text{hcf}(v_1, v_2, v_3, v_4) = 1$ , which is unique up to sign. We denote the reduction of  $\mathbf{v}$  mod  $t$  by  $\underline{\mathbf{V}} = (\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4) \in \mathbb{Z}_4^t$ . To fix things we shall say, arbitrarily, that  $\mathbf{v}$  is positive if the first nonzero entry  $\underline{v}_i$  of  $\underline{\mathbf{V}}$  satisfies  $\underline{v}_i \in \{1, \dots, (t-1)/2\}$  (remember that we have assumed that  $t$  is odd). Then We begin by counting the boundary divisors. These correspond to  $\widetilde{\Gamma}_{\text{bil}}^t$ -orbits of lines in  $\mathbb{Q}_4^t$ : we identify a line by its primitive generator  $\mathbf{v} = (v_1, v_2, v_3, v_4) \in \mathbb{Z}_4^t$  with  $\text{hcf}(v_1, v_2, v_3, v_4) = 1$ , which is unique up to sign. We denote the reduction of  $\mathbf{v}$  mod  $t$  by  $\underline{\mathbf{V}} = (\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4) \in \mathbb{Z}_4^t$ . To fix things we shall say, arbitrarily, that  $\mathbf{v}$  is positive if the first nonzero entry  $\underline{v}_i$  of  $\underline{\mathbf{V}}$  satisfies  $\underline{v}_i \in \{1, \dots, (t-1)/2\}$  (remember that we have assumed that  $t$  is odd). Then

## IV Boundary divisors

**Proof** The stabiliser of any point of  $\mathbb{H}_2$  in  $\Gamma_{\text{bil}}^t$  is cyclic (of order 2), since  $\Gamma_{\text{bil}}^t$  is torsion-free and therefore has no fixed points. A point of  $\mathcal{H}_1 \cap \mathcal{H}_2$  would be the image of a point of  $\mathbb{H}_2$  stabilised by the subgroup generated by  $\zeta$  and  $\xi$ , which is not cyclic.  $\square$

$\mathbf{w}$  to be equivalent under  $\Gamma_{\mathbb{Z}}^t$ . For equivalence under  $\Gamma_{\mathbb{Z}}^t$ , we get the extra element  $\zeta$  which makes  $(v_1, v_2, v_3, v_4)$  equivalent to  $(v_1, -v_2, v_3, -v_4)$ . Since we are interested in orbits of lines, not primitive generators, we may restrict ourselves to positive generators  $\mathbf{v}$ .  $\square$

The irreducible components of the boundary divisor of  $\mathcal{A}_{\text{bil}}^t$  correspond to the  $\Gamma_{\mathbb{Z}}^t$ -orbits (or equivalently to  $\widetilde{\Gamma}_{\text{bil}}^t$ -orbits) of lines in  $\mathbb{Q}^4$ . We denote the boundary component corresponding to  $\mathbb{Q}\mathbf{v}$  by  $D_{\mathbf{v}}$ . We shall be chiefly interested in the cases  $r = t$  and  $r = 1$ . We refer to these as the standard components. They are represented by vectors  $(0, a, 0, b)$  and  $(a, 0, b, 0)$  respectively, in either case with  $\text{hcf}(a, b) = 1$ ,  $0 \leq a \leq t - 1$  and  $0 \leq b < t$ . Note that there are  $\nu$  of each of these.

**Corollary IV.2** *If  $t$  is odd then the number of irreducible boundary divisors of  $\mathcal{A}_{\text{bil}}^t$  with  $t$ -divisor  $r$  is  $\#\Phi_2(h) \# \Phi_2(r)$ , where  $h = t/r$ . For  $r \neq 1, t$ , this is equal to  $\frac{1}{2} \phi_2(h) \phi_2(r)$ .*

**Proof** See above for the standard cases. In general, the  $\Gamma_{\mathbb{Z}}^t$ -orbit of a primitive vector  $\mathbf{v}$  is determined by the classes of  $(v_1/r, v_3/r)$  in  $\Phi_2(h)$  and of  $(v_2, v_4) \in \Phi_2(r)$ . The extra element  $\zeta$  and the freedom to multiply  $\mathbf{v}$  by  $-1 \in \mathbb{Q}$  allow us to multiply either of these classes by  $-1$  and the choices therefore lie in  $\Phi_2(h)$  and  $\Phi_2(r)$ .  $\square$

## V Jacobi forms

This section describes the behaviour of a modular form  $F \in \mathcal{G}_{3n}^*(\Gamma_{\text{bil}}^t)$  near a boundary divisor  $D_{\mathbf{v}}$ . The standard boundary divisors are best treated separately, since it is in those cases only that the torsion plays a role: on the other hand, the standard boundary divisors occur for all  $t$  and their behaviour does not depend very much on the factorisation of  $t$ .

We assume at first, then, that  $D_{\mathbf{v}}$  is a nonstandard boundary divisor. Since all the divisors of given  $t$ -divisor are equivalent under the action of  $\mathbb{Z}_2 \times \text{SL}(2, \mathbb{Z}_t)$  (because the  $t$ -divisor is the only invariant of a boundary divisor of  $\mathcal{A}_{\mathbf{v}}$ ; see [5]), it will be enough to calculate the number of conditions imposed by one divisor of each type. That is to say, we only need consider boundary components in  $\mathcal{A}_{\mathbf{v}}^*$ .

In view of this we may take  $\mathbf{v} = (0, 0, r, 1)$  for some  $r \mid t$  with  $1 < r < t$ . We write  $(0, 0, 0, 1) = \mathbf{v}_{(0,1)}$  (for consistency with [11]) and we put  $h = t/r$ . Since we want to work with  $\Gamma_{\text{bil}}^t$  rather than  $\widetilde{\Gamma}_{\text{bil}}^t$  (so as to use fractional linear transformations) we must consider the lines  $\mathbb{Q}\mathbf{v}H_t = \mathbb{Q}\mathbf{v}'$ , where  $\mathbf{v}' = (0, 0, 1, h)$  and  $\mathbb{Q}\mathbf{v}_{(0,1)}H_t = \mathbb{Q}\mathbf{v}_{(0,1)}$ .

