Cascades of projections from log del Pezzo surfaces

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To Peter Swinnerton-Dyer, in admiration

Abstract

One of the best-loved tales in algebraic geometry is the saga of the blowup of $\mathbb{P}^2$ in $d \le 8$ general points and its anticanonical embedding. If a del Pezzo surface F with log terminal singularities has a large anticanonical system $|-K_F|$, it can likewise be blown up many times to produce cascades of del Pezzo surfaces; as in the ancient fable, a blowup can be viewed as a projection from a bigger weighted projective space to a smaller one, leading in nice cases to weighted hypersurfaces or other low codimension Gorenstein constructions. The simplest examples already give several beautiful cascades, that we exploit as test cases for practice in the study of various kinds of projections and unprojections. We believe that these calculations will eventually have more serious applications to Fano 3-folds of Fano index ≥ 2, involving 1001 lovely and exotic adventures.

1 The story of $\overline{\mathbb{F}}_3$

Once upon a time, there was a surface $F = \overline{\mathbb{F}}_3$, known to all as the cone over the twisted cubic, or as $\mathbb{P}(1,1,3) = \operatorname{Proj} k[u_1, u_2, v]$, where $\operatorname{wt} u_1, u_2 = 1$, $\operatorname{wt} v = 3$. The anticanonical class of F is $-K_F = \mathcal{O}_{\overline{\mathbb{F}}_3}(5)$, so that its anticanonical ring $R(F, -K_F)$ is the fifth Veronese embedding or truncation $k[u_1, u_2, v]^{(5)}$. We see that this ring is generated by

$$\begin{aligned} x_1, \dots, x_9 &= S^5(u_1, u_2), S^2(u_1, u_2)v && \text{in degree 1,} \\ y_1, y_2 &= u_1 v^3, u_2 v^3 && \text{in degree 2,} \\ z &= v^5 && \text{in degree 3,} \end{aligned}$$

where, as usual, we write $S^d(u_1, u_2) = \{u_1^d, u_1^{d-1}u_2, \dots, u_2^d\}$ for the set of monomials of degree d in u_1, u_2.

Note that the two generators y_1, y_2 in degree 2 are essential as orbifold coordinates or *orbinates* at the singular point. This point is simple and well known, but we spell it out, as it is essential for the enjoyment of our narrative: at $P = P_v = (0,0,1) \in \mathbb{P}(1,1,3)$, only $v \neq 0$. We take a cube root $\xi = \sqrt[3]{v}$, thus introducing a $\mathbb{Z}/3$ Galois extension of the homogeneous coordinate ring. The homogeneous ratios $u_1/\xi, u_2/\xi$ are coordinates on a copy of $\mathbb{C}^2$, which is a $\mathbb{Z}/3$ cover of an affine neighbourhood of P; hence P is a quotient singularity of type $\frac{1}{3}(1,1)$. In our truncated subring $R(F, -K_F)$, only $z \neq 0$ at P, and the same orbinates are provided by the homogeneous ratios $y_1/z^{2/3}, y_2/z^{2/3}$. In the projective embedding given by $R(F, -K_F)$, since the orbinates are naturally forms of degree 2, we think of P as a quotient singularity of type $\frac{1}{3}(2,2)$.

There are many ways of seeing that the Hilbert function of $R(F, -K_F)$ is given by

$$P_n = h^0(F, -nK_F) = 1 + \frac{25}{3}\binom{n+1}{2} - \begin{cases} \frac{1}{3} & \text{if } n \equiv 1 \bmod 3 \\ 0 & \text{otherwise} \end{cases}$$

for all $n \geq 0$, and thus the Hilbert series is

$$P_F(t) := \sum P_n t^n = \frac{1 + 7t + 9t^2 + 7t^3 + t^4}{(1-t)^2(1-t^3)}.$$

You can do this as an exercise in orbifold RR ([YPG], Chapter III); or another way is to multiply the Hilbert series $1/(1-s)^2(1-s^3)$ of $k[u_1, u_2, v]$ through by $(1-s^5)^2(1-s^{15})$, truncate it to the polynomial consisting only of terms of degree divisible by 5, and substitute $s^5 = t$.

