On obstructions to the Hasse principle

Per Salberger

to Sir Peter Swinnerton-Dyer

Introduction

A basic problem in arithmetic geometry is to decide if a variety defined over a number field k has a k-rational point. This is only possible if there is a k_v-point on the variety for each completion k_v of k. It remains to decide if there is a k-point on a variety with a k_v-point at each place v of k. The first positive results were obtained by Hasse for quadrics and varieties defined by means of certain norm forms. A class of varieties, therefore, is said to satisfy the Hasse principle if each variety in the class has a k-point as soon as it has k_v-points for all places v. The corresponding property for the smooth locus is called the smooth Hasse principle. It is also natural to ask if weak approximation holds. This means that the set of k-points is dense in the topological space of adelic points on the smooth locus.

There are counterexamples to the Hasse principle and weak approximation already for smooth cubic curves and cubic surfaces. These counterexamples can be explained by means of a general obstruction to the Hasse principle due to Manin based on the Brauer group of the variety and the reciprocity law in class field theory. Most but not all of the known counterexamples can be explained by this obstruction (Skorobogatov [Sk]). It is likely that Manin's obstruction is the only obstruction to the (smooth) Hasse principle for rational varieties. But it has only been proved in very special cases.

It is more reasonable to study the Hasse principle for 0-cycles of degree one. For curves it is possible to relate the uniqueness of Manin's obstruction to the finiteness of the Tate–Shafarevich group of the jacobian, which has been proved for some elliptic curves by Kolyvagin and Rubin. Another fairly general result is due to the author [Sa] and concerns conic bundle surfaces over the projective line. There we proved a difficult conjecture of Colliot-Thélène and Sansuc (Conjecture B on p. 443 in [CT/S1]). It says that a new kind of Shafarevich group $Ш^1(k, M)$ defined by means of K theory vanishes for rational surfaces. This result has several consequences. One corollary concerns the size of the Chow group of degree zero cycles (cf. [CT/S1] and [Sa]). Another corollary obtained in 1987 and announced in [Sa] is the following

0.1 Theorem *Let k be a number field and X a conic bundle surface over $\mathbb{P}^1_k$. Then Manin's obstruction is the only obstruction to the Hasse principle for* 0*-cycles of degree one.*

The author included in [Sa] a proof when the Brauer group $H^2_{\text{ét}}(X, \mathbb{G}_m)$ of X contains no other elements than those coming from the Brauer group of k. Then the Manin obstruction vanishes so that one obtains the simpler statement that the Hasse principle holds for 0-cycles of degree one. One of the motivations for the present paper is to present a proof of Theorem 0.1, by deducing it from our result on $\text{Ш}^1(k, M)$. This is an improved version of the proof found in 1987.

It is based on a generalization of the descent theory of Colliot-Thélène and Sansuc [CT/S2] for rational points to 0-cycles of degree one. The rest of the proof is to show that certain diagrams commute. This is done using techniques similar to those in Bloch [Bl] and [CT/S1].

The descent theory developed by Colliot-Thélène and Sansuc is an analog of the classical descent theory for elliptic curves developed by Fermat, Euler, Mordell and Weil. If $p_\alpha \colon \mathcal{T}_\alpha \to X$ is a class of such descent varieties and K is an overfield of k, then the sets $p_\alpha(\mathcal{T}_\alpha(K))$ form a partition of $X(K)$. The descent varieties we consider are torsors over X under commutative algebraic groups.

For varieties with finitely generated torsion-free Picard groups, Colliot-Thélène and Sansuc [CT/S2] introduced a special kind of descent varieties called universal torsors. These are torsors under the Néron–Severi torus of the variety having a certain universal property among other torsors. One of the most important results in their paper is the following

0.2 Theorem *Let X be a smooth proper rational variety with a k_v-point P_v in each completion of k. Suppose that the set of these k_v-points satisfies Manin's Brauer group condition. Then there exists a universal X-torsor $p \colon \mathcal{T} \to X$ under the Néron–Severi torus T of X (see (1.2)) such that the k_v-torsors under $T \times_k k_v$ at P_v obtained by base extension are trivial for each place v of k_v.*

This means that there are k_v-points Q_v on $\mathcal{T}$ such that $p(Q_v) = P_v$ for each place v of k. Therefore, if the universal torsors over X satisfy the Hasse principle, then Manin's obstruction is the only obstruction to the Hasse principle for X. There are many applications of this result. For some classes of rational varieties X it is possible to establish the Hasse principle for the universal torsors either directly or by means of some intermediate torsors.

The proof of Theorem 0.2 in [CT/S2] uses explicit computations of cocycles. The aim of Section 1 is to offer a proof based on simple functoriality properties of étale cohomology. It is not necessary to assume that X is rational. It suffices to assume (just as in the proof in *op. cit.*) that the Picard group of $X \times_k \overline{k}$ is finitely generated and torsion-free for an algebraic closure $\overline{k}$ of k. Only Brauer classes in the "algebraic part" $\widetilde{H}^2_{\text{ét}}(X, \mathbb{G}_m)$ of the Brauer group of X occur. This is the kernel of the functorial map from $H^2_{\text{ét}}(X, \mathbb{G}_m)$ to $H^2_{\text{ét}}(X \times_k \overline{k}, \mathbb{G}_m)$. If X is smooth and rational, then $\widetilde{H}^2_{\text{ét}}(X, \mathbb{G}_m)$ is the full Brauer group of X.

The basic idea of the proof is to "kill" the nonconstant algebraic part of the Brauer group of X by considering a fibre product Π of a finite number of Severi–Brauer schemes over X which are trivial at the specializations at the given k_v-points. The vanishing of Manin's obstruction for the algebraic part of the Brauer group implies that $\widetilde{H}^2_{\text{ét}}(\Pi, \mathbb{G}_m)$ contains no other elements than those coming from the Brauer group of k. The given k_v-points can be lifted to k_v-points on Π. It is now easy to show that there exists a universal Π-torsor which is trivial at these k_v-points on Π and from this, construct the desired universal X-torsor. (Use (1.4) and its functoriality under $\Pi \to X$.) This gives a natural proof of Theorem 0.2.

There is no direct extension of this proof to 0-cycles of degree one since such cycles cannot be lifted to the Severi–Brauer schemes over X. We therefore replace the Severi–Brauer X-schemes by X-torsors under tori. This makes the proof less transparent. But the rôle of the auxiliary torsors is the same. They are used to simplify the cohomological obstructions. The X-torsors denoted by $\mathcal{S}$ are in fact chosen in such a way that they give rise to universal torsors over Π after pull-back of their base with respect to the morphism $\Pi \to X$.

The advantage of this approach is that we can generalize Theorem 0.2 to a statement where the k_v-points P_v are replaced by 0-cycles of degree one (see Theorem 1.27). Any 0-cycle z_v on $X \times_k k_v$ defines a natural specialization map $\rho(z_v)$ from $H^1_{\text{ét}}(X, T)$ to $H^1_{\text{ét}}(k_v, T_v)$. Our generalization of Theorem 0.2 says that there exists a universal X-torsor $p\colon \mathcal{T} \to X$ such that the class $[\mathcal{T}] \in H^1_{\text{ét}}(X, T)$ of $\mathcal{T}$ belongs to the kernel of $\rho(z_v)$ for each place v of k. This generalization is more difficult to prove and apply than Theorem 0.2, since the triviality of $\rho(z_v)([\mathcal{T}])$ in $H^1_{\text{ét}}(k_v, T_v)$ does not guarantee that z_v can be lifted to a 0-cycle of degree one on $\mathcal{T}$ as in the case of k_v-points.

The results in Section 1 are the following. We first give precise criteria for when there exists a universal torsor for a large class of varieties over a number field k. One necessary condition is that there are universal torsors over the k_v-varieties that are obtained by base extension from k to k_v. A second necessary condition is given by considering the elements in the Brauer group of the variety that become constant after all the base extensions to local fields. We first formulate one criterion (Proposition 1.12) without assuming that there are 0-cycles of degree one over the local fields k_v and then, as an application, a second criterion (Proposition 1.26) under the assumptions that such 0-cycles exist over each completion k_v. Such criteria were first established in [CT/S2] in the case when the 0-cycles are k_v-points on X.

In Theorem 1.27 we then prove our generalization of Theorem 0.2 discussed above. It is worth noting that the result also applies to varieties with $H^1(X, \mathcal{O}_X) = 0$ and torsion-free Néron–Severi group, such as K3 surfaces. But the rationality assumption in [CT/S2, Section 3] remains essential for the conjecture that the universal torsors satisfy the Hasse principle. The converse (ii) $\Rightarrow$ (i) of Theorem 1.27 tells us that the universal torsors contain all the information about the obstruction coming from the algebraic part of the Brauer group.

To prove Theorem 0.1 we need a strange corollary of Theorem 1.27 (Corollary 1.45) for torsors defined over an open subset of X. To prove this result, we use arguments related to the "description locale des torseurs" in [CT/S2]. This corollary plays an important rôle in the proof of Theorem 0.1 in Section 2.

In Section 2 we first recall the K-theoretic construction of Bloch [Bl] for rational surfaces as well as some refinements in [CT/S1] and [Sa]. A fundamental tool in [Bl] is a characteristic homomorphism ϕ' for rational surfaces from the group $Z_0(X)^0$ of 0-cycles of degree zero to $H^1_{\text{ét}}(k, T)$ where T is the Néron–Severi torus of X. In order to prove Theorem 0.1 we need that this map behaves well under specializations. This is not immediate for Bloch's map, but easy to show for another map ϕ of Colliot-Thélène and Sansuc defined by means of universal torsors. We shall therefore make use of the fact that $\phi = \phi'$ for rational surfaces. We then prove that the vanishing of $\text{Ш}^1(k, M)$ implies that the Manin obstruction is the only obstruction to the Hasse principle for 0-cycles of degree one. This is proved for rational surfaces and, more generally, for the class of varieties satisfying certain axioms (2.3) and (2.4). In particular, we deduce Theorem 0.1 from the deep arithmetical result on $\text{Ш}^1(k, M)$ for rational conic bundle surfaces in [Sa].

This paper is a slightly revised version of a manuscript from 1993 in which I prove Theorem 0.1 for a more general class of rational varieties with a pencil of Severi–Brauer varieties. There is also a proof of this more general result in the paper of Colliot-Thélène and Swinnerton-Dyer [CT/SwD]. Their approach is different and not based on descent theory.

I would like to express my gratitude to the referee for his careful reading of the paper.

1 Universal torsors, Brauer groups and obstructions to the Hasse principle

Let k be a field, $\overline{k}$ a separable closure of k and $\mathcal{G} := \mathrm{Gal}(\overline{k}/k)$ the absolute Galois group of k. There is a contravariant equivalence (cf. Borel [Bo]) between the categories of k-tori and the category of finitely generated torsion-free discrete $\mathcal{G}$-modules. If S is a k-torus, then there is a natural $\mathcal{G}$-action on the character group $\widehat{S} := \mathrm{Hom}(\overline{S}, \mathbb{G}_{m,\overline{k}})$ of the $\overline{k}$-torus $\overline{S} = \overline{k} \times_k S$ such that $\widehat{S}$ becomes a finitely generated torsion-free discrete $\mathcal{G}$-module. Conversely, if M is a finitely generated torsion-free discrete $\mathcal{G}$-module, then $D(M) := \mathrm{Hom}_{\mathbb{Z}}(M, \overline{k}^*)$ is a $\overline{k}$-torus with a natural k-structure induced by the $\mathcal{G}$-action on M, thereby defining a k-torus. In the sequel we identify M with its bidual $\widehat{D(M)}$ and write $\mathrm{id}\colon \widehat{D(M)} \xrightarrow{\cong} M$ for the canonical $\mathcal{G}$-isomorphism.