Note that  $\mathcal{V}^{\mathcal{Q}_r} = \mathcal{V}^{(0,1)}$ , where

$$\mathcal{Q}_r = \begin{pmatrix} 1 & 1 & 0 & 0 \\ h-1 & h & 0 & 0 \\ 0 & 0 & 1-h & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in \mathrm{Sp}(4, \mathbb{Z}).$$

**Proposition V.1** *If  $\mathcal{V}$  has  $t$ -divisor  $r \neq t$ , 1 and  $F \in \mathcal{G}_*(\mathbb{P}_{\mathrm{bil}}^t)$  is a cusp form of weight  $k$ , then there are coordinates  $\tau_v^t$  such that  $F$  has a Fourier expansion near  $D_v$  as*

$$F = \sum_{w \geq 0} \theta_v^w(\tau_v^1, \tau_v^2) \exp \frac{2\pi i w \tau_3^t}{rt}.$$

**Proof** As usual (cf. [11]) we write  $\mathcal{P}_v^t$  for the stabiliser of  $\mathcal{V}$  in  $\mathrm{Sp}(4, \mathbb{R})$ , so  $\mathcal{P}_v^t = \mathcal{Q}_{-1}^r \mathcal{P}_v^{(0,1)} \mathcal{Q}_r$ . We take  $\mathcal{P}_v^t = \mathcal{P}_v^t \cap \mathbb{P}_{\mathrm{bil}}^t$ : this group determines the structure of  $\mathcal{A}_{\mathrm{bil}}^t$  near  $D_v$ . It is shown in [11, Proposition I.3.87] that  $\mathcal{P}_v^{(0,1)}$  is generated by  $g_1(\gamma)$  for  $\gamma \in \mathrm{SL}(2, \mathbb{R})$ ,  $g_2 = \zeta$ ,  $g_3(m, n)$  and  $g_4(s)$  for  $m, n, s \in \mathbb{R}$ , where

$$g_1(\gamma) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ for } \gamma = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

and  $g_3$  and  $g_4$  are given by

$$g_3(m, n) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ m & 1 & n & 0 \\ n & 0 & 0 & 1 \end{pmatrix}, \quad g_4(s) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & s \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

So  $\mathcal{P}_v^t$  includes the subgroup generated by all elements of the form  $\mathcal{Q}_{-1}^r g_i \mathcal{Q}_r$  with  $a, b, c, d, m, n, s \in \mathbb{Z}$  which lie in  $\mathbb{P}_{\mathrm{bil}}^t$ . In particular it includes the lattice  $\{\mathcal{Q}_{-1}^r g_4^t(s) \mathcal{Q}_r \mid s \in \mathbb{Z}\}$ . If we take  $Z_v^t = \mathcal{Q}_{-1}^r(Z)$  for  $Z = \begin{pmatrix} \tau_1^t & \tau_2^t \\ \tau_1^t & \tau_2^t \end{pmatrix}$  then we obtain

$$Z_v^t = \begin{pmatrix} h^2 \tau_1 - 2h\tau_2 + \tau_3 & h(\tau_1 + 1) - 2h\tau_2 + \tau_3 \\ -h(\tau_1 + 1) - 2h\tau_2 + \tau_3 & h(\tau_1 + 1) - 2h\tau_2 + \tau_3 \end{pmatrix}.$$

One easily checks that

$$\mathcal{Q}_{-1}^r g_4^t(s) \mathcal{Q}_r : Z_v^t \mapsto \begin{pmatrix} \tau_1^t & \tau_2^t \\ \tau_1^t & \tau_2^t \end{pmatrix} \mapsto \begin{pmatrix} \tau_1^t & \tau_2^t \\ \tau_1^t + s & \tau_2^t + st \end{pmatrix}$$

and this proves the result.  $\square$

We define a subgroup  $\Gamma(t, r)$  of  $\text{SL}(2, \mathbb{Z})$  by

$$\Gamma(t, r) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a \equiv d \equiv 1 \pmod t, b \equiv 0 \pmod{t^2}, c \equiv 0 \pmod r \right\}.$$

**Lemma V.2** *If  $D_\nu$  is nonstandard then  $F'_\nu$  is torsion-free.*

**Proof** The only torsion in  $\Gamma^t$  is 2-torsion and a simple calculation shows that if  $\mathbf{1}_4 \neq g \in P^{\nu(0,1)}$  and  $g^2 = \mathbf{1}_4$ , then  $Q_r^{-1}gQ_r \notin \Gamma^t$  for  $r \neq 1, t$ .  $\square$

**Proposition V.3** *If  $D_\nu$  is nonstandard and  $F \in \mathfrak{S}_k^*(\Gamma^t)$  then  $\theta_\nu^m(r\tau_\nu^1, t\tau_\nu^2)$  is a Jacobi form of weight  $k$  and index  $w$  for  $\Gamma(t, r)$ .*

**Proof** By direct calculation we find that  $Q_r^{-1}g_1(\gamma)Q_r \in \Gamma^t$  if  $\gamma \in \Gamma(t, r)$  and  $Q_r^{-1}g_3(rm, tn)Q_r \in \Gamma^t$  for  $m, n \in \mathbb{Z}$ . Using these two elements, another elementary calculation verifies that the transformation laws for Jacobi forms given in [4] are satisfied, since

$$Q_r^{-1}g_3(rm, tn)Q_r : Z_\nu \mapsto \begin{pmatrix} \tau_\nu^1 & r\tau_\nu^1 + rm\tau_\nu^1 + tn \\ \tau_\nu^2 + r\tau_\nu^1 + tm & r\tau_\nu^2 + 2rm\tau_\nu^1 + r^2m^2\tau_\nu^1 + tn \end{pmatrix}$$

and

$$Q_r^{-1}g_1(\gamma)Q_r : Z_\nu \mapsto \begin{pmatrix} \gamma(\tau_\nu^1) & \gamma(\tau_\nu^2)/(c\tau_\nu^1 + d) \\ \tau_\nu^3 - c\tau_\nu^2/(c\tau_\nu^1 + d) & \tau_\nu^3 + d \end{pmatrix}.$$

**Lemma V.4** *The index of  $\Gamma(t, r)$  in  $\Gamma(1)$  is equal to  $rt\phi_2(t)$  for  $r \neq 1, t$ .*

**Proof** Consider the chain of groups

$$\Gamma(1) = \text{SL}(2, \mathbb{Z}) > \Gamma^0(t) > \Gamma^0(t)(r) > \Gamma(t, r)$$

and the normal subgroup  $\Gamma_1(t) > \Gamma^0(t)$ , where

$$\Gamma^0(t) = \left\{ \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}) \mid a \equiv d \equiv 1 \pmod t, b \equiv 0 \pmod t, c \equiv 0 \pmod h \right\}$$

$$\Gamma_1(t) = \left\{ \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}) \mid a \equiv d \equiv 1 \pmod t, b \equiv c \equiv 0 \pmod t \right\}$$

Thus  $\Gamma_0(t)(r)$  is the kernel of reduction mod  $r$  in  $\Gamma_0(t)$ . By Corollary II.2,  $[\Gamma(1) : \Gamma_1(t)] = t\phi_2(t)$ . By the exact sequence

$$0 \rightarrow \Gamma_1(t) \rightarrow \Gamma_0(t) \rightarrow \left\{ \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \mid c \in \mathbb{Z}_t \right\} \cong \mathbb{Z}_t \rightarrow 0$$

we have  $[\Gamma_0(t) : \Gamma_1(t)] = t$ , and similarly

$$0 \rightarrow \Gamma_0(t)(r) \rightarrow \Gamma_0(t) \rightarrow \left\{ \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \mid c \in \mathbb{Z}_r \right\} \cong \mathbb{Z}_r \rightarrow 0$$

gives  $[\Gamma_0(t) : \Gamma_0(t)(r)] = r$ .