Now let $S = S^{(d)} \to F$ be the blowup of F in d general points P_i, for $d \leq 8$. Write E_i for the -1-curves over P_i. Since $K_S = K_F + \sum E_i$, the anticanonical ring $R(S, -K_S)$ consists of elements of $R(F, -K_F)$ of degree n passing n times through P_i. Thus each point imposes one condition in degree 1, 3 in degree 2, etc. Therefore the Hilbert series of S is

$$P_S(t) = P_F(t) - d \times \frac{t}{(1-t)^3} = \frac{1 + (7-d)t + (9-d)t^2 + (7-d)t^3 + t^4}{(1-t)^2(1-t^3)}.$$

In particular $S^{(d)}$ has anticanonical degree $\frac{25-3d}{3} = (8-d) + \frac{1}{3}$. The first cases are listed in Table 1.1; the first three models suggested by the Hilbert function work without trouble. For $S^{(6)}$, the Hilbert function requires 3 generators in degree 1, 2 in degree 2, and 1 in degree 3, and the corresponding Hilbert numerator is

$$(1-t)^3(1-t^2)^2(1-t^3)P_S(t) = 1 - 2t^3 - 3t^4 + 3t^5 + 2t^3 - t^8.$$

$d=8$	$1/3$	$P_S(t)=\frac{1+t^5}{(1-t)(1-t^2)(1-t^3)}$	$S_{10}\subset\mathbb{P}(1,2,3,5)$
$d=7$	$4/3$	$P_S(t)=\frac{1+2t^2+t^4}{(1-t)^2(1-t^3)}$	$S_{4,4}\subset\mathbb{P}(1,1,2,2,3)$
$d=6$	$7/3$	$P_S(t)=\frac{1+2t^2-2t^3-t^5}{(1-t)^3(1-t^3)}$	$S_{\mathrm{Pf}}\subset\mathbb{P}(1^3,2^2,3)$
$d=5$	$10/3$	$P_S(t)=\frac{1+t^2-4t^3+t^4+t^6}{(1-t)^4(1-t^3)}$	codim 4
$d=4$	$13/3$	$P_S(t)=\frac{1-t^2-4t^3+4t^4+t^5-t^7}{(1-t)^5(1-t^3)}$	codim 5

Table 1.1: The cascade above $S_{10}\subset\mathbb{P}(1,2,3,5)$

This indicates that $S^{(6)}\subset\mathbb{P}(1^3,2^2,3)$ should be defined (in coordinates x_1,x_2,x_3,y_1,y_2,z) by the Pfaffians of a 5×5 skew matrix

$$A(S^{(6)})=\begin{pmatrix} x_1 & x_2 & b_{14} & b_{15}\\ & x_3 & b_{24} & b_{25}\\ & & b_{34} & b_{35}\\ & & & z\end{pmatrix}\quad\text{of degrees}\quad\begin{pmatrix}1&1&2&2\\ &1&2&2\\ & &2&2\\ & & &3\end{pmatrix}. \tag{1.1}$$

We see that this works: thus the 3 Pfaffians involving z give $x_iz=\cdots$, so that at the point $P_z=(0,\dots,0,1)$ the three x_i are eliminated as implicit functions, and P_z is a $\frac{1}{3}(2,2)$ singularity with orbinates y_1,y_2.