We recall some basic notions and results from the descent theory of Colliot-Thélène and Sansuc [CT/S2]. We will consider k-varieties over a perfect field k satisfying the following assumptions.

$$X \text{ is a smooth proper } k\text{-variety such that } \overline{X} := \overline{k} \times X \text{ is connected and } \mathrm{Pic}\,\overline{X} := H^1_{\text{ét}}(\overline{X}, \mathbb{G}_m) \text{ is finitely generated and torsion-free.} \tag{1.1}$$

Let $\pi\colon \mathcal{S} \to X$ be a k-morphism from a k-variety $\mathcal{S}$ which is faithfully flat and locally of finite type over X. Let S be a k-torus. Then $\pi\colon \mathcal{S} \to X$ is said to be a (left) X-torsor under S if there is a (left) action $\sigma\colon S \times \mathcal{S} \to \mathcal{S}$ such that the k-morphism

$$(\sigma, \mathrm{pr}_2)\colon S \times_X \mathcal{S} \longrightarrow \mathcal{S} \times_X \mathcal{S}$$

induced by σ and the second projection $\mathrm{pr}_2\colon S \times_X \mathcal{S} \to \mathcal{S}$ is an isomorphism. We usually write $\mathcal{S}$ rather than $\pi\colon \mathcal{S} \to X$ for the X-torsor. An X-torsor under a k-torus is locally trivial in the étale topology by a theorem of Grothendieck. The isomorphism classes of X-torsors under S correspond to elements of $H^1_{\text{ét}}(X, S)$.

Now let $\chi\colon H^1_{\text{ét}}(X, S) \to \mathrm{Hom}_{\mathcal{G}}(\widehat{S}, \mathrm{Pic}\,\overline{X})$ be the homomorphism induced by the additive pairing $H^1_{\text{ét}}(\overline{X}, \overline{S}) \times \mathrm{Hom}(\overline{S}, \mathbb{G}_{m,\overline{k}}) \to H^1_{\text{ét}}(\overline{X}, \mathbb{G}_{m,\overline{k}})$.

1.2 Definition

(a) Let $\mathcal{S}$ be an X-torsor under S and $[\mathcal{S}]$ its class in $H^1_{\text{ét}}(X, S)$. Then $\chi([\mathcal{S}]) \in \mathrm{Hom}_{\mathcal{G}}(\widehat{S}, \mathrm{Pic}\,\overline{X})$ is called the *type* of $\mathcal{S}$.

(b) The *Néron–Severi torus* T of X is the k-torus $D(\mathrm{Pic}\,\overline{X})$ associated to the discrete $\mathcal{G}$-module $\mathrm{Pic}\,\overline{X}$.

(c) A *universal torsor* over X is an X-torsor under the Néron–Severi torus T whose type is $\mathrm{id}\colon \widehat{T} \to \operatorname{Pic} \overline{X}$.

By considering the spectral sequence

$$\operatorname{Ext}^p_{k_{\text{ét}}}(\widehat{S}, R^q p_* \mathbb{G}_{m,X}) \Rightarrow \operatorname{Ext}^{p+q}_{X_{\text{ét}}}(p^* \widehat{S}, \mathbb{G}_{m,X}) \tag{1.3}$$

for a k-torus S and the structure morphism $p\colon X \to \operatorname{Spec} k$ (see [CT/S2, 1.5.1]), Colliot-Thélène and Sansuc obtained the exact sequence:

$$0 \to H^1_{\text{ét}}(k,S) \to H^1_{\text{ét}}(X,S) \xrightarrow{\chi} \operatorname{Hom}_{\mathcal{G}}(\widehat{S}, \operatorname{Pic} \overline{X}) \xrightarrow{\delta} H^2_{\text{ét}}(k,S) \to H^2_{\text{ét}}(X,S). \tag{1.4}$$

The homomorphisms $H^i_{\text{ét}}(k,S) \to H^i_{\text{ét}}(X,S)$ are the functorial contravariant maps in étale cohomology. We shall not give any explicit description of δ. All we need in the proofs is that the sequence (1.4) is functorial under field extensions of k and homomorphisms of k-tori.

Let $\widetilde{H}^2_{\text{ét}}(X,S) := \operatorname{Ker}\big(H^2_{\text{ét}}(X,S) \to H^2_{\text{ét}}(\overline{X},\overline{S})\big)$. By analysing (1.3) further, one extends the end of (1.4) to an exact sequence:

$$\operatorname{Hom}_{\mathcal{G}}(\widehat{S}, \operatorname{Pic} \overline{X}) \xrightarrow{\delta} H^2_{\text{ét}}(k,S) \to \widetilde{H}^2_{\text{ét}}(X,S) \to \operatorname{Ext}_{\mathcal{G}}(\widehat{S}, \operatorname{Pic} \overline{X}) \to H^3_{\text{ét}}(k,S). \tag{1.5}$$

In particular for $S = \mathbb{G}_{m,k}$, one obtains the well-known sequence:

$$H^2_{\text{ét}}(k, \mathbb{G}_{m,k}) \to \widetilde{H}^2_{\text{ét}}(X, \mathbb{G}_{m,k}) \to \operatorname{Ext}_{\mathcal{G}}(\mathbb{Z}, \operatorname{Pic} \overline{X}) \to H^3_{\text{ét}}(k, \mathbb{G}_{m,k}). \tag{1.6}$$

The next result is also in [CT/S2]. We include a proof, since *op. cit.* does not prove the implication (iii) $\Rightarrow$ (ii) *directly.*

1.7 Proposition *Let k, X be as in (1.1) and let T be the Néron–Severi torus of X. Then the following conditions are equivalent.*

(i) $H^2_{\text{ét}}(k,T) \to H^2_{\text{ét}}(X,T)$ *is injective for the Néron–Severi torus T.*

(ii) $H^2_{\text{ét}}(k,S) \to H^2_{\text{ét}}(X,S)$ *is injective for any k-torus S.*

(iii) There exists a universal torsor over X.

Proof (ii) ⇒ (i) is trivial and (i) ⇒ (iii) is immediate from (1.4). To prove (iii) ⇒ (ii), let S be a k-torus and $\chi \in \mathrm{Hom}_{\mathcal{G}}(\widehat{S}, \mathrm{Pic}\,\overline{X})$. Then there is a dual homomorphism $D(\chi)$ of k-tori $T \to S$ inducing a commutative diagram

$$\begin{array}{ccccccc} H^1_{\text{ét}}(X,T) & \to & \mathrm{Hom}_{\mathcal{G}}(\widehat{T}, \mathrm{Pic}\,\overline{X}) & \to & H^2_{\text{ét}}(k,T) & \to & H^2_{\text{ét}}(X,T) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^1_{\text{ét}}(X,S) & \to & \mathrm{Hom}_{\mathcal{G}}(\widehat{S}, \mathrm{Pic}\,\overline{X}) & \to & H^2_{\text{ét}}(k,S) & \to & H^2_{\text{ét}}(X,S) \end{array}$$

such that $\mathrm{id} \in \mathrm{Hom}_{\mathcal{G}}(\widehat{T}, \mathrm{Pic}\,\overline{X})$ goes to χ in $\mathrm{Hom}_{\mathcal{G}}(\widehat{S}, \mathrm{Pic}\,\overline{X})$. Hence $\delta(\chi) = 0$, thereby proving (iii) ⇒ (ii). □

Recall that a 0-*cycle* on X is a finite formal sum $z = \sum n_i P_i$ where the P_i are closed points on X and the n_i integers. The integer $n := \sum n_i [k(P_i) : k]$ is called the *degree* of z. Denote by $Z_0(X)$ the free abelian group of 0-cycles on X. For each k-torus S and each positive integer i, there is a natural additive pairing

$$\rho\colon Z_0(X) \times H^i_{\text{ét}}(X,S) \to H^i_{\text{ét}}(k,S) \tag{1.8}$$

sending a pair consisting of a closed point $P \in Z_0(X)$ and an element $\varepsilon \in H^i_{\text{ét}}(X,S)$ to the corestriction in $H^i_{\text{ét}}(k,S)$ of the pullback $\varepsilon(P)$ of ε in $H^i_{\text{ét}}(k(P),S)$. It can be proved that this pairing factorizes through rational equivalence, but we do not need this.

If $z = \sum n_i P_i$ is a 0-cycle, write $\rho(z)\colon H^i_{\text{ét}}(X,T) \to H^i_{\text{ét}}(k,T)$ for the homomorphism sending $\varepsilon \in H^i_{\text{ét}}(X,T)$ to $\rho(z,\varepsilon) \in H^i_{\text{ét}}(k,T)$. This gives a retraction of the functorial map from $H^i_{\text{ét}}(k,T)$ to $H^i_{\text{ét}}(X,T)$ when z is of degree one. Then by Proposition 1.7, there exists a universal torsor over X.

Let T be the Néron–Severi torus of X and $\mathcal{T}$ a universal X-torsor. Let $\phi_{\mathcal{T}}\colon Z_0(X) \to H^1_{\text{ét}}(k,T)$ be the homomorphism which sends $z \in Z_0(X)$ to $\rho(z,[\mathcal{T}])$ (see (1.8)), and $Z_0(X)^0$ the subgroup of $Z_0(X)$ consisting of 0-cycles of degree zero.

1.9 Proposition *The restriction of $\phi_{\mathcal{T}}$ to $Z_0(X)^0$ is independent of the choice of universal torsor $\mathcal{T}$.*

Proof Use (1.4) and the fact that

$$Z_0(X)^0 \times \mathrm{Im}(H^1_{\text{ét}}(k,T) \to H^1_{\text{ét}}(X,T)) \subseteq \mathrm{Ker}(\rho). \quad \square$$

We therefore drop the index and write ϕ for this map $Z_0(X)^0 \to H^1_{\text{ét}}(k,T)$. For other constructions of ϕ that do not depend on the assumption that a universal torsor exists, see [CT/S1, Section 1] and the next section.

The following almost trivial lemma from homological algebra will be useful.

1.10 Lemma *Let L be a finitely generated torsion-free discrete $\mathcal{G}$-module. Then*

(a) $H^1(\mathcal{G},\mathbb{Z})=0$.

(b) $H^1(\mathcal{G},L)=\mathrm{Ext}_{\mathcal{G}}(\mathbb{Z},L)$ *is finite.*

(c) Let $\varepsilon_1,\varepsilon_2,\dots,\varepsilon_r\in\mathrm{Ext}_{\mathcal{G}}(\mathbb{Z}^{(r)},L)$ *and* $\varepsilon\in\mathrm{Ext}_{\mathcal{G}}(\mathbb{Z}^{(r)},L)$ *correspond to* $\{\varepsilon_j\}_{j=1}^r\in\bigoplus\mathrm{Ext}_{\mathcal{G}}(\mathbb{Z},L)$.

Then there is an extension of discrete $\mathcal{G}$-modules

$$0\to L\to M\to\mathbb{Z}^{(r)}\to 0 \qquad (*)$$

such that

(i) $\mathrm{Ker}(\mathrm{Ext}_{\mathcal{G}}(\mathbb{Z},L)\to\mathrm{Ext}_{\mathcal{G}}(\mathbb{Z},M))$ *is the subgroup generated by* $\varepsilon_1,\varepsilon_2,\dots,\varepsilon_r$,

(ii) the connecting homomorphism $\mathrm{Hom}_{\mathcal{G}}(L,L)\to\mathrm{Ext}_{\mathcal{G}}(\mathbb{Z}^{(r)},L)$ *induced by* $(*)$ *sends* $\mathrm{id}\in\mathrm{Hom}_{\mathcal{G}}(L,L)$ *to* ε.

Now let k be a number field. Denote by Ω_k the set of places of k, and by k_v the v-adic completion of k for a place v. Choose an algebraic closure $\overline{k}_v$ of k_v and an embedding $\overline{k}\subset\overline{k}_v$ for each $v\in\Omega_k$. We may then regard the Galois group $\mathcal{G}_v:=\mathrm{Gal}(\overline{k}_v/k_v)$ as a subgroup of $\mathcal{G}=\mathrm{Gal}(\overline{k}/k)$ for each $v\in\Omega_k$.

If M is a discrete $\mathcal{G}$-module and i a positive integer, write

$$Ш^i(k,M):=\mathrm{Ker}\Big(H^i(\mathcal{G},M)\to\prod_{\text{all } v}H^i(\mathcal{G}_v,M)\Big).$$

In particular, if M is the group $S(\overline{k})$ of $\overline{k}$-points on a k-torus S, we write $Ш^i(k,S):=Ш^i(k,S(\overline{k}))$. Finally, set

$$Ч^1(k,S):=\mathrm{Coker}\Big(H^1(\mathcal{G},S(\overline{k}))\to\bigoplus_{\text{all } v}H^1(\mathcal{G}_v,S(\overline{k}_v))\Big).$$

The following result from class field theory is due to Nakayama and Tate [Ta1]. It plays an important rôle in [CT/S2].