To calculate  $[\Gamma(t)(r) : \Gamma(t, r)]$ , we let  $\Gamma_0(t)(r)$  act on  $\mathbb{Z}_t \times \mathbb{Z}_{t^2}$  by right multi-

plication  $\gamma : (x, y) \mapsto (ax + cy, bx + dy)$ . The stabiliser of  $(1, 0) \in \mathbb{Z}_t \times \mathbb{Z}_{t^2}$  is then  $\{ \bar{\gamma} \in \Gamma_0(t)(r) \mid a \equiv 1 \pmod{t}, b \equiv 0 \pmod{t^2} \}$ , which is  $\Gamma(t, r)$ . However, the orbit of  $(1, 0) \in \mathbb{Z}_t \times \mathbb{Z}_{t^2}$  is  $\{ (a, b) \in \mathbb{Z}_t \times \mathbb{Z}_{t^2} \mid \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in \Gamma_0(t)(r) \}$ : that is, the set of possible first rows of a matrix in  $\Gamma_0(t)(r)$  taken mod  $t$  in the first column and mod  $t^2$  in the second. This is evidently equal to  $\{ (1, tb) \mid b \in \mathbb{Z}_t \}$ , and hence of size  $t$ . Thus  $[\Gamma(t)(r) : \Gamma(t, r)] = t$ , which completes the proof.  $\square$

The standard case is only slightly different, but now there is torsion.

**Proposition V.5** *If  $D_v$  is standard and  $F \in \mathcal{G}_*^k(\Gamma_{\text{bil}}^t)$  then  $\theta_v^m(r\tau_v^1, t\tau_v^2)$  is a Jacobi form of weight  $k$  and index  $w$  for a group  $\Gamma'(t, r)$ , which contains  $\Gamma(t, r)$  as a subgroup of index 2.*

**Proof** Although the standard boundary components are most obviously given by  $(0, 0, 0, 1)$  for  $r = t$  and  $(0, 0, 1, 0)$  for  $r = 1$ , we choose to take advantage of the calculations that we have already performed by working instead with  $(0, 0, t, 1)$  and  $(0, 0, 1, 1)$ . Lemma V.3 is still true, but we also have  $\mathcal{Q}_{-1}^t \zeta \mathcal{Q}_t \in \Gamma_{\text{bil}}^t$  and  $\mathcal{Q}_{-1}^1(-\zeta) \mathcal{Q}_1 \in \Gamma_{\text{bil}}^t$ . These give rise to the stated extra invariance.  $\square$

**Lemma V.6** *The dimension of the space  $J_{3k, w}(\Gamma'(t, r))$  of Jacobi forms of weight  $3k$  and index  $w$  for  $\Gamma'(t, r)$  is given as a polynomial in  $k$  and  $w$  by*

$$\dim J_{3k, w}(\Gamma'(t, r)) = \delta r t v \left( \frac{7}{kw} + \frac{6}{w^2} \right) + \text{linear terms}$$

where  $\delta = \frac{1}{2}$  if  $r = 1$  or  $r = t$  and  $\delta = 1$  otherwise.

**Proof** By [4, Theorem 3.4] we have

$$(4) \quad \sum_{2w}^{z=0} \dim \mathfrak{S}_{3k+z}(\Gamma'(t, r)) \leq \dim J_{3k,w}(\Gamma'(t, r)).$$

Since  $\Gamma'(t, r)$  is torsion-free, the corresponding modular curve has genus  $1 + \frac{\mu(t, r)}{12} - \frac{\nu(t, r)}{2}$ , where  $\mu(t, r)$  is the index of  $\Gamma'(t, r)$  in  $\mathrm{PSL}(2, \mathbb{Z})$  and  $\nu(t, r)$  is the number of cusps (see [18, Proposition 1.40]). Hence by [18, Theorem 2.23] the space of modular forms satisfies

$$(5) \quad \dim \mathfrak{S}_k(\Gamma'(t, r)) = k \left( \frac{\mu(t, r)}{12} - \frac{\nu(t, r)}{2} \right) + \frac{2}{k} \nu(t, r) + O(1) = \frac{k\mu(t, r)}{12} + O(1)$$

as a polynomial in  $k$ . By Lemma V.4 we have  $\mu(t, r) = \frac{7}{1} r t \phi_2(t) = r t \nu$  for the nonstandard cases,  $\mu(t, 1) = \frac{7}{2} t \nu$  and  $\mu(t, t) = \frac{7}{1} t^2 \nu$ . Now the result follows from equations (5) and (4).  $\square$

If  $F \in \mathfrak{S}_{3k}^t(\Gamma_{\mathrm{bil}}^t)$  then  $F \cdot (d\tau_1 \wedge d\tau_2 \wedge d\tau_3)_{\otimes^k}$  extends over the component  $D^v$  if and only if  $\theta_x^w = 0$  for all  $w > k$ : see [1, Chapter IV, Theorem 1]. Hence the obstruction  $\Omega^v$  coming from the boundary component  $D^v$  is

$$(6) \quad \Omega^v = \sum_{k-1}^{w=0} \dim J_{3k,w}(\Gamma'(t, r))$$

where  $\Gamma'(t, r) = \Gamma(t, r)$  if  $D^v$  is nonstandard.

By Corollary IV.2 the total obstruction from the boundary is

$$\Omega^\infty = \sum_{r|t} \# \Phi \# \Phi(h) \# \Phi(r) \sum_{k-1}^{w=0} \dim J_{3k,w}(\Gamma'(t, r)),$$

and we may assume that  $k$  is even.

**Corollary V.7** *The obstruction coming from the boundary is*

$$\Omega^\infty \leq \left( \sum_{r|t} \delta r t \nu \# \Phi \# \Phi(h) \# \Phi(r) \right) \left( \frac{11}{36} k^3 + O(k^2) \right).$$

**Proof** Summing the expression in Lemma V.6 for  $0 \leq w \leq k$ , as required by equation (6) gives the coefficient of  $\frac{36}{11}$  and the rest comes directly from Lemma V.6 and Corollary IV.2.  $\square$

## VI Intersection numbers

We need to know the degrees of the normal bundles of the curves that generate  $\text{Pic } H_1$  and  $\text{Pic } H_2$ . For this we first need to describe the surfaces  $H_1$  and  $H_2$ . The statements and the proofs are very similar to the corresponding results for the case of  $\mathcal{A}_{\text{lev}}^p$ , given in [11] and [12]. Therefore we simply refer to those sources for proofs, pointing out such differences as there are.