Remark 1.1 For $S^{(5)}$ and $S^{(4)}$, innocently putting in only the generators required by the Hilbert series suggests the similar codimension 3 Pfaffian models of Table 1.2. However, experience says that they cannot possibly

$d=5$	$\frac{1-4t^3-t^4+t^4+4t^5-t^8}{(1-t)^4(1-t^2)(1-t^3)}$	$S^{(5)}\subset\mathbb{P}(1^4,2,3)$	$\begin{pmatrix}1&1&1&1\\ &2&2&2\\ & &2&2\\ & & &2\end{pmatrix}$
$d=4$	$\frac{1-t^2-4t^3+4t^4+t^5-t^7}{(1-t)^5(1-t^3)}$	$S^{(4)}\subset\mathbb{P}(1^5,3)$	$\begin{pmatrix}1&1&1&2\\ &1&1&2\\ & &1&2\\ & & &2\end{pmatrix}$

Table 1.2: Candidate Pfaffian models that don't work

work: each of these is a *mirage* of a type encountered many times in the course of previous adventures. For one thing, there is nowhere for a variable of degree 3 to appear in the matrix, so that its Pfaffians define a weighted projective cone with vertex $(0,\dots,0,1)$ over a base $C\subset\mathbb{P}(1^4,2)$ (respectively, $C\subset\mathbb{P}^4$) that is a projectively Gorenstein curve C with $K_C=\mathcal{O}(2)$; the cone point is not log terminal. For another, the anticanonical ring needs two generators of degree 2 to provide orbinates at the singularity of type $\frac{1}{3}(2,2)$. The conclusion is that we have not yet put in enough generators for the graded ring (or, in other contexts, that the variety we seek does not exist). Mirages of this type appear all over the study of graded rings, as discussed in 3.3.

As we see below, $S^{(d)}$ is an explicit construction from $\overline{\mathbb{F}}_3$, and has projections down to $S_{10} \subset \mathbb{P}(1,2,3,5)$ or $S_{4,4} \subset \mathbb{P}(1,1,2,2,3)$, so that we can find out anything we want to know about the rings $R(S,-K_S)$ by working in birational terms, either from above by projecting from $\overline{\mathbb{F}}_3$, or from below by unprojecting from one of the low codimension cases. We first relate without proof what happens. Listen and attend!

Consider $S = S^{(5)}$ first. First, $R(S,-K_S)$ has two generators y_1, y_2 and one relation in degree 2; the Hilbert series on its own cannot detect this, because the relation masks the second generator. Once you know about the additional generator, the anticanonical model of $S^{(5)}$ is a codimension 4 construction $S^{(5)} \subset \mathbb{P}(1^4, 2^2, 3)$, with Hilbert numerator

$$(1-t)^4(1-t^2)^2(1-t^3)P_S(t) = 1 - t^2 - 4t^3 + 8t^5 - 4t^7 - t^8 + t^{10};$$

however, there is still more masking going on: although the Hilbert series only demands one relation in degree 2 and 4 in degree 3, there are in fact also 4 relations and 4 syzygies in degree 4, and the ring has the 9×16 minimal resolution

$$\begin{aligned} \mathcal{O}_S \leftarrow \mathcal{O}_{\mathbb{P}} \leftarrow \mathcal{O}_{\mathbb{P}}(-2) \oplus 4\mathcal{O}(-3) \oplus 4\mathcal{O}(-4) \\ \leftarrow 4\mathcal{O}_{\mathbb{P}}(-4) \oplus 8\mathcal{O}(-5) \oplus 4\mathcal{O}(-6) \leftarrow \cdots \text{(sym.)} \end{aligned} \quad (1.2)$$

The syzygy matrixes in this complex have 4×4 blocks of zeros (of degree 0). We represent this by writing out the Hilbert numerator as the expression

$$1 \quad - t^2 - 4t^3 - 4t^4 \quad + 4t^4 + 8t^5 + 4t^6 \quad - 4t^6 - 4t^7 - t^8 \quad + t^{10},$$

where the spacing is significant. Likewise, $S^{(4)}$ is the codimension 5 construction $S^{(4)} \subset \mathbb{P}(1^5, 2^2, 3)$, with 14×35 resolution represented by

$$\begin{aligned} 1 \quad - 3t^2 - 6t^3 - 5t^4 \quad + 2t^3 + 12t^4 + 15t^5 + 6t^6 \\ - 6t^5 - 15t^6 - 12t^7 - 2t^8 \quad + 5t^7 + 6t^8 + 3t^9 \quad - t^{11}. \end{aligned} \quad (1.3)$$