1.11 Theorem *Let k be a number field and S a k-torus. Then there is a perfect pairing*

$$Ш^2(k,S)\times Ш^1(k,\widehat{S})\longrightarrow\mathbb{Q}/\mathbb{Z}$$

which is functorial under homomorphisms of k-tori. The kernel of the induced epimorphism from $\operatorname{Hom}(H^1(\mathcal{G},\widehat{S}),\mathbb{Q}/\mathbb{Z})$ *to* $Ш^2(k,S)$ *is isomorphic to* $Ч^1(k,S)$. *Moreover,* $H^3_{\text{ét}}(k,S)=0$ *for any split k-torus S.*

The next proposition generalizes a result in Section 3.3 in [CT/S2]. If we use the word "locally" for a property which holds for $X_v := k_v \times X$ for each place $v \in \Omega_k$, then we can express Proposition 1.12 in the following way. There exists a universal torsor over X if and only if there exists one locally, and moreover every locally constant Azumaya algebra over X is Brauer equivalent to a product of a locally trivial Azumaya algebra and a constant Azumaya algebra. We can replace $\widetilde{H}^2_{\text{ét}}$ by $H^2_{\text{ét}}$ in (i), since any "locally" constant Brauer class belongs to $\widetilde{H}^2_{\text{ét}}(X,\mathbb{G}_m)$. We prefer the formulation here since the universal torsors are related to $\widetilde{H}^2_{\text{ét}}(X,\mathbb{G}_m)$ rather than $H^2_{\text{ét}}(X,\mathbb{G}_m)$.

1.12 Proposition *Let k be a number field and X a smooth proper geometrically connected variety over k for which* $\operatorname{Pic}\overline{X}$ *is finitely generated and torsion-free. Then the following statements are equivalent.*

(i) The map from $\operatorname{Ker}(\widetilde{H}^2_{\text{ét}}(X,\mathbb{G}_m)\to\prod_{\text{all } v}\widetilde{H}^2_{\text{ét}}(X_v,\mathbb{G}_m))$ *to*

$$\operatorname{Ker}\Big(\widetilde{H}^2_{\text{ét}}(X,\mathbb{G}_m)/\operatorname{Im}H^2_{\text{ét}}(k,\mathbb{G}_m)\to\prod_{\text{all } v}\widetilde{H}^2_{\text{ét}}(X_v,\mathbb{G}_m)/\operatorname{Im}H^2_{\text{ét}}(k_v,\mathbb{G}_m)\Big)$$

is surjective, and for each place $v\in\Omega_k$ *there exists a universal torsor over* X_v.

(ii) There exists a universal torsor over X.

Proof We apply Lemma 1.10 for the $\mathcal{G}$-module $L=\operatorname{Pic}\overline{X}$ and choose a set of generators $\varepsilon_1,\varepsilon_2,\dots,\varepsilon_r$ of $\operatorname{Ker}(\operatorname{Ext}_{\mathcal{G}}(\mathbb{Z},\operatorname{Pic}\overline{X})\to\prod_{\text{all } v}\operatorname{Ext}_{\mathcal{G}_v}(\mathbb{Z},\operatorname{Pic}\overline{X}))$. Let $\varepsilon\in\operatorname{Ext}_{\mathcal{G}}(\mathbb{Z}^{(r)},L)$ correspond to $\bigoplus_{j=1}^r\varepsilon_j\in\operatorname{Ext}_{\mathcal{G}}(\mathbb{Z},L)$. We then obtain an exact sequence of discrete $\mathcal{G}$-modules

$$0\to\operatorname{Pic}\overline{X}\to M\to\mathbb{Z}^{(r)}\to 0 \tag{1.13}$$

such that

$$Ш^1(k,\operatorname{Pic}\overline{X})=\operatorname{Ker}(H^1(\mathcal{G},\operatorname{Pic}\overline{X})\to H^1(\mathcal{G},M));\quad\text{and} \tag{1.14}$$

id $\in\operatorname{Hom}_{\mathcal{G}}(\operatorname{Pic}\overline{X},\operatorname{Pic}\overline{X})$ maps to ε under the connecting homomorphism $\operatorname{Hom}_{\mathcal{G}}(\operatorname{Pic}\overline{X},\operatorname{Pic}\overline{X})\to\operatorname{Ext}_{\mathcal{G}}(\mathbb{Z}^{(r)},\operatorname{Pic}\overline{X})$ induced by (1.13). (1.15)

the extension (1.13) is split as a sequence of $\mathcal{G}_v$-modules for each $v\in\Omega_k$. (1.16)

Now apply $D(\dots)$ to (1.13) and consider the dual sequence of k-tori:

$$1 \to R \to S \to T \to 1, \tag{1.17}$$

where T is the Néron–Severi torus of X and $R = \prod_{j=1}^{r} \mathbb{G}_{m,k}$. From (1.14) and the arithmetical duality result in Theorem 1.11 we obtain that:

$$Ш^2(k,S) \subseteq \operatorname{Ker}(H^2_{\text{ét}}(k,S) \to H^2_{\text{ét}}(k,T)) \tag{1.18}$$

and from (1.16) that the sequences of k_v-tori

$$1 \to R_v \to S_v \to T_v \to 1 \tag{1.19}$$

induced from (1.17) split for all places v of k.

Proof of (i) $\Rightarrow$ (ii) Consider the following commutative diagram with exact rows and columns

$$\begin{array}{ccccc}
H^2_{\text{ét}}(k,R) & \to & \widetilde{H}^2_{\text{ét}}(X,R) & \to & \operatorname{Ext}_{\mathcal{G}}(\widehat{R}, \operatorname{Pic}\overline{X}) \\
\downarrow & & \downarrow & & \downarrow \\
H^2_{\text{ét}}(k,S) & \to & \widetilde{H}^2_{\text{ét}}(X,S) & \to & \operatorname{Ext}_{\mathcal{G}}(\widehat{S}, \operatorname{Pic}\overline{X}) \\
\downarrow & & \downarrow & & \downarrow \\
H^2_{\text{ét}}(k,T) & \to & \widetilde{H}^2_{\text{ét}}(X,T) & \to & \operatorname{Ext}_{\mathcal{G}}(\widehat{T}, \operatorname{Pic}\overline{X}) \\
\downarrow & & & & \\
0 & & & &
\end{array} \tag{1.20}$$

where the horizontal sequences are those in (1.5) and the vertical sequences are induced by (1.17). The complex in the second column is exact since (1.17) splits over $\overline{k}$. The map $H^2_{\text{ét}}(k,S) \to H^2_{\text{ét}}(k,T)$ is surjective since $H^3_{\text{ét}}(k,R) = 0$ for a number field k (see (1.8)).

In order to prove that there is a universal torsor, it suffices by Proposition 1.7 to show that $H^2_{\text{ét}}(k,T) \to \widetilde{H}^2_{\text{ét}}(X,T)$ is injective. So let $\kappa \in \operatorname{Ker}(H^2_{\text{ét}}(k,T) \to \widetilde{H}^2_{\text{ét}}(X,T))$ and lift κ to an element $\beta \in H^2_{\text{ét}}(k,S)$. Then, by exactness of (1.20), there exists γ in $\operatorname{Ker}(\widetilde{H}^2_{\text{ét}}(X,R) \to \operatorname{Ext}_{\mathcal{G}}(\widehat{S}, \operatorname{Pic}\overline{X}))$ with the same image as β in $\widetilde{H}^2_{\text{ét}}(X,S)$. Let γ_v be the image of γ in $\widetilde{H}^2_{\text{ét}}(X_v,R)$ and consider the following commutative diagram with exact rows and columns.

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & H^2_{\text{ét}}(k_v,R_v) & \to & \widetilde{H}^2_{\text{ét}}(X_v,R_v) & \to & \operatorname{Ext}_{\mathcal{G}_v}(\widehat{R_v}, \operatorname{Pic}\overline{X}_v) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & H^2_{\text{ét}}(k_v,S_v) & \to & \widetilde{H}^2_{\text{ét}}(X_v,S_v) & \to & \operatorname{Ext}_{\mathcal{G}_v}(\widehat{S_v}, \operatorname{Pic}\overline{X}_v)
\end{array} \tag{1.21}$$

The zeros in the columns come from the splitting property in (1.16) and (1.19), and the zeros in the rows from the existence of universal torsors over X_v (see Proposition 1.7). Since γ goes to zero in $\mathrm{Ext}_{\mathcal{G}}(\widehat{S}, \mathrm{Pic}\,\overline{X})$, we conclude from (1.21) that $\gamma_v \in \mathrm{Im}(H^2_{\text{ét}}(k_v, R_v) \to \widetilde{H}^2_{\text{ét}}(X_v, R_v))$ for each $v \in \Omega_k$. On considering the images of γ in $\widetilde{H}^2_{\text{ét}}(X, \mathbb{G}_m)$ under the maps from $\widetilde{H}^2_{\text{ét}}(X, R)$ induced by the r projections from $R = \prod_{j=1}^r \mathbb{G}_{m,k}$ to $\mathbb{G}_m$, we deduce from the first assumption in (i) that there exists $\alpha \in H^2_{\text{ét}}(k, R)$ that maps to γ_v in $H^2_{\text{ét}}(X_v, R_v)$ for each $v \in \Omega_k$. Let $\tilde{\alpha}$ be the image of α in $H^2_{\text{ét}}(k, S)$. By the choice of γ, we conclude that $\beta - \tilde{\alpha}$ goes to 0 in $\prod_{\text{all } v} \widetilde{H}^2_{\text{ét}}(X_v, S_v)$ and by the injectivity of the functorial maps $H^2_{\text{ét}}(k_v, S_v) \to \widetilde{H}^2_{\text{ét}}(X_v, S_v)$ that $\beta - \tilde{\alpha} \in Ш^2(k, S)$. But then the image κ of $\beta - \tilde{\alpha}$ in $H^2_{\text{ét}}(k, T)$ is equal to zero (see (1.18)). This completes the proof of (i) $\Rightarrow$ (ii).

Proof of (ii) $\Rightarrow$ (i) Let $\mathcal{T}$ be a universal torsor over X. Then $\mathcal{T}_v := k_v \times \mathcal{T}$ is a universal torsor over X_v for each $v \in \Omega_k$. To prove the first part of (i), consider the following commutative diagram with exact rows

$$\begin{array}{ccccccc} H^1_{\text{ét}}(k,T) & \to & H^1_{\text{ét}}(X,T) & \to & \mathrm{Hom}_{\mathcal{G}}(\widehat{T}, \mathrm{Pic}\,\overline{X}) & \to & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ H^2_{\text{ét}}(k,R) & \to & \widetilde{H}^2_{\text{ét}}(X,R) & \to & \mathrm{Ext}_{\mathcal{G}}(\widehat{R}, \mathrm{Pic}\,\overline{X}) & \to & 0 \end{array} \qquad (1.22)$$

Let $\gamma \in \widetilde{H}^2_{\text{ét}}(X, R)$ be the image of $[\mathcal{T}] \in H^1_{\text{ét}}(X, T)$ and $\gamma_1, \gamma_2, \ldots, \gamma_r$ the images of γ in $\widetilde{H}^2_{\text{ét}}(X, \mathbb{G}_m)$ under the maps from $\widetilde{H}^2_{\text{ét}}(X, R)$ induced by the r projections from $R = \prod_{j=1}^r \mathbb{G}_{m,k}$ to $\mathbb{G}_m$. Then $\gamma_1, \gamma_2, \ldots, \gamma_r$ have images $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r$ in $\mathrm{Ext}_{\mathcal{G}}(\widehat{R}, \mathrm{Pic}\,\overline{X})$. Thus by the choice of ε_j (see (1.6)) we get that the kernel of the map

$$\widetilde{H}^2_{\text{ét}}(X, \mathbb{G}_m)/\mathrm{Im}\, H^2_{\text{ét}}(k, \mathbb{G}_m) \to \prod_{\text{all } v} \widetilde{H}^2_{\text{ét}}(X_v, \mathbb{G}_m)/\mathrm{Im}\, H^2_{\text{ét}}(k_v, \mathbb{G}_m)$$

is generated by the images of $\gamma_1, \gamma_2, \ldots, \gamma_r$ in $\widetilde{H}^2_{\text{ét}}(X, \mathbb{G}_m)/\mathrm{Im}\, H^2_{\text{ét}}(k, \mathbb{G}_m)$. To verify the first condition in (i), it thus suffices to show that the elements $\gamma_1, \gamma_2, \ldots, \gamma_r$ belong to $\mathrm{Ker}(\widetilde{H}^2_{\text{ét}}(X, \mathbb{G}_m) \to \prod_{\text{all } v} \widetilde{H}^2_{\text{ét}}(X_v, \mathbb{G}_m))$. That is, we must prove that $[\mathcal{T}]$ belongs to the kernel of the composite map:

$$H^1_{\text{ét}}(X, T) \longrightarrow \widetilde{H}^2_{\text{ét}}(X, R) \longrightarrow \prod_{\text{all } v} \widetilde{H}^2_{\text{ét}}(X_v, R_v).$$