**Proposition VI.1**  $H_1$  is isomorphic to  $X(t) \times X(t)$ .

**Proof** Identical to [11, I.5.53].  $\square$

**Proposition VI.2**  $H_2$  is the minimal resolution of a surface  $\underline{H}_2$  which is given by two  $\text{SL}(2, \mathbb{Z}_2)$ -covering maps

$$X(2t) \times X(2t) \times X(2t) \longrightarrow \underline{H}_2 \longrightarrow X(t) \times X(t).$$

The singularities that are resolved are  $\nu^2$  ordinary double points, one over each point  $(\alpha, \beta) \in X(t) \times X(t)$  for which  $\alpha$  and  $\beta$  are cusps.

**Proof** Similar to [11, Proposition I.5.55] and the discussion before [12, Proposition 4.21].  $X(2)$  and  $X(2p)$  are both replaced by  $X(2t)$  and  $X(1)$  and  $X(p)$  by  $X(t)$ . Since  $t > 3$  there are no elliptic fixed points and hence no other singularities in this case.  $\square$

**Proposition VI.3**  $H_1^0$  and  $H_2^0$  meet the standard boundary components  $D_\nu$  transversally in irreducible curves  $C_\nu \cong X^\circ(t)$  and  $C'_\nu \cong X^\circ(2t)$  respectively.  $D_\nu$  is isomorphic to the (open) Kummer modular surface  $K^\circ(t)$ ,  $C_\nu$  is the zero section and  $C'_\nu$  is the 3-section given by the 2-torsion points of the universal elliptic curve over  $X(t)$ .

**Proof** This is essentially the same as [11, Proposition I.5.49], slightly simpler in fact. We may work with  $\mathbf{v} = (0, 0, 1, 0)$  and copy the proof for the central boundary component in  $\mathcal{A}_{\text{lev}}^p$ , replacing  $p$  by  $t$  (again the fact that  $p$  is prime is not used).  $\square$

We do not claim that the closure of  $D_\nu$  is the Kummer modular surface  $K(t)$ . They are, however, isomorphic near  $H_1$  and  $H_2$ . We remark that  $H_1$  and  $H_2$  do not meet the nonstandard boundary divisors, because of Lemma V.2.

**Proposition VI.4**  $\mathcal{A}_{\text{bil}^*}^t$  is smooth near  $H_1$  and  $H_2$ .

**Proof** Certainly  $\mathcal{A}_{\text{bil}}^t$  is smooth since the only torsion in  $\Gamma_{\text{bil}}^t$  is 2-torsion fixing a divisor in  $\mathbb{H}_2$ . There can in principle be singularities at infinity, but such singularities must lie on corank 2 boundary components not meeting  $H_1$  nor  $H_2$  (this follows again from Lemma V.2).  $\square$

**Corollary VI.5**  $H_1$  does not meet  $H_2$ .

**Proof** Since  $\mathcal{A}_{\text{bil}}^t$  and the divisors  $H_1$  and  $H_2$  are smooth at the relevant points, the intersection must be empty or contain a curve. However, the intersection also lies in the corank 2 boundary components. These components consist entirely of rational curves, and if  $t > 5$  then  $H_1 \cong X(t) \times X(t)$  contains no rational curves. Hence  $H_1 \cap H_2 = \emptyset$ .

With a little more work one can check that this is still true for  $t \leq 5$ , but we are in any case not concerned with that.  $\square$

**Proposition VI.6** The Picard group  $\text{Pic } H_1$  is generated by the classes of  $\Sigma_1 = \overline{C}^{0010}$  and  $\Psi_1 = \overline{C}^{0001}$ . The intersection numbers are  $\Sigma_1^2 = \Psi_1^2 = 0$ ,  $\Sigma_1 \cdot \Psi_1 = 1$  and  $\Sigma_1 \cdot H_1 = \Psi_1 \cdot H_1 = -\mu/6$ .

**Proof** As in [12, Proposition 4.18] (but one has to use the alternative indicated in the remark that follows).  $\square$

**Proposition VI.7** The Picard group  $\text{Pic } H_2$  is generated by the classes of  $\Sigma_2$  and  $\Psi_2$ , which are the inverse images of general fibres of the two projections in  $X(t) \times X(t)$ , and of the exceptional curves  $R_{\alpha\beta}$  of the resolution  $H_2 \rightarrow \underline{H}_2$ . The intersection numbers in  $H_2$  are  $\Sigma_2^2 = \Psi_2^2 = \Sigma_2 \cdot R_{\alpha\beta} = \Psi_2 \cdot R_{\alpha\beta} = 0$ ,  $R_{\alpha\beta} \cdot R_{\alpha'\beta'} = -2\delta^{\alpha\alpha'}\delta^{\beta\beta'}$  and  $\Sigma_2 \cdot \Psi_2 = 6$ . In  $\mathcal{A}_{\text{bil}}^t$  we have  $\Sigma_2 \cdot H_2 = \Psi_2 \cdot H_2 = -\mu$  and  $R_{\alpha\beta} \cdot H_2 = -4$ .

**Proof** The same as the proofs of [12, Proposition 4.21] and [12, Lemma 4.24]. The curves  $R_{(a,b)}$  arise from elliptic fixed points so they are absent here.  $\square$

Notice that  $\Sigma_2$  and  $\Psi_2$  are also images of the general fibres in  $X(2t) \times X(2t)$  and are themselves isomorphic to  $X(2t)$ .

## VII Branch locus

The closure of the branch locus of the map  $\mathbb{H}_2 \rightarrow \mathcal{A}_{\text{bil}}^t$  is  $H_1 \cup H_2$  and modular forms of weight  $3k$  (for  $k$  even) give rise to  $k$ -fold differential forms with poles of order  $k/2$  along  $H_1$  and  $H_2$ . We have to calculate the number of conditions imposed by these poles.