These assertions can be justified either by viewing $S^{(d)}$ as projected from

$F = \overline{\mathbb{F}}_3$, or as unprojected from $S^{(d+1)}$. For convenience, we do $S^{(5)}$ from below, and $S^{(4)}$ from above (but we could do either case by the other method, with slightly longer computations).

Projecting from a general $P \in S^{(5)}$ blows P up to a -1-curve $l = \mathbb{P}^1$ contained in the Pfaffian model of $S^{(6)} \subset \mathbb{P}(1^3, 2^2, 3)$. Inversely, $S^{(5)}$ is obtained as the Kustin–Miller unprojection of $l \subset S^{(6)}$ (see Papadakis and Reid [PR]): the ring of $S^{(5)}$ is generated over that of $S^{(6)}$ by adjoining 1 unprojection variable $x = x_4$ of degree $k_S - k_l = -1 - (-2) = 1$, with unprojection equations $x \cdot g_i = \cdots$, for the generators g_i of I_l. Now l is clearly a complete intersection of 4 hypersurfaces of degrees $1, 2, 2, 3$ (it is $x_3 = y_1 = y_2 = z = 0$ up to a coordinate change). The ring of $S^{(5)}$ thus has equations the old equations of $S^{(6)}$ of degrees $3, 3, 4, 4, 4$ (the Pfaffians (1.1) defining $A(S^{(6)})$), together with 4 unprojection equations of degrees $2, 3, 3, 4$. The numerical shape of the resolution (1.2) comes from this and Gorenstein symmetry. The same result can be obtained by applying the Kustin–Miller construction directly: the projective resolution of the ring of $S^{(6)}$ is the Buchsbaum–Eisenbud complex $L_\bullet$ of the matrix $A(S^{(6)})$, and that of l is the Koszul complex $M_\bullet$ of the regular sequence defining l. Then $R(S^{(5)})$ arises from a homomorphism $L_\bullet \to M_\bullet$ extending the map $\mathcal{O}_{S^{(6)}} \twoheadrightarrow \mathcal{O}_l$. For details, see Papadakis [P2].

We justify $S^{(4)}$ in the other direction, by projecting down from F. We can choose coordinates to put a general set of 4 points in the form

$$\{P_1, \dots, P_4\} \subset F = \mathbb{P}(1,1,3) \quad \text{given by} \quad f_4(u_1, u_2) = v = 0.$$

The anticanonical ring of the 4-point blowup $S^{(4)}$ is then generated by

$$\begin{aligned} x_1, \dots, x_5 &= \{u_1 f, u_2 f, S^2(u_1, u_2) v\} && \text{in degree 1,} \\ y_1, y_2 &= u_1 v^3, u_2 v^3 && \text{in degree 2,} \\ z &= v^5 && \text{in degree 3.} \end{aligned}$$

The ideal of relations between these can be studied by explicit elimination (we used computer algebra, but it is not at all essential); one finds that it is generated by

$$\operatorname{rank} \begin{pmatrix} * & x_1 & x_2 & y_0 \\ x_1 & x_3 & x_4 & y_1 \\ x_2 & x_4 & x_5 & y_2 \\ y_0 & y_1 & y_2 & z \end{pmatrix} \le 1, \quad \text{where} \quad y_0 = q(x_3, x_4, x_5). \qquad (1.4)$$

Taking y_0 as a variable gives the second Veronese embedding of the one point blowup of the 3-fold wps $\mathbb{P}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2})$. Thus $S^{(4)}$ is a hypersurface of weighted degree 2 in this curious weighted quasihomogenous variety. The second Veronese embedding of the one point blowup of $\mathbb{P}^3$ is a well known codimension 5 del Pezzo variety appearing in other myths, and its equations have a 14×35 resolution. We check that this agrees with (1.3).