But $[\mathcal{T}_v] \in H^1_{\text{ét}}(X_v, T_v)$ maps to zero in $\widetilde{H}^2_{\text{ét}}(X_v, R_v)$ since the sequence $1 \to R_v \to S_v \to T_v \to 1$ splits. This completes the proof of Proposition 1.12. $\square$

Now suppose that we are given a 0-cycle z_v on X_v for each place $v \in \Omega_k$. If S is a k-torus, let S_v be the k_v-torus obtained by base extension and let

$$\rho_v \colon Z_0(X_v) \times H^i_{\text{ét}}(X_v, S_v) \longrightarrow H^i_{\text{ét}}(k_v, S_v) \tag{1.23}$$

be the pairing described in (1.8). We denote this map by ρ_v for all k-tori S and all positive integers i. Let $\rho_v(z_v)\colon H^i_{\text{ét}}(X_v, S_v) \to H^i_{\text{ét}}(k_v, S_v)$ be the homomorphism sending $\varepsilon_v \in H^i_{\text{ét}}(X_v, S_v)$ to $\rho_v(z_v, \varepsilon_v) \in H^i_{\text{ét}}(k_v, S_v)$.

Now recall the fundamental exact sequence of Hasse (see, for example, Tate [Ta2])

$$0 \to H^2_{\text{ét}}(k, \mathbb{G}_m) \to \bigoplus_{\text{all } v} H^2_{\text{ét}}(k_v, \mathbb{G}_m) \to \mathbb{Q}/\mathbb{Z} \to 0. \tag{1.24}$$

The map from $H^2_{\text{ét}}(k, \mathbb{G}_m)$ is the direct sum over $v \in \Omega_k$ of the functorial maps $H^2_{\text{ét}}(k, \mathbb{G}_m) \to H^2_{\text{ét}}(k_v, \mathbb{G}_m)$. The map to $\mathbb{Q}/\mathbb{Z}$ is the direct sum of the local maps $\text{inv}_v\colon H^2_{\text{ét}}(k_v, \mathbb{G}_m) \to \mathbb{Q}/\mathbb{Z}$ which are isomorphisms for non-archimedean places. The fact that the sum of all local invariants is 0 for an element of the Brauer group $H^2_{\text{ét}}(k, \mathbb{G}_m)$ of k is called the reciprocity law.

Manin [Ma] noticed that the reciprocity law gives rise to the following necessary condition for the existence of a 0-cycle of degree r on X.

There exists a set of 0-cycles z_v of degree r on X_v indexed by $v \in \Omega_k$ s.t. $\sum_{\text{all } v} \text{inv}_v(\rho_v(z_v))(\mathcal{A}_v) = 0$ for all $\mathcal{A} \in H^2_{\text{ét}}(X, \mathbb{G}_m)$. (1.25)

We now relate the Brauer group obstruction to the Hasse principle for 0-cycles of degree one to another obstruction based on universal torsors. The following result is an immediate corollary of Proposition 1.12.

1.26 Proposition *Let k be a number field and X a smooth proper geometrically connected k-variety for which $\text{Pic}\,\overline{X}$ is finitely generated and torsion-free. Suppose given a 0-cycle of degree one z_v on X_v for each place $v \in \Omega_k$. Then the following statements are equivalent.*

(i) Manin's reciprocity condition $\sum_{\text{all } v} \text{inv}_v(\rho_v(z_v))(\mathcal{A}_v) = 0$ holds for all $\mathcal{A} \in \text{Ker}\big(\widetilde{H}^2_{\text{ét}}(X, \mathbb{G}_m) \to \prod_{\text{all } v} \widetilde{H}^2_{\text{ét}}(X_v, \mathbb{G}_m)/\text{Im}\, H^2_{\text{ét}}(k_v, \mathbb{G}_m)\big)$.

(ii) There exists a universal torsor over X.

Proof Given 0-cycles z_v of degree one on X_v for each place $v \in \Omega_k$, we have to prove that the conditions (1.12i) and (1.26i) are equivalent. It was already noticed after (1.8) that the existence of a 0-cycle of degree one on X_v implies the existence of a universal torsor over X_v. It thus suffices to show that the subgroup of

$$\operatorname{Ker}\Big(\widetilde{H}^2_{\text{ét}}(X,\mathbb{G}_m) \to \prod_{\text{all } v} \widetilde{H}^2_{\text{ét}}(X_v,\mathbb{G}_m)/\operatorname{Im} H^2_{\text{ét}}(k_v,\mathbb{G}_m)\Big)$$

generated by $\operatorname{Ker}\big(\widetilde{H}^2_{\text{ét}}(X,\mathbb{G}_m) \to \prod_{\text{all } v} \widetilde{H}^2_{\text{ét}}(X_v,\mathbb{G}_m)\big)$ and $\operatorname{Im}(H^2_{\text{ét}}(k,\mathbb{G}_m))$ equals the subgroup of elements $\mathcal{A}$ satisfying $\sum_{\text{all } v} \operatorname{inv}_v(\rho_v(z_v))(\mathcal{A}_v) = 0$. This is a formal consequence of the Hasse exact sequence of Brauer groups (1.24) and the fact that for all places v of k, the map $\rho_v(z_v)$ defines a retraction of $H^2_{\text{ét}}(k_v,\mathbb{G}_m) \to \widetilde{H}^2_{\text{ét}}(X_v,\mathbb{G}_m)$. □

We now consider Manin's obstruction to the Hasse principle for 0-cycles of degree one given by arbitrary elements in $\widetilde{H}^2_{\text{ét}}(X,\mathbb{G}_m)$ and relate it to the existence of universal torsors with certain properties. The following result was proved in [CT/S2, 3.5.1] in the case of rational points.

1.27 Theorem *Let k be a number field and X a smooth proper geometrically connected k-variety for which $\operatorname{Pic}\overline{X}$ is finitely generated and torsion-free. Suppose given a 0-cycle z_v of degree one on X_v for each place $v \in \Omega_k$. Then the following statements are equivalent.*

(i) Manin's reciprocity condition $\sum_{\text{all } v} \operatorname{inv}_v(\rho_v(z_v))(\mathcal{A}_v) = 0$ holds for all $\mathcal{A} \in \widetilde{H}^2_{\text{ét}}(X,\mathbb{G}_m)$.

(ii) There exists a universal torsor $\mathcal{T}$ over X such that $\rho_v(z_v)([\mathcal{T}_v]) = 0$ in $H^1_{\text{ét}}(k_v,T)$ for each $v \in \Omega_k$.

Proof We again apply Lemma 1.10 for the $\mathcal{G}$-module $L = \operatorname{Pic}\overline{X}$. Let $\varepsilon_1,\varepsilon_2,\dots,\varepsilon_r$ be generators of $\operatorname{Ext}_{\mathcal{G}}(\mathbb{Z},\operatorname{Pic}\overline{X})$, and let $\varepsilon \in \operatorname{Ext}_{\mathcal{G}}(\mathbb{Z}^{(r)},L)$ correspond to $\{\varepsilon_j\}_{j=1}^r \in \operatorname{Ext}_{\mathcal{G}}(\mathbb{Z},\operatorname{Pic}\overline{X})$. We then obtain an exact sequence of discrete $\mathcal{G}$-modules:

$$0 \to \operatorname{Pic}\overline{X} \to M \to \mathbb{Z}^{(r)} \to 0 \tag{1.28}$$

such that

$$H^1(\mathcal{G},M) = 0; \quad \text{and} \tag{1.29}$$

$$\begin{array}{l}\text{id} \in \operatorname{Hom}_{\mathcal{G}}(\operatorname{Pic}\overline{X},\operatorname{Pic}\overline{X}) \text{ maps to } \varepsilon \in \operatorname{Ext}_{\mathcal{G}}(\mathbb{Z}^{(r)},\operatorname{Pic}\overline{X}) \\ \text{under the connecting homomorphism induced by (1.28).}\end{array} \tag{1.30}$$

Now apply $D(\dots)$ to (1.28) and consider the dual sequence of k-tori.

$$1 \to R \to S \to T \to 1, \tag{1.31}$$

where T is the Néron–Severi torus of X and $R = \prod_{j=1}^{r} \mathbb{G}_{m,k}$. From (1.29) and the arithmetical duality result in Theorem 1.11 we obtain

$$Ч^1(k,S) = 0 \quad \text{and} \quad Ш^2(k,S) = 0. \tag{1.32}$$

Proof of (i) ⇒ (ii) Consider the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccc}
H^1_{\text{ét}}(k,S) & \to & H^1_{\text{ét}}(X,S) & \to & \mathrm{Hom}_{\mathcal{G}}(\widehat{S}, \mathrm{Pic}\,\overline{X}) \\
\downarrow & & \downarrow & & \downarrow \\
H^1_{\text{ét}}(k,T) & \to & H^1_{\text{ét}}(X,T) & \to & \mathrm{Hom}_{\mathcal{G}}(\widehat{T}, \mathrm{Pic}\,\overline{X}) \\
\downarrow & & \downarrow & & \downarrow \\
H^2_{\text{ét}}(k,R) & \to & \widetilde{H}^2_{\text{ét}}(X,R) & \to & \mathrm{Ext}_{\mathcal{G}}(\widehat{R}, \mathrm{Pic}\,\overline{X}) \\
\downarrow & & \downarrow & & \downarrow \\
H^2_{\text{ét}}(k,S) & \to & \widetilde{H}^2_{\text{ét}}(X,S) & \to & \mathrm{Ext}_{\mathcal{G}}(\widehat{S}, \mathrm{Pic}\,\overline{X})
\end{array} \tag{1.33}$$

deduced from (1.31) and the spectral sequence in (1.3). We know from Proposition 1.26 that there exists a universal torsor over X. Let $[\mathcal{T}] \in H^1_{\text{ét}}(X,T)$ be the class of one such torsor $\mathcal{T}$ and consider the images γ in $\widetilde{H}^2_{\text{ét}}(X,R)$ and $\gamma_v \in \widetilde{H}^2_{\text{ét}}(X_v,R_v)$, $v \in \Omega_k$, of $[\mathcal{T}]$. Then, since $R = \prod_{j=1}^{r} \mathbb{G}_{m,k}$, we deduce from Manin's reciprocity condition (i) and the Hasse exact sequence (1.24) that there exists $\beta \in H^2_{\text{ét}}(k,R)$ that maps to $\rho_v(z_v)(\gamma_v)$ in $H^2_{\text{ét}}(k_v,R_v)$ for each $v \in \Omega_k$. But $\rho_v(z_v)(\gamma_v) \in \mathrm{Ker}(H^2_{\text{ét}}(k_v,R_v) \to H^2_{\text{ét}}(k_v,S_v))$ since it is the image of $\rho_v(z_v)([\mathcal{T}_v]) \in H^1_{\text{ét}}(k_v,T_v)$ in $H^2_{\text{ét}}(k_v,R_v)$. Therefore, $\beta \in \mathrm{Ker}(H^2_{\text{ét}}(k,R) \to H^2_{\text{ét}}(k,S))$ since $Ш^2(k,S) = 0$ (cf. (1.32)). Let $\alpha \in H^1_{\text{ét}}(k,T)$ be a lifting of β and α_v the image of α in $H^1_{\text{ét}}(k_v,T_v)$. Then $\rho_v(z_v)([\mathcal{T}_v]) - \alpha_v$ vanishes for all but finitely many $v \in \Omega_k$ and maps to 0 in $H^2_{\text{ét}}(k_v,R_v)$ for all $v \in \Omega_k$. This combined with the fact that $Ч^1(k,S) = 0$ implies that there exists $\sigma \in H^1_{\text{ét}}(k,S)$ whose image σ in $H^1_{\text{ét}}(k_v,T_v)$ is $\rho_v(z_v)([\mathcal{T}_v]) - \alpha_v$ for each $v \in \Omega_k$. Let $\tilde{\sigma}$ be the image of σ in $H^1_{\text{ét}}(X,T)$ and $\tilde{\alpha}$ the image of α in $H^1_{\text{ét}}(X,T)$. Then, since $\tilde{\alpha} + \tilde{\sigma}$ belongs to the image of $H^1_{\text{ét}}(k,T) \to H^1_{\text{ét}}(X,T)$ it follows that $[\widetilde{\mathcal{T}}] := [\mathcal{T}] + \tilde{\alpha} + \tilde{\sigma}$ is the class of a torsor $\widetilde{\mathcal{T}}$ of the same type as $\mathcal{T}$. Further, $\rho_v(z_v)([\widetilde{\mathcal{T}}_v]) = 0$ for all $v \in \Omega_k$. This completes the proof of (i) ⇒ (ii).