**Proposition VIII.1** *The obstruction from  $H_1$  to extending modular forms of weight  $3k$  to  $k$ -fold holomorphic differential forms is*

$$\Omega_1 \leq \nu_2 \left( \frac{1}{1} + t^2 \left( \frac{1}{1} + \frac{24}{864} \right) \right) k^3 + O(k^2).$$

**Proof** If  $F$  is a modular form of weight  $3k$  for  $k$  even, vanishing to sufficiently high order at infinity, and  $\omega = d\tau_1 \wedge d\tau_2 \wedge d\tau_3$ , then  $F\omega^{\otimes k}$  determines a section of  $kK + \frac{3}{2}H_1 + \frac{3}{2}H_2$ , where  $K$  denotes the canonical sheaf of  $\mathcal{A}_{\text{bl}^*}$ . From

$$0 \rightarrow \mathcal{O}(-H_1) \rightarrow \mathcal{O} \rightarrow \mathcal{O}_{H_1} \rightarrow 0$$

we get, for  $0 \leq j < k/2$

$$\begin{aligned} 0 &\rightarrow H_0(kK + \frac{2}{k}H_1 + \frac{2}{k}H_2) \rightarrow H_0(kK + \frac{2}{k}H_1 + \frac{2}{k}H_2) \rightarrow H_0(kK + \frac{2}{k}H_2) \\ &\rightarrow H_0(kK + \frac{2}{k}H_1 + \frac{2}{k}H_2) \rightarrow H_0(kK + \frac{2}{k}H_2) \rightarrow H_0(kK + \frac{2}{k}H_2) \rightarrow 0 \end{aligned}$$

Thus

$$\begin{aligned} h_0(kK + \frac{2}{k}H_1 + \frac{2}{k}H_2) &\leq h_0(kK + \frac{2}{k}H_1 + \frac{2}{k}H_2) + h_0(kK + \frac{2}{k}H_2) \\ &\quad + h_0(kK + \frac{2}{k}H_1 + \frac{2}{k}H_2) \end{aligned}$$

Note that, by Lemma VI.5,  $H_2|_{H_1} = 0$ . Therefore

$$h_0(kK + \frac{2}{k}H_2) \geq h_0(kK + \frac{2}{k}H_1 + \frac{2}{k}H_2) + \sum_{k/2-1}^j h_0(kK + \frac{2}{k}H_2) + h_0(kK + \frac{2}{k}H_1 + \frac{2}{k}H_2).$$

so

$$\Omega_1 \leq \sum_{k/2-1}^j h_0(kK + \frac{2}{k}H_2) + \sum_{k/2-1}^j h_0(kK + \frac{2}{k}H_1 + \frac{2}{k}H_2).$$

(7)

By Lemma VI.6,  $K_{H_1}$  and  $H_1|_{H_1}$  are both multiples of  $\Sigma_1 + \Phi_{H_1}$ , and any positive multiple of  $\Sigma_1 + \Phi_{H_1}$  is ample. Suppose  $H_1|_{H_1} = a_1(\Sigma_1 + \Phi_{H_1})$  and  $K_{H_1} = b_1(\Sigma_1 + \Phi_{H_1})$ . Then

$$- \frac{6}{t} \Sigma_1 + a_1 \Sigma_1 + b_1 \Phi_{H_1} = a_2 \Sigma_1 + a_3 \Phi_{H_1} = a_4$$