Exercise 1.2 Chronicle the fate of $\overline{\mathbb{F}}_5$ and its d-point blowup $S^{(d)} \to \overline{\mathbb{F}}_5$ for $d \le 9$. [Hint: the Hilbert series is

$$\begin{aligned} P(t) &= \frac{1+9t+9t^2+11t^3+9t^4+9t^5+t^6}{(1-t)^2(1-t^5)} - d \times \frac{t}{(1-t)^3} \\ &= \frac{1+(9-d)t+(9-d)t^2+(11-d)t^3+(9-d)t^4+(9-d)t^5+t^6}{(1-t)^2(1-t^5)}. \end{aligned}$$

The singularity polarised by $-K = A$ is of type $\frac{1}{5}(3,3)$, so that $S^{(d)}$ is in $\mathbb{P}(1^{11-d},3,3,5)$. Thus $d=9$ gives $S_{6,6} \subset \mathbb{P}(1,1,3,3,5)$ and $d=8$ gives a nice Pfaffian in $\mathbb{P}(1,1,1,3,3,5)$, with Hilbert numerator

$$1 - 2t^4 - 3t^6 + 3t^7 + 2t^9 - t^{13},$$

etc.]

These surfaces have a singularity of type $\frac{1}{5}(3,3)$; we were disappointed at first to observe that none of these is a hyperplane section $S \in |A|$ for a Mori Fano 3-fold X of Fano index 2. For then X would have a quotient singularity of type $\frac{1}{5}(1,3,3)$, which is unfortunately not terminal. For further disappointment, see 3.2.2.

2 The ingenious history of $\frac{1}{5}(2,4)$

Let T be a del Pezzo surface polarised by $-K_T = \mathcal{O}_T(A)$ with a quotient point $P \in T$ of type $\frac{1}{5}(2,4)$ as its only singularity. (Up to isomorphism, P is the quotient singularity $\frac{1}{5}(1,2)$, but to give sections of $-K_T$ weight 1, and make $\mathcal{O}_T(A) = -K_T$ the preferred generator of the local class group, we twist μ_5 by an automorphism so that $d\xi \wedge d\eta$ is in the $\varepsilon \mapsto \varepsilon$ character space, and thus $\operatorname{wt}\xi = 2, \operatorname{wt}\eta = 4 \bmod 5$.) By an exercise in the style of [YPG], Chapter III, we see that

$$P_n(T) = 1 + \binom{n+1}{2}A^2 - \begin{cases} 0 & n \equiv 0 \mod 5 \\ 2/5 & n \equiv 1 \mod 5 \\ 1/5 & n \equiv 2 \mod 5 \\ 2/5 & n \equiv 3 \mod 5 \\ 0 & n \equiv 4 \mod 5 \end{cases}$$

Trying $n = 1$ gives $A^2 \equiv 2/5 \bmod \mathbb{Z}$. Putting these values in a Hilbert series as usual and setting $A^2 = k + \frac{2}{5}$ gives

$$\begin{aligned}
P(t) &= \frac{1}{1-t} + \frac{t}{(1-t)^3}A^2 - \frac{1}{5}\cdot\frac{2t+t^2+2t^3}{1-t^5}\\
&= \frac{1}{1-t} + \frac{t}{(1-t)^3}k\\
&\qquad + \frac{1}{5}\cdot\frac{2t(1+t+t^2+t^3+t^4)-(1-t)^2(2t+t^2+2t^3)}{(1-t)^2(1-t^5)}\\
&= \frac{1-t+t^2+t^4-t^5+t^6}{(1-t)^2(1-t^5)} + \frac{t}{(1-t)^3}k.
\end{aligned}$$

The case $k = 0$ gives

$$\begin{aligned}
\frac{1-t+t^2+t^4-t^5+t^6}{(1-t)^2(1-t^5)} &= \frac{1+t^3+t^4+t^7}{(1-t)(1-t^2)(1-t^5)}\\
&= \frac{1-t^6-t^8+t^{14}}{(1-t)(1-t^2)(1-t^3)(1-t^4)(1-t^5)},
\end{aligned}$$

that is, $T_{6,8} \subset \mathbb{P}(1,2,3,4,5)$.