Proof of (ii) ⇒ (i) Let $\mathcal{T}$ be a universal torsor over X with the property that $\rho_v(z_v)([\mathcal{T}_v]) = 0$ in $H^1_{\text{ét}}(k_v,T)$ for all $v \in \Omega_k$. We now proceed as in the proof of Proposition 1.12, (ii) ⇒ (i) and consider the image γ of $[\mathcal{T}] \in H^1_{\text{ét}}(X,T)$ in $\widetilde{H}^2_{\text{ét}}(X,R)$ under the vertical map in (1.33), and the images $\gamma_1, \gamma_2, \dots, \gamma_r$ of γ in $\widetilde{H}^2_{\text{ét}}(X,\mathbb{G}_m)$ under the maps from $\widetilde{H}^2_{\text{ét}}(X,R)$ induced by the r projections from $R = \prod_{j=1}^{r} \mathbb{G}_{m,k}$ to $\mathbb{G}_m$. Then $\rho_v(z_v)(\gamma_j) = 0$ for all

$j = 1, \dots, r$ and all places v of k. This together with the reciprocity law (1.24) implies that $(z_v)_{v\in\Omega_k}$ satisfies Manin's condition for any $\mathcal{A} \in \widetilde{H}^2_{\text{ét}}(X, \mathbb{G}_m)$ in the subgroup Γ generated by $\gamma_1, \gamma_2, \dots, \gamma_r$ and the image of $H^2_{\text{ét}}(k, \mathbb{G}_m)$. But the images $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_r$ in $\operatorname{Ext}_{\mathcal{G}}(\mathbb{Z}, \operatorname{Pic}\overline{X})$ of $\gamma_1, \gamma_2, \dots, \gamma_r$ were chosen to generate $\operatorname{Ext}_{\mathcal{G}}(\mathbb{Z}, \operatorname{Pic}\overline{X})$. Thus, $\Gamma = \widetilde{H}^2_{\text{ét}}(X, \mathbb{G}_m)$, as was to be proved.

1.34 Corollary *Let k be a number field and X a smooth proper geometrically connected k-variety for which $\operatorname{Pic}\overline{X}$ is finitely generated and torsion-free. Suppose given a 0-cycle z_v of degree one on X_v for each place v such that Manin's reciprocity condition $\sum_{\text{all } v} \operatorname{inv}_v(\rho_v(z_v))(\mathcal{A}_v) = 0$ holds for all $\mathcal{A} \in \widetilde{H}^2_{\text{ét}}(X, \mathbb{G}_m)$. Then for each k-torus S and each element τ in $\operatorname{Hom}_{\mathcal{G}}(\widehat{S}, \operatorname{Pic}\overline{X})$ there exists an X-torsor $\mathcal{S}$ under S of type τ such that $\rho_v(z_v)([\mathcal{S}_v]) = 0$ in $H^1_{\text{ét}}(k_v, S)$ for all $v \in \Omega_k$.*

Proof We know from (1.24) that there exists a universal torsor $\mathcal{T}$ over X such that $\rho_v(z_v)([\mathcal{T}_v]) = 0$ in $H^1_{\text{ét}}(k_v, T_v)$ for each $v \in \Omega_k$. Let $\mathcal{S} := \mathcal{T} \times^T S$ be the torsor under S induced from $\mathcal{T}$ by the k-homomorpism $D(\tau)\colon T \to S$ dual to τ. Then $\mathcal{S}$ satisfies the above conditions.

1.35 Theorem *Let k be a number field and X a smooth proper geometrically connected k-variety for which $\operatorname{Pic}\overline{X}$ is finitely generated and torsion-free. Let T be the Néron–Severi torus of X and r an integer. Let z_v be a 0-cycle of degree r on X_v for each place $v \in \Omega_k$ such that Manin's reciprocity condition $\sum_{\text{all } v} \operatorname{inv}_v(\rho_v(z_v))(\mathcal{A}_v) = 0$ holds for all $\mathcal{A} \in \widetilde{H}^2_{\text{ét}}(X, \mathbb{G}_m)$. Then for each X-torsor under T there exists another X-torsor $\mathcal{T}$ of the same type such that $\rho_v(z_v)([\mathcal{T}_v]) = 0$ for all $v \in \Omega_k$.*

Proof An examination of the proof of (i) $\Rightarrow$ (ii) in (1.24) reveals that we only used the hypothesis that $r = 1$ to prove that there exists a universal torsor $\mathcal{T}$. The rest of the arguments is valid for any r and any T-torsor $\mathcal{T}$. □

We now make use of the ideas of [CT/S2, 2.3]. Let k be a perfect field and let X be as in (1.1). Let U be an open k-subvariety of X with $\operatorname{Pic}\overline{U} = 0$. If $\mathcal{S}$ is an X-torsor, let $\mathcal{S}_U$ be the U-torsor obtained by restriction.

Consider the exact sequence of $\mathcal{G}$-modules for the absolute Galois group $\mathcal{G} := \operatorname{Gal}(\overline{k}/k)$.

$$0 \to \overline{k}[U]^*/\overline{k}^* \to \operatorname{Div}_{\overline{Z}}\overline{X} \to \operatorname{Pic}\overline{X} \to 0, \qquad (1.36)$$

where Z is the complement of U in X, and $\operatorname{Div}_{\overline{Z}}\overline{X}$ the group of Weil divisors on $\overline{X}$ with support in $\overline{Z}$. On applying $D(\dots)$ we obtain a dual exact sequence of k-tori

$$1 \to T \to N \to V \to 1. \tag{1.37}$$

The spectral sequence (1.3) and the exact sequence (1.37) give rise to the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{G}}(\widehat{V}, \overline{k}[U]^*) & \xrightarrow{\delta} & \mathrm{Ext}_{\mathcal{G}}(\widehat{T}, \overline{k}[U]^*) \\ \downarrow \simeq & & \downarrow \simeq \\ H^0_{\text{ét}}(U, V) & \xrightarrow{\delta} & H^1_{\text{ét}}(U, T) \end{array} \tag{1.38}$$

The second vertical map is onto since $\mathrm{Pic}\,\overline{U} = 0$ (see [CT/S2, 1.5.1]).

1.39 Proposition *Let $\varepsilon \in H^1_{\text{ét}}(U,T)$. Then the following two conditions are equivalent.*

(i) There exists a universal X-torsor $\mathcal{T}$ such that $[\mathcal{T}_U] = \varepsilon$.

(ii) There is a section $\sigma \in \mathrm{Hom}_{\mathcal{G}}(\widehat{V}, \overline{k}[U]^*)$ of the obvious map $\psi\colon \overline{k}[U]^* \to \overline{k}[U]^*/\overline{k}^*$ that maps to ε in $H^1_{\text{ét}}(U,T)$.

Proof See the "description locale des torseurs" in Section 2.3 of [CT/S2].

Now assume that (1.39ii) holds. Then for any k-torus S there is a commutative diagram

$$\begin{array}{ccccc} \mathrm{Ext}_{\mathcal{G}}(\widehat{S}, \overline{k}^*) & \to & \mathrm{Ext}_{\mathcal{G}}(\widehat{S}, \overline{k}[U]^*) & \to & \mathrm{Ext}_{\mathcal{G}}(\widehat{S}, \overline{k}^*) \\ \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\ H^1_{\text{ét}}(k, S) & \to & H^1_{\text{ét}}(U, S) & \to & H^1_{\text{ét}}(k, S) \end{array} \tag{1.40}$$

defined in the following way. The vertical isomorphisms come from the spectral sequence (1.3) (see [CT/S2, 1.5.1]). The horizontal maps in the first square are the functorial maps and the horizontal map in the second square is induced by the $\mathcal{G}$-retraction $\sigma\psi/\mathrm{id}\colon \overline{k}[U]^* \to \overline{k}^*$ of the inclusion $\overline{k}^* \subset \overline{k}[U]^*$.

By completing the second square we obtain a homomorphism:

$$r_U\colon H^1_{\text{ét}}(U,S) \to H^1_{\text{ét}}(k,S) \tag{1.41}$$

which is a retraction of the functorial map from $H^1_{\text{ét}}(k,S)$ to $H^1_{\text{ét}}(U,S)$.

Let $r\colon H^1_{\text{ét}}(X,S) \to H^1_{\text{ét}}(k,S)$ be the composite of the restriction map from $H^1_{\text{ét}}(X,S)$ to $H^1_{\text{ét}}(U,S)$ and r_U.

1.42 Proposition *Let $\mathcal{T}$ be a universal X-torsor and $\sigma \in \mathrm{Hom}_{\mathcal{G}}(\widehat{V}, \overline{k}[U]^*)$ a section of $\psi\colon \overline{k}[U]^* \to \widehat{V}$ such that σ maps to the class $[\mathcal{T}_U]$ of $\mathcal{T}_U$ in $H^1_{\text{ét}}(U,T)$ under the map in (1.38). Then the following hold:*

(a) r is a retraction of the functorial map from $H^1_{\text{ét}}(k,S)$ to $H^1_{\text{ét}}(X,S)$;

(b) r is functorial under homomorphisms of k-tori;

(c) $r([\mathcal{T}]) = 0$;

(d) r depends only on $[\mathcal{T}]$ and not on the choice of σ.

Proof (c) To do this, we use the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{G}}(\widehat{V}, \overline{k}[U]^*) & \xrightarrow{\delta} & \mathrm{Ext}_{\mathcal{G}}(\widehat{T}, \overline{k}[U]^*) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\mathcal{G}}(\widehat{V}, \overline{k}[U]^*/\overline{k}^*) & \xrightarrow{\delta} & \mathrm{Ext}_{\mathcal{G}}(\widehat{T}, \overline{k}[U]^*/\overline{k}^*) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\mathcal{G}}(\widehat{V}, \overline{k}[U]^*) & \xrightarrow{\delta} & \mathrm{Ext}_{\mathcal{G}}(\widehat{T}, \overline{k}[U]^*)
\end{array} \qquad (1.43)$$

where the horizontal maps are induced by (1.37) and the vertical maps by ψ and σ. Then $\delta(\sigma)$ corresponds to $[\mathcal{T}_U]$ under the isomorphism between $\mathrm{Ext}_{\mathcal{G}}(\widehat{T}, \overline{k}[U]^*)$ and $H^1_{\text{ét}}(U,T)$. Therefore, $r_U([\mathcal{T}_U]) = 0$ if and only if $\delta(\sigma)$ maps to itself under the endomorphism of $\mathrm{Ext}_{\mathcal{G}}(\widehat{T}, \overline{k}[U]^*)$ induced by $\sigma\psi$. But this is clear from the commutative diagram (1.43).