and

$$- \frac{6}{t} \nu = a_2 g(\Sigma_1) - 2 = a_3 + a_4 = a_5 + a_6 + a_7 + a_8 + a_9 + a_{10} + a_{11} + a_{12} + a_{13} + a_{14} + a_{15} + a_{16} + a_{17} + a_{18} + a_{19} + a_{20} + a_{21} + a_{22} + a_{23} + a_{24} + a_{25} + a_{26} + a_{27} + a_{28} + a_{29} + a_{30} + a_{31} + a_{32} + a_{33} + a_{34} + a_{35} + a_{36} + a_{37} + a_{38} + a_{39} + a_{40} + a_{41} + a_{42} + a_{43} + a_{44} + a_{45} + a_{46} + a_{47} + a_{48} + a_{49} + a_{50} + a_{51} + a_{52} + a_{53} + a_{54} + a_{55} + a_{56} + a_{57} + a_{58} + a_{59} + a_{60} + a_{61} + a_{62} + a_{63} + a_{64} + a_{65} + a_{66} + a_{67} + a_{68} + a_{69} + a_{70} + a_{71} + a_{72} + a_{73} + a_{74} + a_{75} + a_{76} + a_{77} + a_{78} + a_{79} + a_{80} + a_{81} + a_{82} + a_{83} + a_{84} + a_{85} + a_{86} + a_{87} + a_{88} + a_{89} + a_{90} + a_{91} + a_{92} + a_{93} + a_{94} + a_{95} + a_{96} + a_{97} + a_{98} + a_{99} + a_{100} + a_{101} + a_{102} + a_{103} + a_{104} + a_{105} + a_{106} + a_{107} + a_{108} + a_{109} + a_{110} + a_{111} + a_{112} + a_{113} + a_{114} + a_{115} + a_{116} + a_{117} + a_{118} + a_{119} + a_{120} + a_{121} + a_{122} + a_{123} + a_{124} + a_{125} + a_{126} + a_{127} + a_{128} + a_{129} + a_{130} + a_{131} + a_{132} + a_{133} + a_{134} + a_{135} + a_{136} + a_{137} + a_{138} + a_{139} + a_{140} + a_{141} + a_{142} + a_{143} + a_{144} + a_{145} + a_{146} + a_{147} + a_{148} + a_{149} + a_{150} + a_{151} + a_{152} + a_{153} + a_{154} + a_{155} + a_{156} + a_{157} + a_{158} + a_{159} + a_{160} + a_{161} + a_{162} + a_{163} + a_{164} + a_{165} + a_{166} + a_{167} + a_{168} + a_{169} + a_{170} + a_{171} + a_{172} + a_{173} + a_{174} + a_{175} + a_{176} + a_{177} + a_{178} + a_{179} + a_{180} + a_{181} + a_{182} + a_{183} + a_{184} + a_{185} + a_{186} + a_{187} + a_{188} + a_{189} + a_{190} + a_{191} + a_{192} + a_{193} + a_{194} + a_{195} + a_{196} + a_{197} + a_{198} + a_{199} + a_{200} + a_{201} + a_{202} + a_{203} + a_{204} + a_{205} + a_{206} + a_{207} + a_{208} + a_{209} + a_{210} + a_{211} + a_{212} + a_{213} + a_{214} + a_{215} + a_{216} + a_{217} + a_{218} + a_{219} + a_{220} + a_{221} + a_{222} + a_{223} + a_{224} + a_{225} + a_{226} + a_{227} + a_{228} + a_{229} + a_{230} + a_{231} + a_{232} + a_{233} + a_{234} + a_{235} + a_{236} + a_{237} + a_{238} + a_{239} + a_{240} + a_{241} + a_{242} + a_{243} + a_{244} + a_{245} + a_{246} + a_{247} + a_{248} + a_{249} + a_{250} + a_{251} + a_{252} + a_{253} + a_{254} + a_{255} + a_{256} + a_{257} + a_{258} + a_{259} + a_{260} + a_{261} + a_{262} + a_{263} + a_{264} + a_{265} + a_{266} + a_{267} + a_{268} + a_{269} + a_{270} + a_{271} + a_{272} + a_{273} + a_{274} + a_{275} + a_{276} + a_{277} + a_{278} + a_{279} + a_{280} + a_{281} + a_{282} + a_{283} + a_{284} + a_{285} + a_{286} + a_{287} + a_{288} + a_{289} + a_{290} + a_{291} + a_{292} + a_{293} + a_{294} + a_{295} + a_{296} + a_{297} + a_{298} + a_{299} + a_{300} + a_{301} + a_{302} + a_{303} + a_{304} + a_{305} + a_{306} + a_{307} + a_{308} + a_{309} + a_{310} + a_{311} + a_{312} + a_{313} + a_{314} + a_{315} + a_{316} + a_{317} + a_{318} + a_{319} + a_{320} + a_{321} + a_{322} + a_{323} + a_{324} + a_{325} + a_{326} + a_{327} + a_{328} + a_{329} + a_{330} + a_{331} + a_{332} + a_{333} + a_{334} + a_{335} + a_{336} + a_{337} + a_{338} + a_{339} + a_{340} + a_{341} + a_{342} + a_{343} + a_{344} + a_{345} + a_{346} + a_{347} + a_{348} + a_{349} + a_{350} + a_{351} + a_{352} + a_{353} + a_{354} + a_{355} + a_{356} + a_{357} + a_{358} + a_{359} + a_{360} + a_{361} + a_{362} + a_{363} + a_{364} + a_{365} + a_{366} + a_{367} + a_{368} + a_{369} + a_{370} + a_{371} + a_{372} + a_{373} + a_{374} + a_{375} + a_{376} + a_{377} + a_{378} + a_{379} + a_{380} + a_{381} + a_{382} + a_{383} + a_{384} + a_{385} + a_{386} + a_{387} + a_{388} + a_{389} + a_{390} + a_{391} + a_{392} + a_{393} + a_{394} + a_{395} + a_{396} + a_{397} + a_{398} + a_{399} + a_{400} + a_{401} + a_{402} + a_{403} + a_{404} + a_{405} + a_{406} + a_{407} + a_{408} + a_{409} + a_{410} + a_{411} + a_{412} + a_{413} + a_{414} + a_{415} + a_{416} + a_{417} + a_{418} + a_{419} + a_{420} + a_{421} + a_{422} + a_{423} + a_{424} + a_{425} + a_{426} + a_{427} + a_{428} + a_{429} + a_{430} + a_{431} + a_{432} + a_{433} + a_{434} + a_{435} + a_{436} + a_{437} + a_{438} + a_{439} + a_{440} + a_{441} + a_{442} + a_{443} + a_{444} + a_{445} + a_{446} + a_{447} + a_{448} + a_{449} + a_{450} + a_{451} + a_{452} + a_{453} + a_{454} + a_{455} + a_{456} + a_{457} + a_{458} + a_{459} + a_{460} + a_{461} + a_{462} + a_{463} + a_{464} + a_{465} + a_{466} + a_{467} + a_{468} + a_{469} + a_{470} + a_{471} + a_{472} + a_{473} + a_{474} + a_{475} + a_{476} + a_{477} + a_{478} + a_{479} + a_{480} + a_{481} + a_{482} + a_{483} + a_{484} + a_{485} + a_{486} + a_{487} + a_{488} + a_{489} + a_{490} + a_{491} + a_{492} + a_{493} + a_{494} + a_{495} + a_{496} + a_{497} + a_{498} + a_{499} + a_{500} + a_{501} + a_{502} + a_{503} + a_{504} + a_{505} + a_{506} + a_{507} + a_{508} + a_{509} + a_{510} + a_{511} + a_{512} + a_{513} + a_{514} + a_{515} + a_{516} + a_{517} + a_{518} + a_{519} + a_{520} + a_{521} + a_{522} + a_{523} + a_{524} + a_{525} + a_{526} + a_{527} + a_{528} + a_{529} + a_{530} + a_{531} + a_{532} + a_{533} + a_{534} + a_{535} + a_{536} + a_{537} + a_{538} + a_{539} + a_{540} + a_{541} + a_{542} + a_{543} + a_{544} + a_{545} + a_{546} + a_{547} + a_{548} + a_{549} + a_{550} + a_{551} + a_{552} + a_{553} + a_{554} + a_{555} + a_{556} + a_{557} + a_{558} + a_{559} + a_{560} + a_{561} + a_{562} + a_{563} + a_{564} + a_{565} + a_{566} + a_{567} + a_{568} + a_{569} + a_{570} + a_{571} + a_{572} + a_{573} + a_{574} + a_{575} + a_{576} + a_{577} + a_{578} + a_{579} + a_{580} + a_{581} + a_{582} + a_{583} + a_{584} + a_{585} + a_{586} + a_{587} + a_{588} + a_{589} + a_{590} + a_{591} + a_{592} + a_{593} + a_{594} + a_{595} + a_{596} + a_{597} + a_{598} + a_{599} + a_{600} + a_{601} + a_{602} + a_{603} + a_{604} + a_{605} + a_{606} + a_{607} + a_{608} + a_{609} + a_{610} + a_{611} + a_{612} + a_{613} + a_{614} + a_{615} + a_{616} + a_{617} + a_{618} + a_{619} + a_{620} + a_{621} + a_{622} + a_{623} + a_{624} + a_{625} + a_{626} + a_{627} + a_{628} + a_{629} + a_{630} + a_{631} + a_{632} + a_{633} + a_{634} + a_{635} + a_{636} + a_{637} + a_{638} + a_{639} + a_{640} + a_{641} + a_{642} + a_{643} + a_{644} + a_{645} + a_{646} + a_{647} + a_{648} + a_{649} + a_{650} + a_{651} + a_{652} + a_{653} + a_{654} + a_{655} + a_{656} + a_{657} + a_{658} + a_{659} + a_{660} + a_{661} + a_{662} + a_{663} + a_{664} + a_{665} + a_{666} + a_{667} + a_{668} + a_{669} + a_{670} + a_{671} + a_{672} + a_{673} + a_{674} + a_{675} + a_{676} + a_{677} + a_{678} + a_{679} + a_{680} + a_{681} + a_{682} + a_{683} + a_{684} + a_{685} + a_{686} + a_{687} + a_{688} + a_{689} + a_{690} + a_{691} + a_{692} + a_{693} + a_{694} + a_{695} + a_{696} + a_{697} + a_{698} + a_{699} + a_{700} + a_{701} + a_{702} + a_{703} + a_{704} + a_{705} + a_{706} + a_{707} + a_{708} + a_{709} + a_{710} + a_{711} + a_{712} + a_{713} + a_{714} + a_{715} + a_{716} + a_{717} + a_{718} + a_{719} + a_{720} + a_{721} + a_{722} + a_{723} + a_{724} + a_{725} + a_{726} + a_{727} + a_{728} + a_{729} + a_{730} + a_{731} + a_{732} + a_{733} + a_{734} + a_{735} + a_{736} + a_{737} + a_{738} + a_{739} + a_{740} + a_{741} + a_{742} + a_{743} + a_{744} + a_{745} + a_{746} + a_{747} + a_{748} + a_{749} + a_{750} + a_{751} + a_{752} + a_{753} + a_{754} + a_{755} + a_{756} + a_{757} + a_{758} + a_{759} + a_{760} + a_{761} + a_{762} + a_{763} + a_{764} + a_{765} + a_{766} + a_{767} + a_{768} + a_{769} + a_{770} + a_{771} + a_{772} + a_{773} + a_{774} + a_{775} + a_{776} + a_{777} + a_{778} + a_{779} + a_{780} + a_{781} + a_{782} + a_{783} + a_{784} + a_{785} + a_{786} + a_{787} + a_{788} + a_{789} + a_{790} + a_{791} + a_{792} + a_{793} + a_{794} + a_{795} + a_{796} + a_{797} + a_{798} + a_{799} + a_{800} + a_{801} + a_{802} + a_{803} + a_{804} + a_{805} + a_{806} + a_{807} + a_{808} + a_{809} + a_{810} + a_{811} + a_{812} + a_{813} + a_{814} + a_{815} + a_{816} + a_{817} + a_{818} + a_{819} + a_{820} + a_{821} + a_{822} + a_{823} + a_{824} + a_{825} + a_{826} + a_{827} + a_{828} + a_{829} + a_{830} + a_{831} + a_{832} + a_{833} + a_{834} + a_{835} + a_{836} + a_{837} + a_{838} + a_{839} + a_{840} + a_{841} + a_{842} + a_{843} + a_{844} + a_{845} + a_{846} + a_{847} + a_{848} + a_{849} + a_{850} + a_{851} + a_{852} + a_{853} + a_{854} + a_{855} + a_{856} + a_{857} + a_{858} + a_{859} + a_{860} + a_{861} + a_{862} + a_{863} + a_{864} + a_{865} + a_{866} + a_{867} + a_{868} + a_{869} + a_{870} + a_{871} + a_{872} + a_{873} + a_{874} + a_{875} + a_{876} + a_{877} + a_{878} + a_{879} + a_{880} + a_{881} + a_{882} + a_{883} + a_{884} + a_{885} + a_{886} + a_{887} + a_{888} + a_{889} + a_{890} + a_{891} + a_{892} + a_{893} + a_{894} + a_{895} + a_{896} + a_{897} + a_{898} + a_{899} + a_{900} + a_{901} + a_{902} + a_{903} + a_{904} + a_{905} + a_{906} + a_{907} + a_{908} + a_{909} + a_{910} + a_{911} + a_{912} + a_{913} + a_{914} + a_{915} + a_{916} + a_{917} + a_{918} + a_{919} + a_{920} + a_{921} + a_{922} + a_{923} + a_{924} + a_{925} + a_{926} + a_{927} + a_{928} + a_{929} + a_{930} + a_{931} + a_{932} + a_{933} + a_{934} + a_{935} + a_{936} + a_{937} + a_{938} + a_{939} + a_{940} + a_{941} + a_{942} + a_{943} + a_{944} + a_{945} + a_{946} + a_{947} + a_{948} + a_{949} + a_{950} + a_{951} + a_{952} + a_{953} + a_{954} + a_{955} + a_{956} + a_{957} + a_{958} + a_{959} + a_{960} + a_{961} + a_{962} + a_{963} + a_{964} + a_{965} + a_{966} + a_{967} + a_{968} + a_{969} + a_{970} + a_{971} + a_{972} + a_{973} + a_{974} + a_{975} + a_{976} + a_{977} + a_{978} + a_{979} + a_{980} + a_{981} + a_{982} + a_{983} + a_{984} + a_{985} + a_{986} + a_{987} + a_{988} + a_{989} + a_{990} + a_{991} + a_{992} + a_{993} + a_{994} + a_{995} + a_{996} + a_{997} + a_{998} + a_{999} + a_{1000}.$$