This surface turns out to be the bottom of a cascade of six projections, whose head is the surface $T = T_6 \subset \mathbb{P}(1,1,3,5)$ with $-K_T = A = \mathcal{O}(4)$. We guessed this as follows: by the standard dimension count for del Pezzo surfaces, we expect $T_{6,8}$ to contain a finite number of -1-curves not passing through the singularity. Contracting k disjoint -1-curves gives a surface with $K_T^2 = A^2 = k + \frac{2}{5}$ and the above Hilbert series. For $k = 6$, we see that $A^2 = 6 + \frac{2}{5} = \frac{32}{5}$ is divisible by 4^2, and we guess that $A = 4B$, leading to a surface with the Hilbert series of $T = T_6 \subset \mathbb{P}(1,1,3,5)$. Hindsight is the only justification for this guesswork.

One sees that the minimal resolution $\widetilde{T} \to T$ is the scroll $\mathbb{F}_3$ blown up in two points on a fibre, and that T is obtained from this by contracting the chain of $\mathbb{P}^1$s with self-intersection $(-3,-2)$ coming from the negative section and the birational transform of the fibre (see Figure 2.1).

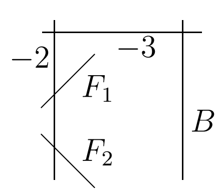

Figure 2.1: Resolution of $T = T_6 \subset \mathbb{P}(1,1,3,5)$

We start by calculating the anticanonical ring of the head of the cascade, $T = T_6 \subset \mathbb{P}(1,1,3,5)$. Take coordinates u_1, u_2, v, w in $\mathbb{P}(1,1,3,5)$, and take the defining equation of T to be

$$u_2w = f_6(u_1, v) = av^2 + bvu_1^3 + cu_1^6 = l_1(v, u_1^3)l_2(v, u_1^3); \tag{2.1}$$

we could normalise the right-hand side to $(v - u_1^3)(v + u_1^3)$. We use this relation to eliminate any monomial divisible by u_2w. Write B for the divisor class corresponding to $\mathcal{O}_{\mathbb{P}}(1)$ or its restriction to T. Since $-K_T = 4B$, the anticanonical embedding of T is the 4th Veronese embedding of $T \subset \mathbb{P}(1, 1, 3, 5)$; one checks that the anticanonical ring is generated by

$$\begin{array}{rcll} x_1, \dots, x_7 &=& S^4(u_1, u_2), (u_1, u_2)v & \text{in degree 1,} \\ y_1, y_2 &=& u_1^3w, vw & \text{in degree 2,} \\ z &=& u_1^2w^2 & \text{in degree 3,} \\ t &=& u_1w^3 & \text{in degree 4,} \\ u &=& w^4 & \text{in degree 5,} \end{array} \tag{2.2}$$

and that its relations are given by the 2×2 minors of

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_6 & y_1 & z & t \\ x_2 & x_3 & x_4 & x_5 & x_7 & A & B & C \\ y_1 & A & B & C & y_2 & z & t & u \end{pmatrix}, \tag{2.3}$$

with

$$\begin{aligned} A &= ax_6^2 + bx_1x_6 + cx_1^2, \\ B &= ax_6x_7 + bx_2x_6 + cx_1x_2, \\ C &= ax_7^2 + bx_3x_6 + cx_1x_3. \end{aligned}$$