(d) Let $\mathcal{S}$ be an X-torsor under S of type $\chi([\mathcal{S}]) \in \mathrm{Hom}_{\mathcal{G}}(\widehat{S}, \mathrm{Pic}\,\overline{X})$. Then $\mathcal{S}$ is of the same type as the X-torsor $\mathcal{T} \times^T S$ obtained from the k-homomorpism $D(\tau)\colon T \to S$ dual to $\tau = \chi([\mathcal{S}])$. Therefore, $[\mathcal{S}] - [\mathcal{T} \times^T S] \in H^1_{\text{ét}}(X,S))$ is the image of a unique element α in $H^1_{\text{ét}}(k,S)$ by (1.4). Also, $r([\mathcal{T} \times^T S]) = 0$ by (b) and (c). Hence, $r([\mathcal{S}]) = \alpha$ by (a), thereby completing the proof. □

Now suppose there is a 0-cycle z of degree one on X. Then (cf. (1.8)) there is a natural retraction $\rho(z)\colon H^1_{\text{ét}}(X,S) \to H^1_{\text{ét}}(k,S)$ associated to z for each k-torus S which is functorial under homomorphisms of k-tori.

1.44 Proposition *Let k, X be as above and suppose that there exists a universal X-torsor $\mathcal{T}$ such that $\rho(z)([\mathcal{T}]) = 0$. Let S be a k-torus and r the retraction from $H^1_{\text{ét}}(X,S)$ to $H^1_{\text{ét}}(k,S)$ defined by $[\mathcal{T}]$ (see (1.42d)). Then the two maps $\rho(z)$ and r coincide.*

Proof The map $\rho(z)$ satisfies the same axioms (1.42a–c) as r. It therefore follows from the proof of (1.42d) that the two maps coincide. □

One can give another proof of Proposition 1.44 based on the $\mathcal{G}$-retraction from $\overline{k}[U]^*$ to $\overline{k}^*$ associated to z.

The following result will be used in the next section in the case $S = T$.

1.45 Corollary *Let k be a number field and X a smooth proper geometrically connected k-variety for which $\operatorname{Pic}\overline{X}$ is finitely generated and torsion-free. Let S be a k-torus. Suppose that for each v we are given a 0-cycle z_v of degree one on $X_v := X \times_k k_v$ and an X_v-torsor $\mathcal{S}_v$ under S_v such that the following hold.*

(i) Manin's reciprocity condition $\sum_{\text{all } v} \operatorname{inv}_v(\rho_v(z_v))(\mathcal{A}_v) = 0$ holds for all $\mathcal{A} \in \widetilde{H}^2_{\text{ét}}(X, \mathbb{G}_m)$.

(ii) There exists an element η of $H^1_{\text{ét}}(k(X), S \times_k k(X))$ having the same image as $[\mathcal{S}_v]$ in $H^1_{\text{ét}}(k_v(X), S_v \times_{k_v} k_v(X))$ for each $v \in \Omega_k$.

Then there exists an element $\alpha \in H^1_{\text{ét}}(k, S)$ with image equal to $\rho_v(z_v)([\mathcal{S}_v])$ in $H^1_{\text{ét}}(k_v, S_v)$ for every $v \in \Omega_k$.

Proof Let U be an open nonempty subset of X and $v \in \Omega_k$ any place of k. We first show that there exists a 0-cycle u_v of degree one on $U_v := U \times_k k_v$ with $\rho_v(u_v)(\mathcal{A}_v) = \rho_v(z_v)(\mathcal{A}_v)$ for all $\mathcal{A}_v \in H^2_{\text{ét}}(X_v, \mathbb{G}_m)$ and such that $\rho_v(u_v)([\mathcal{S}_v]) = \rho_v(z_v)([\mathcal{S}_v])$. By the additivity and functoriality of ρ_v under corestrictions it suffices to do this in the case where z_v is a k_v-point P_v.

Let O_v be an affine open neighbourhood of P_v. We may then represent each element in $H^2_{\text{ét}}(X_v, \mathbb{G}_m)$ by an Azumaya algebra over O_v (see [Mi, p. 149]) and consider the corresponding Severi–Brauer scheme over O_v (cf. *op. cit.*). We shall only consider elements in the finite kernel of the specialization map from $\widetilde{H}^2_{\text{ét}}(X_v, \mathbb{G}_m)$ to $H^2_{\text{ét}}(k_v(P_v), \mathbb{G}_m)$. Let Π_v be the fibre product over O_v of the Severi–Brauer schemes corresponding to restrictions of these elements in $\widetilde{H}^2_{\text{ét}}(X_v, \mathbb{G}_m)$. Then Π_v is a smooth proper O_v-scheme and its fibre over P_v is a multiprojective space over k_v.

Let W_v be the restriction over O_v of an X_v-torsor of the same type as $\mathcal{S}_v$ which is trivial over P_v. It then follows from the v-adic implicit function theorem applied to the fibre product of Π_v and W_v over O_v that there exists a k_v-point on $U_v \cap O_v$ that can be lifted to k_v-points on Π_v and W_v. This k_v-point has all the desired properties. We may therefore replace z_v by a 0-cycle on U_v for each v without changing the hypothesis in Corollary 1.45.

Now choose an open subset U of X such that $\operatorname{Pic}\overline{U} = 0$ and such that η is the restriction of an element $\varepsilon \in H^1_{\text{ét}}(U, S)$. Assume, as we may, that z_v is a 0-cycle on U_v for each $v \in \Omega_k$.

Now apply Theorem 1.27. Then there exists a universal torsor $\mathcal{T}$ over X such that $\rho_v(z_v)([\mathcal{T}_v]) = 0$ in $H^1_{\text{ét}}(k_v, T)$ for all $v \in \Omega_k$. Also, let σ be a $\mathcal{G}$-module homomorphism from $\overline{k}[U]^*/\overline{k}^*$ to $\overline{k}[U]^*$ as in Proposition 1.39. Finally, let r_U be the retraction from $H^1_{\text{ét}}(U, S)$ to $H^1_{\text{ét}}(k, S)$ in (1.41) defined by means of σ.

Then $\alpha = r_U(\varepsilon) \in H^1_{\text{ét}}(k, S)$ is the desired element with image $\rho_v(z_v)([\mathcal{S}_v])$ in $H^1_{\text{ét}}(k_v, S_v)$ for all $v \in \Omega_k$. To show this, we fix one place v and change the notation so that $k = k_v$. We also omit the index v for all varieties, morphisms, cohomology groups defined over $k = k_v$. Thus U, resp. $\rho(z)([\mathcal{S}])$, will mean U_v, resp. $\rho_v(z_v)([\mathcal{S}_v])$, and ε, $\mathcal{T}$ will now mean the images after base extension to k_v. We shall also make use of the functoriality of r and $\rho(z)$ under extensions of the base field without further comments.

Then we get an element $\varepsilon \in H^1_{\text{ét}}(U, S)$, a 0-cycle z of degree one on U, a universal X-torsor $\mathcal{T}$ with $\rho(z)([\mathcal{T}]) = 0$ and an X-torsor $\mathcal{S}$ under S satisfying the following condition:

> The image of $\varepsilon \in H^1_{\text{ét}}(U, S)$ in $H^1_{\text{ét}}(k(X), S)$ equals that of the class $[\mathcal{S}] \in H^1_{\text{ét}}(X, S)$ in $H^1_{\text{ét}}(k(X), S)$. (∗)

But it follows from the commutative diagram (cf. (1.40))

$$\begin{array}{ccc} \mathrm{Ext}_{\mathcal{G}}(\widehat{S}, \overline{k}[U]^*) & \rightarrow & \mathrm{Ext}_{\mathcal{G}}(\widehat{S}, \overline{k}(X)^*) \\ \downarrow\simeq & & \downarrow\simeq \\ H^1_{\text{ét}}(U, S) & \rightarrow & H^1_{\text{ét}}(k(X), S) \end{array}$$

that the restriction map from $H^1_{\text{ét}}(U, S)$ to $H^1_{\text{ét}}(k(U), S)$ is injective. Therefore, $\varepsilon = [\mathcal{S}_U]$, and hence $r_U(\varepsilon) = r([\mathcal{S}])$. Moreover, $r([\mathcal{S}]) = \rho(z)([\mathcal{S}])$ by Proposition 1.44. Hence $r_U(\varepsilon) = \rho(z)([\mathcal{S}])$, as was to be proved.

In Corollary 1.45 and some other results in this section we have assumed that the functorial maps from $\mathrm{Pic}\,\overline{X}$ to $\mathrm{Pic}(\overline{k}_v \times X)$ are isomorphisms for all $v \in \Omega_k$. This was used to guarantee that the base extensions of universal X-torsors to torsors over X_v remain universal. We therefore include the following result for which we could find no reference.

1.46 Proposition *Let k be an algebraically closed field, and let X be a smooth and proper k-variety for which* $\mathrm{Pic}\,X$ *is finitely generated. Then the functorial map from* $\mathrm{Pic}\,X$ *to* $\mathrm{Pic}(X \times E)$ *is an isomorphism for any extension field E of k.*

Proof The assumption implies that $H^1(X, \mathcal{O}_X) = 0$. Thus $\mathrm{Pic}(X \times V) = \mathrm{Pic}\,X \times \mathrm{Pic}\,V$ for any (integral) k-variety V by the exercise on p. 292 in [Ha]. (The assumption that X is projective is not necessary since Grothendieck's theorem on pp. 290–291 in *op. cit.* also holds for proper morphisms.) Now make use of the fact that E is the union of its finitely generated k-subalgebras A. Therefore, there are canonical isomorphisms

$$\varinjlim \operatorname{Pic}(\operatorname{Spec} A) = \operatorname{Pic}(E) = 0 \quad \text{and}$$
$$\operatorname{Pic}(X \times E) = \varinjlim \operatorname{Pic}(X \times \operatorname{Spec} A) = \operatorname{Pic} X \oplus \varinjlim \operatorname{Pic}(\operatorname{Spec} A) = \operatorname{Pic} X,$$

as was to be proved.

2 K theory and obstructions to the Hasse principle

Let k be a perfect field, $\overline{k}$ an algebraic closure of k and $\mathcal{G} := \operatorname{Gal}(\overline{k}/k)$ the absolute Galois group of k. Let X be a smooth proper k-variety such that $\overline{X} := \overline{k} \times X$ is connected.

Then there is a complex of discrete $\mathcal{G}$-modules (cf. [Bl])

$$\bigoplus_{\sigma \in \overline{X}_2} K_2(\overline{k}(\sigma)) \xrightarrow{\text{tame}} \bigoplus_{\gamma \in \overline{X}_1} \overline{k}(\gamma)^* \xrightarrow{\text{div}} \bigoplus_{\overline{X}_0} \mathbb{Z}, \tag{2.1}$$

where $\overline{X}_i$ denotes the set of points of dimension i. The first map is given by tame symbols and the second is the usual divisor map. Let M be the cokernel of the first map and $\bigoplus^0_{\overline{X}_0} \mathbb{Z}$ the image of the second. (This notation will become natural later after (2.4).) Then (2.1) induces a short exact sequence of discrete $\mathcal{G}$-modules

$$0 \to \operatorname{Ker}(\text{div})/\operatorname{Im}(\text{tame}) \to M \to {\textstyle\bigoplus^0_{\overline{X}_0}} \mathbb{Z} \to 0. \tag{2.2}$$

Let $Z_i(\overline{X})$ be the free abelian group of cycles of dimension i on $\overline{X}$; write $R_i(\overline{X})$ for the subgroup of i-cycles rationally equivalent to zero and $\operatorname{Ch}_i(\overline{X}) := Z_i(\overline{X})/R_i(\overline{X})$ for the Chow group of cycles of dimension i on $\overline{X}$. The degree of a 0-cycle on X depends only on its rational equivalence class since $\overline{X}$ is proper. Let $A_0(\overline{X})$ be the subgroup of $\operatorname{Ch}_0(\overline{X})$ of 0-cycles of degree 0. Finally, define the map

$$\pi\colon\ \operatorname{Ch}_1(\overline{X}) \otimes_{\mathbb{Z}} \overline{k}^* \longrightarrow \operatorname{Ker}(\text{div})/\operatorname{Im}(\text{tame})$$

by the inclusions:

$$Z_1(\overline{X}) \otimes_{\mathbb{Z}} \overline{k}^* = \bigoplus_{\overline{X}_1} \overline{k}^* \subset \operatorname{Ker}(\text{div}) \quad \text{and} \quad R_1(\overline{X}) \otimes_{\mathbb{Z}} \overline{k}^* \subset \operatorname{Im}(\text{tame}).$$

Now let k, $\overline{k}$, $\mathcal{G}$, X, $\overline{X}$ be as above and assume in addition that the following holds.