Hence, using equation (7)

$$\Omega_1 \leq \sum_{k/2-1}^j h_0 \left( \frac{6}{k\mu} - k\nu + \frac{6}{12} + \frac{6}{j\mu} \right) (\Sigma_1 + \Phi_1) \\ = \sum_{k/2-1}^j h_0 \left( \frac{6}{k\nu} - k\nu + \frac{6}{j\nu} \right) (\Sigma_1 + \Phi_1).$$

Since  $t \geq 7$  (we know from [14] that  $\mathcal{A}_{\text{bil}}^t$  is rational for  $t \leq 5$ ), we have  $\frac{6}{k\nu} - k\nu + \frac{6}{j\nu} + \nu > 0$  for all  $j$  and hence  $\left( \frac{6}{k\nu} - k\nu + \frac{6}{j\nu} \right) (\Sigma_1 + \Phi_1) - K_{H_1}$  is ample. So by vanishing we have

$$\Omega_1 \leq \sum_{k/2-1}^j \frac{6}{k\nu} - k\nu + \frac{6}{j\nu} (\Sigma_1 + \Phi_1) + O(k_2^2) \\ = \sum_{k/2-1}^j \left( \frac{6}{k\nu} - k\nu + \frac{6}{j\nu} \right) (\Sigma_1 + \Phi_1) + O(k_2^2) \\ = \frac{6}{j\nu} (\Sigma_1 + \Phi_1) + O(k_2^2). \quad \square$$

Next we carry out the same calculation for  $H_2$ .