Theorem 2.1 *For $d \le 6$, write $\sigma\colon T^{(d)} \dashrightarrow T$ for the blowup of T in d general points $P_1, \dots, P_d$. (We elucidate what "general" means in (2.6) below.) Write E_i for the -1-curves over P_i and $A^{(d)} = \sigma^*A - \sum E_i$ for the anticanonical class of $T^{(d)}$. Then $T^{(d)}$ is a log del Pezzo surface with only singularity of type $\frac{1}{5}(2,4)$ and $(-K_S)^2 = 6 - d + \frac{2}{5}$.*

For $d \le 5$, the anticanonical ring of $T^{(d)}$ needs $12-d$ generators of degrees $1^{7-d}, 2^2, 3, 4, 5$, and gives an embedding $T^{(d)} \subset \mathbb{P}(1^{7-d}, 2^2, 3, 4, 5)$ that takes the E_i to disjoint projectively normal lines

$$E_i \cong \mathbb{P}^1 \subset T^{(d)} \subset \mathbb{P}(1^{7-d}, 2^2, 3, 4, 5).$$

The anticanonical ring of $T^{(6)}$ needs 5 generators of degrees $1, 2, 3, 4, 5$, and embeds $T^{(6)}$ as the complete intersection $T_{6,8} \subset \mathbb{P}(1, 2, 3, 4, 5)$, taking the E_i to disjoint -1-curves in $T_{6,8}$ (of course, the $E_i \subset \mathbb{P}(1, 2, 3, 4, 5)$ cannot be projectively normal).

Each inclusion $R(T^{(d)}, A^{(d)}) \subset R(T^{(d-1)}, A^{(d-1)})$ for $d \le 5$ is a Kustin–Miller unprojection in the sense of [PR]. That is, it introduces precisely one new generator of degree 1 with pole along E_d, subject only to linear relations. For $d = 6$, see Remark 2.5.

Proof As in the analogous recitations for nonsingular del Pezzo surfaces, the proof consists for the most part of restricting to the general curve $C \in |A^{(d)}|$. The restriction $R(T^{(d)}, A^{(d)}) \to R(C, A^{(d)})$ is a surjective ring homomorphism, and is the quotient by the principal ideal (x_C), where x_C is the equation of C. Thus the hyperplane section principle applies, and we only have to prove the appropriate generation results for $R(C, A^{(d)})$. In the antique tale, C is a nonsingular elliptic curve, and we win because we know everything about linear systems on it. In our case $C \in |-K_{T^{(d)}}|$ is an elephant, so is again a projectively Gorenstein curve with $K_C = 0$, but it is an orbifold nodal rational curve in a sense we are about to study. Our proof will then boil down to a monomial calculation.

The general curve $C \in |A|$ on T is irreducible and has an ordinary node at P, and the two orbinates of $P \in T$ restrict to respective local analytic coordinates on the two branches of the node. In other words, $P \in C$ is locally analytically equivalent to the quotient $\Big((\xi\eta = 0) \subset \mathbb{C}^2\Big)/(\frac{1}{5}(2,4))$, where ξ, η are as in Remark 2.4. To make formal sense of this, we need to work with the affine cone over $T \supset C$ along the u-axis, and the $\mathbb{C}^*$ action on them. The cone over T is nonsingular along the u-axis outside the origin, with transverse coordinates y_2, t (see (2.8)) – the $\frac{1}{5}(2,4)$ singularity arises from the $\mathbb{Z}/5$ isotropy. The coefficient of x_6 in the equation (x_C) of C is nonzero in general, corresponding to $u_1 v$ in (2.2). Therefore, along the u-axis, the cone over C is given locally by $y_2 t =$ higher order terms.