2.3 Assumptions

(i) $\mathrm{Ch}_1(\overline{X})$ and $\mathrm{Pic}(\overline{X}) = \mathrm{Ch}_{n-1}(\overline{X})$ are finitely generated and torsion-free.

(ii) The intersection pairing $\cup\colon \mathrm{Ch}_1(\overline{X}) \times \mathrm{Ch}_{n-1}(\overline{X}) \to \mathbb{Z}$ is perfect.

(iii) $\pi\colon \mathrm{Ch}_1(\overline{X}) \otimes_{\mathbb{Z}} \overline{k}^* \to \mathrm{Ker}(\mathrm{div})/g\,\mathrm{Im}(\mathrm{tame})$ is an isomorphism.

Then $\cup$ and π define an isomorphism between the Néron–Severi torus $T = D(\mathrm{Pic}\,\overline{X})$ and $\mathrm{Ker}(\mathrm{div})/\,\mathrm{Im}(\mathrm{tame})$. Suppose further that

$$A_0(\overline{X}) = 0. \qquad (2.4)$$

Then the Galois cohomology of (2.2) gives rise to an exact sequence

$$Z_0(X)^0 \to H^1(\mathcal{G}, T(\overline{k})) \to H^1(\mathcal{G}, M) \to \mathbb{Z}/\deg(Z_0(X)) \to H^2(\mathcal{G}, T(\overline{k})), \qquad (2.5)$$

where $Z_0(X)$ is the group of 0-cycles of degree 0. Denote by ϕ' the map from $Z_0(X)^0$ to $H^1_{\text{ét}}(k,T)$ obtained from (2.5) by identifying $H^1(\mathcal{G}, T(\overline{k}))$ with $H^1_{\text{ét}}(k,T)$.

2.6 Example Let k be a perfect field and X a smooth proper rational geometrically connected k-surface. Then Bloch [Bl] showed that (2.3) and (2.4) hold and from that deduced the map ϕ' described above. He also noticed that the values of ϕ' only depend on the rational equivalence class in $Z_0(X)$.

2.7 Proposition *Let k, $\overline{k}$, $\mathcal{G}$, X, $\overline{X}$ be as above and assume in addition that* (2.3) *and* (2.4) *hold. Suppose that there exists a universal torsor over X. Then the maps ϕ (see Proposition 1.9) and ϕ' coincide.*

Proof This is stated and proved in [CT/S1, Section 1] for rational surfaces, but the proof uses no other properties of rational surfaces than (2.3) and (2.4).

Now consider a discrete valuation ring A containing k; let K be its field of fractions and F its residue field, and suppose that these fields are perfect. For a closed point P on X_K, write $A(P)$ for the integral closure of A in $K(P)$. The valuative criterion of properness for $X_A \to \mathrm{Spec}\,A$ implies that there is a unique A-morphism $g\colon \mathrm{Spec}\,A(P) \to X_A$ extending $P \to X_K$. Let

$$\mathrm{sp}\colon Z_0(X_K) \to Z_0(X_F)$$

be the specialization homomorphism that sends a closed point P to the cycle associated to the 0-dimensional closed subscheme $\operatorname{Spec} A(P) \times_{\operatorname{Spec} A} F$ of X_F. Then extend sp to arbitrary 0-cycles by additivity.

It is easy to see that sp sends 0-cycles of degree zero to 0-cycles of degree zero. Denote by sp^0 the associated map from $Z_0(X_K)^0$ to $Z_0(X_F)^0$. Then the obvious diagram

$$\begin{array}{ccc} Z_0(X)^0 & \xrightarrow{\mathrm{id}} & Z_0(X)^0 \\ \downarrow & & \downarrow \\ Z_0(X_K)^0 & \xrightarrow{\mathrm{sp}^0} & Z_0(X_F)^0 \end{array} \tag{2.8}$$

commutes and sp and sp^0 have the expected functoriality properties under field extensions of k. It can be shown that sp induces a specialization map of Chow groups of 0-cycles, but we shall not need this.

2.9 Proposition *Suppose that there exists a universal torsor $\mathcal{T}$ over X, and let $\phi_{\mathcal{T}}$ be the map described in Proposition 1.9. Then the following holds.*

(a) The functorial map from $H^1_{\text{ét}}(\operatorname{Spec} A, T_A)$ to $H^1_{\text{ét}}(K, T_K)$ is injective.

(b) $\phi_{\mathcal{T}}(Z_0(X_K)) \subseteq \operatorname{Im}(H^1_{\text{ét}}(\operatorname{Spec} A, T_A) \to H^1_{\text{ét}}(K, T_K))$.

(c) The following diagram commutes

$$\begin{array}{ccc} Z_0(X_K) & \xrightarrow{\mathrm{sp}} & Z_0(X_F) \\ \downarrow \phi_{\mathcal{T}} & & \downarrow \phi_{\mathcal{T}} \\ \operatorname{Im}(H^1_{\text{ét}}(\operatorname{Spec} A, T_A) \to H^1_{\text{ét}}(K, T_K)) & \xrightarrow{\Theta} & H^1_{\text{ét}}(F, T_F) \end{array}$$

for the functorial map Θ from $H^1_{\text{ét}}(\operatorname{Spec} A, T_A)$ (cf. (a)).

Proof (a) See [CT/S3, Section 4].

(b) The argument is well known (see, for example, [CT/S1, p. 428]). The X_K-torsor $\mathcal{T}_K$ extends to an X_A-torsor $\mathcal{T}_A$ under T_A, and any closed point P on X_K can be extended to a morphism $\operatorname{Spec} A(P) \to X_A$ (see the construction of sp). Combined with the existence of corestriction maps from $H^1_{\text{ét}}(\operatorname{Spec} A(P), T_{A(P)})$ to $H^1_{\text{ét}}(\operatorname{Spec} A, T_A)$, this implies that $\phi(Z_0(X_K)) \subseteq H^1_{\text{ét}}(\operatorname{Spec} A, T_A)$.

(c) The horizontal maps factorize over the completion of K. We may thus assume that A is complete and hence that $A(P)$ is discrete for each closed point P. By using obvious functoriality properties under corestriction of the maps involved, one reduces to prove that $\Theta(\phi_{\mathcal{T}}(P)) = \phi_{\mathcal{T}}(\sigma(P))$ for a rational point P. To see this, note that both composites give the pullback of $\mathcal{T}_A$ at the closed point on X_F determined by $\operatorname{Spec} A(P) \to X_A$.

2.10 Lemma *Let k be a field of characteristic 0, and X, Y two smooth, proper, geometrically connected k-varieties. Suppose that (2.3) and (2.4) hold for $\overline{X} := X \times_k \overline{k}$ for any algebraically closed field $\overline{k}$ containing k, and that there exists a universal torsor over X. Then for any 0-cycle y on Y, the following holds:*

(a) The map $\rho(y)\colon H^1_{\text{ét}}(Y,T) \to H^1_{\text{ét}}(k,T)$ factorizes through a map $\rho'(y)$ from $\operatorname{Im}\big(H^1_{\text{ét}}(Y,T) \to H^1_{\text{ét}}(k(Y), T \times_k k(Y)\big)$ to $H^1_{\text{ét}}(k,T)$.

(b) $\phi'(Z_0(X \times_k k(Y))^0) \subseteq \operatorname{Im}\big(H^1_{\text{ét}}(Y,T) \to H^1_{\text{ét}}(k(Y), T \times_k k(Y))\big)$

(c) $\phi'(Z_0(X \times_k k(Y))^0)$ maps to $\phi'(Z_0(X)^0)$ under $\rho'(y)$.

Proof (a) See [CT/S2, 2.7.5].

(b) By [CT/S1, p. 428], it is known that

$$\operatorname{Im}\Big(H^1_{\text{ét}}(Y,T) \to H^1_{\text{ét}}(k(Y), T \times_k k(Y))\Big) = \bigcap_Q \operatorname{Im}\Big(H^1_{\text{ét}}(\mathcal{O}_{Y,Q}, T \times_k \mathcal{O}_{Y,Q}) \to H^1_{\text{ét}}(k(Y), T \times_k k(Y))\Big),$$

where Q runs over all points of codimension one on Y. The desired inclusion is therefore a consequence of (2.9b) and the fact that $\phi = \phi'$ (see Proposition 2.7).

(c) Let $y = \sum n_i y_i$, where the y_i are closed points on Y. Since ρ is additive with respect to $Z_0(Y)$, it suffices to prove the statement for each $\rho'(y_i)$. By factorizing $\rho(y_i)$ through $H^1_{\text{ét}}(Y \times_k k(y_i), T \times_k k(y_i))$ and using the functoriality of ϕ under extensions of the base field, we reduce further to the case when y is a rational point. We now use induction on $\dim Y$ and note that the case $\dim Y = 0$ is trivial. If $\dim Y \geq 1$, let $f\colon \tilde{Y} \to Y$ be the blowup at the k-rational point y, $Z = f^{-1}(y)$ and A the stalk of $\mathcal{O}_{\tilde{Y}}$ at the generic point of Z. Then A is a discrete valuation ring with field of fractions $K := k(\tilde{Y}) = k(Y)$ and residue field $F := k(Z)$. Then by (2.9c) and Proposition 2.7 there is a commutative diagram

$$\begin{array}{ccc} Z_0(X_K)^0 & \xrightarrow{\text{sp}} & Z_0(X_F)^0 \\ \downarrow \phi' & & \downarrow \phi' \\ \operatorname{Im}(H^1_{\text{ét}}(\operatorname{Spec} A, T_A) \to H^1_{\text{ét}}(K, T_K)) & \xrightarrow{\Theta} & H^1_{\text{ét}}(F, T_F) \end{array} \qquad (2.11)$$

Now choose a rational K-point z on the above Z. Then, since $\dim Z = \dim Y - 1$, we obtain from the induction assumption that (c) holds if we consider the pair (Z, z) instead of (Y, y). Further, by using the commutativity of (2.11), we deduce from this that (c) also holds for the pair $(\tilde{Y}, \tilde{y})$, which in turn implies that (c) holds for (Y, y) since $K(Y) = K(\tilde{Y})$ and $f(z) = y$. This finishes the proof. □

We shall in the sequel use the following functoriality properties of (2.1). Let $k \subset k_1$ be an extension of perfect fields with algebraic closures $\overline{k} \subset \overline{k}_1$. Put $\mathcal{G} = \mathrm{Gal}(\overline{k}/k)$, $\mathcal{G}_1 = \mathrm{Gal}(\overline{k}_1/k_1)$, $X_1 = X \times_k k_1$ and $\overline{X}_1 = X \times_k \overline{k}_1$. We may then consider (2.1) as a sequence of $\mathcal{G}_1$-modules through the homomorphism $\mathcal{G}_1 \to \mathcal{G}$ obtained by restricting the $\mathcal{G}_1$-action to $\overline{k}$. This sequence is the upper row in a commutative diagram of discrete $\mathcal{G}_1$-modules where the bottom row is given by (2.1) applied to $\overline{X}_1$. Now suppose that $\overline{X}$ and $\overline{X}_1$ satisfy (2.3) and (2.4). Then we obtain the following commutative diagram with exact rows from the functoriality of (2.5) under extension of the base field:

$$\begin{array}{ccccccccc} Z_0(X)^0 & \to & H^1_{\text{ét}}(k,T) & \to & H^1(\mathcal{G},M) & \to & \mathbb{Z}/\deg Z_0(X) & \to & H^2_{\text{ét}}(k,T) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Z_0(X_1)^0 & \to & H^1_{\text{ét}}(k_1,T_1) & \to & H^1(\mathcal{G}_1,M_1) & \to & \mathbb{Z}/\deg Z_0(X_1) & \to & H^2_{\text{ét}}(k_1,T_1) \end{array} \tag{2.12}$$

where $T_1 = T \times_k k_1$ and M_1 is the cokernel of the tame symbol map in (2.1) for $\overline{X}_1$. Note that T_1 can be identified with the Néron–Severi torus of $\overline{X}_1$ since the functorial map gives an isomorphism from $\mathrm{Pic}(\overline{X})$ to $\mathrm{Pic}(\overline{X}_1)$ by Proposition 1.46.