**Proposition VII.2** *The obstruction from  $H_2$  is*

$$\Omega_2 \leq \nu^2 \left( \frac{1}{1} + \frac{1}{1} \right) \left( t^2 - \left( \frac{1}{1} + \frac{1}{1} \right) t - \frac{3}{7} + \frac{3}{1} \right) k_3 + O(k_2^2).$$

**Proof** By the same argument as above (equation (7)) the obstruction is

$$\Omega_2 \leq \sum_{k/2-1}^j h_0 \left( kK_{H_2} - \frac{k}{2} \right) (\Sigma_2 + \Phi_2).$$

In this case  $H_2|_{H_2} = a_2(\Sigma_2 + \Phi_2) + c_2R$ , where  $R = \sum_{\alpha, \beta} R^{\alpha, \beta}$  is the sum of all the exceptional curves of  $H_2 \rightarrow \overline{H_2}$ , and  $K_{H_2} = b_2(\Sigma_2 + \Phi_2) + d_2R$ . Since  $\Sigma_2 \cong X(2t)$  [18, 1.6.4]

$$2g(\Sigma_2) - 2 = \frac{3}{1} t - 3\nu(2t) = \mu - \frac{2}{\nu}.$$

Hence

$$-\mu = \Sigma_2 \cdot H_2 = a_2 \Sigma_2 + a_2 \Sigma_2 + a_2 \Sigma_2 + c_2 \Sigma_2 + c_2 \Sigma_2 + c_2 \Sigma_2 + d_2 R.$$

so  $a_2 = -\mu/6$ , and

$$-4\nu^2 = R \cdot H_2 = a_2 \Sigma_2 \cdot R + a_2 \Psi_2 \cdot R + c_2 R^2 = -2\nu^2 c_2;$$

so  $c_2 = 2$ . Therefore

$$H_2|_{H_2} = -\frac{6}{\mu}(\Sigma_2 + \Psi_2) + 2R.$$

Similarly

$$\mu - \frac{\nu}{2} = (K_{H_2} + \Sigma_2) \cdot \Sigma_2 = 6b_2$$

so  $b_2 = \mu/6 - \nu/12$ , and  $0 = R \cdot K_{H_2} = d_2 R^2$  so  $d_2 = 0$ . Hence

$$K_{H_2} = \frac{1}{\nu}(\mu - \frac{\nu}{2})(\Sigma_2 + \Psi_2).$$

Moreover  $L_j = (k - 1)K_{H_2} - (\frac{\nu}{k} + j)H_2|_{H_2}$  is ample, as is easily checked using the Nakai criterion and the fact that the cone of effective curves on  $H_2$  is spanned by  $R^{\alpha\beta}$  and by the nonexceptional components of the fibres of the two maps  $H_2 \rightarrow X(t)$ . These components are  $\Sigma^\alpha \equiv \Sigma_2 - \sum^\beta R^{\alpha\beta}$  and  $\Psi^\beta \equiv \Psi_2 - \sum^\alpha R^{\alpha\beta}$ , and it is simple to check that  $L_j \cdot \Sigma^\alpha = L_j \cdot \Psi^\beta$  and  $L_j \cdot R^{\alpha\beta}$  are all positive for the relevant values of  $j$ ,  $k$  and  $t$ . Therefore

$$\begin{aligned} \Omega_2 &\leq \sum_{k/2-1}^j \frac{2}{k} (kK_{H_2} - \frac{\nu}{2} + j)H_2|_{H_2} \\ &= \sum_{k/2-1}^j \frac{2}{k} (\nu(\frac{\nu}{k} + j) + \frac{6}{\nu}(\Sigma_2 + \Psi_2) + (k + 2j)R)_2 \\ &= \nu^2 k_3 (t_2^2 (\frac{8}{3} + \frac{8}{1} + \frac{72}{1} - t(\frac{4}{1} + \frac{24}{1} + \frac{24}{1} - 2 - \frac{3}{1}) + O(k^2)), \\ &\text{since } (\Sigma_2 + \Psi_2)_2 = 12. \quad \square \end{aligned}$$

## VIII Final calculation

In this section we assemble the results of the previous sections into a proof of the main theorem.

**Theorem VIII.1**  $\mathcal{A}^{bil*}$  is of general type for  $t$  odd and  $t \geq 17$ .

**Proof** We put  $n = 3k$  in Theorem II.6, and use  $\phi_2(t) = 2\nu$  and the fact that

$$\phi_4(t) = t^4 \prod_{p|t} (1 - p^{-4}) = t^2 \phi_2(t) \prod_{p|t} (1 + p^{-2}).$$

This gives the expression

$$\dim \mathfrak{S}_*^n(\Gamma_{\text{bil}}^t) = k_3 \nu_2^4 \prod_{p|t} (1 + p^{-2}) + O(k_2^2).$$

From Proposition VII.1 and Proposition VII.2 we have

$$\begin{aligned} \Omega_1 &= k_3 \nu_2^2 \left( \frac{37}{7} t^2 - \frac{24}{7} t + \frac{1}{2} \right) + O(k_2^2), \\ \Omega_2 &= k_3 \nu_2^2 \left( \frac{37}{7} t^2 - \frac{24}{7} t - \frac{55}{24} \right) + O(k_2^2) \end{aligned}$$

and from Corollary V.7 and Corollary IV.2

$$\Omega_\infty = k_3 \nu_2^2 \sum_{p|t} \frac{11}{36r} t^2 \prod_{p|(r,h)} (1 - p^{-2}) + O(k_2^2).$$

since  $\phi_2(r)\phi_2(h) = t^2 \prod_{p|(r,h)} (1 - p^{-2})$ .

It follows that  $\mathcal{A}_{\text{bil}}^t$  is of general type, for odd  $t$ , provided

$$(8) \quad \frac{1}{320} \prod_{p|t} (1 + p^{-2}) t^4 - \frac{481}{864} t^2 + \frac{12}{7} t + \frac{24}{43} - \sum_{p|t} \frac{36r}{11} t^2 \prod_{p|(r,h)} (1 - p^{-2}) > 0.$$

This is simple to check: since either  $r = 1$  or  $r \geq 3$ , and since the sum of the divisors of  $t$  is less than  $t/2$ , the last term can be replaced by  $-\frac{36}{11} t^2 - \frac{108}{11} t^3$  and the  $t$  and constant terms, and the  $p^{-2} t^4$  term, can be discarded as they are positive. The resulting expression is a quadratic in  $t$  whose larger root is less than 40, so we need only consider odd  $t \leq 39$ . We deal with primes, products of two primes and prime powers separately. In the case of primes, the expression on the left-hand side of the inequality (8) becomes  $\frac{320}{1} t^4 - \frac{481}{864} t^2 + \frac{18}{5} t + \frac{24}{43}$ , which is positive for  $t \geq 17$ . The expression in the case of  $t = pq$  is positive if  $t \geq 21$ . For  $t = p^2$  we get an expression which is negative for  $t = 9$  but positive for  $t = 25$ , and for  $t = p^3$  the expression is positive.  $\square$

One can also say something for  $t$  even, though not if  $t$  is a power of 2.

**Corollary VIII.2**  $\mathcal{A}_{\text{bil}}^t$  is of general type unless  $t = 2^a b$  with  $b$  odd and  $b < 17$ .

**Proof**  $\mathcal{A}_{\text{bil}}^{nt}$  covers  $\mathcal{A}_{\text{bil}}^t$  for any  $n$ , and therefore  $\mathcal{A}_{\text{bil}}^{nt}$  is of general type if  $\mathcal{A}_{\text{bil}}^t$  is of general type.  $\square$

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