From now on, let k be a number field and choose algebraic closures $\overline{k}_v$ of k_v, and embeddings $\overline{k} \subset \overline{k}_v$ for each place v of k. Let $\mathcal{G}_v = \mathrm{Gal}(\overline{k}_v/k_v)$, $X_v = X \times_k k_v$, $\overline{X}_v = X \times_k \overline{k}_v$, and let M_v be the cokernel of the tame symbol map in (2.1) for $\overline{X}_v$. Write $Ш^1(k, M)$ for the kernel of the diagonal map from $H^1(\mathcal{G}, M)$ to $\prod_{\text{all } v} H^1(\mathcal{G}_v, M_v)$.

2.13 Theorem *Let k be a number field and X a smooth proper geometrically connected k-variety such that* (2.3) *and* (2.4) *hold for $\overline{X} := X \times_k \overline{k}$ for any algebraically closed field $\overline{k}$ containing k. Suppose that for each $v \in \Omega_k$ we are given a 0-cycle z_v of degree one on X_v and that Manin's reciprocity condition $\sum_{\text{all } v} \mathrm{inv}_v(\rho_v(z_v))(\mathcal{A}_v) = 0$ holds for all $\mathcal{A} \in \widetilde{H}^2_{\text{ét}}(X, \mathbb{G}_m)$.*

Then $Ш^1(k, M)$ maps onto $\mathbb{Z}/\deg(Z_0(X))$ under the map from $H^1(\mathcal{G}, M)$ in (2.5)*. In particular, if $Ш^1(k, M) = 0$, then there is a 0-cycle of degree one on X.*

Proof Let $\overline{k}$ be an algebraic closure of k, and $\overline{K}$ an algebraic closure of the function field $\overline{k}(\overline{X})$ of $\overline{X} := X \times_k \overline{k}$. Then $\overline{K}$ is also an algebraic closure of $K := k(X)$ and we have a natural homomorphism from $\mathcal{H} := \mathrm{Gal}(\overline{K}/K)$ to

$\mathcal{G} := \mathrm{Gal}(\overline{k}/k)$. Now consider (2.12) for $k_1 = K$. Then $\mathbb{Z}/\deg(Z_0(X_1)) = 0$ since the generic point of X defines a K-rational point on $X_1 = X_K$.

By Proposition 1.26, since Manin's reciprocity condition is satisfied, there exists a universal torsor over X. In turn, this implies that (cf. (2.2.5) and (2.2.8) in [CT/S2]) the map from $H^2_{\text{ét}}(k,T)$ to $H^2_{\text{ét}}(k_1,T_1)$ is injective. We thus obtain the following commutative diagram with exact rows from (2.12):

$$\begin{array}{ccccccccc} Z_0(X)^0 & \to & H^1_{\text{ét}}(k,T) & \to & H^1(\mathcal{G},M) & \to & \mathbb{Z}/\deg Z_0(X) & \to & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ Z_0(X_K)^0 & \to & H^1_{\text{ét}}(K,T_K) & \to & H^1(\mathcal{H},M_K) & \to & 0 & & \end{array} \tag{2.14}$$

where M_K is the cokernel of the tame symbol map (see (2.1)) for $X \times_k \overline{K}$. The assertion that $Ш^1(k,M)$ maps onto $\mathbb{Z}/\deg(Z_0(X))$ therefore reduces to the assertion that $H^1(\mathcal{G},M)$ is generated by $Ш^1(k,M)$ and the image of $H^1_{\text{ét}}(k,T)$.

For each place v of k there is a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} Z_0(X)^0 & \to & H^1_{\text{ét}}(k,T) & \to & H^1(\mathcal{G},M) & \to & \mathbb{Z}/\deg Z_0(X) & \to & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ Z_0(X_v)^0 & \to & H^1_{\text{ét}}(k_v,T_v) & \to & H^1(\mathcal{G}_v,M_v) & \to & 0 & & \end{array} \tag{2.15}$$

where the zero in the second row comes from the existence of a 0-cycle of degree one on X_v. Let $K_v := k_v(X_v)$ be the function field of X_v, and $\overline{K}_v$ an algebraic closure of the function field $\overline{k}_v(\overline{X}_v)$ of $\overline{X}_v$ containing $\overline{K}$. Then $\overline{K}_v$ is also an algebraic closure of K_v, and there are natural homomorphisms from $\mathcal{H}_v = \mathrm{Gal}(\overline{K}_v/K_v)$ to $\mathcal{G}_v$ and $\mathcal{H}$. Let M_{K_v} be the cokenel of the tame symbol map (see (2.1)) for $X \times_k \overline{K}_v$. Then there are commutative diagrams with exact rows

$$\begin{array}{ccccccc} Z_0(X_K)^0 & \to & H^1_{\text{ét}}(K,T_K) & \to & H^1(\mathcal{H},M_K) & \to & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ Z_0(X_{K_v})^0 & \to & H^1_{\text{ét}}(K_v,T_{K_v}) & \to & H^1(\mathcal{H}_v,M_{K_v}) & \to & 0 \end{array} \tag{2.16}$$

and

$$\begin{array}{ccccccc} Z_0(X_v)^0 & \to & H^1_{\text{ét}}(k_v,T_v) & \to & H^1(\mathcal{G}_v,M_v) & \to & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ Z_0(X_{K_v})^0 & \to & H^1_{\text{ét}}(K_v,T_{K_v}) & \to & H^1(\mathcal{H}_v,M_{K_v}) & \to & 0 \end{array} \tag{2.17}$$

and (2.14–17) are parts of a three-dimensional commutative diagam that also contains the commutative diagrams

$$\begin{array}{ccc} H^1_{\text{ét}}(k,T) & \to & H^1_{\text{ét}}(k_v,T_v) \\ \downarrow & & \downarrow \\ H^1_{\text{ét}}(K,T_K) & \to & H^1_{\text{ét}}(K_v,T_{K_v}) \end{array} \quad \text{and} \quad \begin{array}{ccc} H^1(\mathcal{G},M) & \to & H^1(\mathcal{G}_v,M_v) \\ \downarrow & & \downarrow \\ H^1(\mathcal{H},M_K) & \to & H^1(\mathcal{H}_v,M_{K_v}). \end{array}$$

Now let μ be an element of $H^1(\mathcal{G}, M)$, μ_K its image in $H^1(\mathcal{H}, M_K)$, μ_v its image in $H^1(\mathcal{G}_v, M_v)$ and μ_{K_v} its image in $H^1(\mathcal{H}_v, M_{K_v})$. Lift μ_K to an element η of $H^1_{\text{ét}}(K, T_K)$ (cf. (2.14)) and μ_v to an element $\beta_v \in H^1_{\text{ét}}(k_v, T_v)$ (cf. (2.15)), and consider the images η_v of η and β_{K_v} of β_v in $H^1_{\text{ét}}(K_v, T_{K_v})$. Then $\eta_v - \beta_{K_v} \in \operatorname{Ker}\big(H^1_{\text{ét}}(K_v, T_{K_v}) \to H^1_{\text{ét}}(\mathcal{H}_v, M_{K_v})\big)$ which by exactness of the second row in (2.17) implies that $\eta_v - \beta_{K_v} \in \phi'(Z_0(X_{K_v})^0)$. Thus by (2.10b), $\eta_v - \beta_{K_v} \in \operatorname{Im}\big(H^1_{\text{ét}}(X_v, T_v) \to H^1_{\text{ét}}(K_v, T_{K_v})\big)$, and hence so does η_v. Choose for each place v an X_v-torsor $\mathcal{T}_v$ under T_v such that $[\mathcal{T}_v] \in H^1_{\text{ét}}(X_v, T_v)$ maps to η_v in $H^1_{\text{ét}}(K_v, T_{K_v})$. Then since $\eta_v - \beta_{K_v} \in \phi'(Z_0(X_{K_v})^0)$ we conclude from (a) and (c) of Lemma 2.10 that $\rho_v(z_v)([\mathcal{T}_v] - \beta_{K_v}) \in \phi'(Z_0(X_v)^0)$. This means that $\rho_v(z_v)([\mathcal{T}_v])$ and $\rho_v(z_v)(\beta_{K_v}) = \beta_v$ have the same image μ_v in $H^1(\mathcal{G}_v, M_v)$.

From the assumption that the 0-cycles $(z_v)_{v\in\Omega_k}$ satisfy Manin's reciprocity condition for all $\mathcal{A} \in \widetilde{H}^2_{\text{ét}}(X, \mathbb{G}_m)$, we deduce from Corollary 1.45 that there exists $\alpha \in H^1_{\text{ét}}(k, T)$ with image $\rho_v(z_v)([\mathcal{T}_v])$ in $H^1_{\text{ét}}(k_v, T_v)$ for each place $v \in \Omega_k$. Therefore, $\alpha \in H^1_{\text{ét}}(k, T)$ maps to an element in $H^1(\mathcal{G}, M)$ with the same image as μ in $H^1(\mathcal{G}_v, M_v)$ for each $v \in \Omega_k$. This completes the proof.

2.18 Theorem *Let k be a number field and X a smooth proper geometrically connected k-surface. Suppose that there exists a rational function $t \in k(X)$ on X such that $k(X)$ is the function field of a Severi–Brauer curve over $k(t)$. Then*

(a) $Ш^1(k, M) = 0$,

(b) Suppose that for each $v \in \Omega_k$ we are given a 0-cycle z_v of degree one on X_v such that Manin's reciprocity condition $\sum_{\text{all } v} \operatorname{inv}_v(\rho_v(z_v))(\mathcal{A}_v) = 0$ holds for all $\mathcal{A} \in H^2_{\text{ét}}(X, \mathbb{G}_m)$. Then there exists a 0-cycle of degree one on X.

Proof (a) $H^1(\mathcal{G}, M)$ and $Ш^1(k, M)$ are k-birational invariants [Sa]. The assumptions on X implies that it is k-birational to a relatively minimal conic bundle surface over $\mathbb{P}^1$. It is therefore sufficient to prove that $Ш^1(k, M) = 0$ for relatively minimal conic bundle surface over $\mathbb{P}^1$. But this is the main result of [Sa].

(b) This is a consequence of (a) and the previous theorem.

References

[Bl] S. Bloch, On the Chow groups of certain rational surfaces, *Ann. Sci. École Norm. Sup.* **14** (1981) 41–59

[Bo] A. Borel, *Linear algebraic groups*, 2nd ed. Graduate texts in Math. 126, Springer-Verlag

[CT/S1] J.-L. Colliot-Thélène and J.-J. Sansuc, On the Chow groups of certain rational surfaces: a sequel to a paper of S. Bloch, *Duke Math. J.* **48** (1981) 421–447

[CT/S2] J.-L. Colliot-Thélène et J.-J. Sansuc, La descente sur les variétés rationnelles. II, *Duke Math. J.* **54** (1987) 375–492. (cf. *C. R. Acad. Sci. Paris* **284** Série A (1977) 1215–1218)

[CT/S3] J.-L. Colliot-Thélène and J.-J. Sansuc, Principal homogeneous spaces under flasque tori: applications, *J. Algebra* **106** (1987) 145–202

[CT/SwD] J.-L. Colliot-Thélène and Sir Peter Swinnerton-Dyer, Hasse principle and weak approximation for pencils of Severi–Brauer varieties and similar varieties, *J. reine angew. Math.* **453** (1994) 49–112

[Ha] R. Hartshorne, *Algebraic Geometry*, Springer-Verlag, 1977

[Ma] Yu.I. Manin, Le groupe de Brauer–Grothendieck en géométrie diophantienne, In: *Actes Congrès Intern. Math.* (Nice 1970), Gauthiers-Villars, Paris 1971, Tome 1, 409–411

[Mi] J.S. Milne, *Étale cohomology*, Princeton University Press, Princeton 1980

[Sa] P. Salberger, Zero-cycles on rational surfaces over number fields, *Invent. Math.* **91** (1988) 505–524

[Sk] A.N. Skorobogatov, Beyond the Manin obstruction, *Invent. Math.* **113** (1999) 399–424

[Ta1] J. Tate, The cohomology groups of tori in finite Galois extensions of number fields, *Nagoya Math. J.* **27** (1966) 709–719

[Ta2] J. Tate, Global class field theory, In: *Algebraic number theory*, J.W.S. Cassels, A. Fröhlich, eds., Academic Press 1967

Per Salberger,
Department of Mathematics,
Chalmers University of Technology,
S–412 96 Göteborg, Sweden
e-mail: salberg@math.chalmers.se