We choose a general curve $C \in |A|$ and $d \le 6$ general points $P_1, \dots, P_d$ contained in C. These points are also independent general points of T, because $|A|$ is a 6-dimensional linear system on T. This choice ensures the existence of an irreducible curve $C \in |A - \sum P_i|$ with the local behaviour at P just described. The birational transform of C on $T^{(d)}$ is an isomorphic curve $C \in |A^{(d)}|$ that we continue to denote by C. It is irreducible, therefore nef, and big since $(A^{(d)})^2 > 0$.

The normalisation $n\colon \widetilde{C} \to C \subset T^{(d)}$ is a conventional orbifold curve: it is a rational curve with two marked point P_1, P_2, the inverse image of the node of C. In calculations, we take $\widetilde{C} = \mathbb{P}^1$, and $P_1 = 0$ and $P_2 = \infty$. It is polarised by $\widetilde{A} = n^*(A^{(d)}) = \frac{3}{5}P_1 + \frac{4}{5}P_2 + (5-d)Q$, where Q is some other point. This is just a notational device to handle the sheaf of graded algebras

$$\mathcal{A} = \bigoplus \mathcal{A}_i \quad \text{with} \quad \mathcal{A}_i = \mathcal{O}_{\mathbb{P}^1}\left(\left[\frac{3i}{5}\right]P_1 + \left[\frac{4i}{5}\right]P_2\right) \otimes \mathcal{O}_{\mathbb{P}^1}((5-d)i).$$

We calculate $R(\widetilde{C}, \widetilde{A})$ in monomial terms (the answer has a nice toric description, see Exercise 2.2).

For $d = 5$, the calculations is as follows: $R(\widetilde{C}, \widetilde{A}) = R(\mathbb{P}^1, \frac{3}{5}P_1 + \frac{4}{5}P_2)$ is generated by

$$\begin{array}{rll} x & \text{in degree 1} & \text{with } \operatorname{div} x = \frac{3}{5}P_1 + \frac{4}{5}P_2, \\ y_1, y_2 & \text{in degree 2} & \text{with } \operatorname{div}(y_1, y_2) = (2P_1, 2P_2) + \frac{1}{5}P_1 + \frac{3}{5}P_2, \\ z & \text{in degree 3} & \text{with } \operatorname{div} z = 3P_1 + \frac{4}{5}P_1 + \frac{2}{5}P_2, \\ t & \text{in degree 4} & \text{with } \operatorname{div} t = 5P_1 + \frac{2}{5}P_1 + \frac{1}{5}P_2, \\ u_1, u_2 & \text{in degree 5} & \text{with } \operatorname{div}(u_1, u_2) = (7P_1, 7P_2). \end{array} \tag{2.4}$$

Here, in each degree, $|iD|$ is the fractional part $\{\frac{3i}{5}\}P_1 + \{\frac{4i}{5}\}P_2$ plus a linear system $|\mathcal{O}_{\mathbb{P}^1}(k_i)|$, based by elements corresponding to the monomials $S^{k_i}(t_1, t_2)$, of which the middle ones are old, and some of the extreme ones are new generators. Thus in degree 2, $k_2 = 2$, and the monomials y_1, x^2, y_2 correspond to t_1^2, t_1t_2, t_2^2.

Exercise 2.2 The generators of $R(\widetilde{C}, \widetilde{A})$ and the relations between them are simply grasped by noting that $u_1, t, z, y_1, x, y_2, u_2$ in (2.4) satisfy

$$u_1 z = t^2, \quad t y_1 = z^2, \quad z x = y_1^2, \quad y_1 y_2 = x^4, \quad x u_2 = y_2^3;$$

this is the Jung–Hirzebruch presentation of the invariant ring of $\mathbb{Z}/(35)$ acting on $\mathbb{C}^2$ by $\frac{1}{35}(1, 12)$, where $[2, 2, 2, 4, 3] = \frac{35}{35-12}$. The case $d = 6$ gives $[2, 2, 4] = \frac{10}{7}$. Generalising this result to the general orbifold curve $(\mathbb{P}^1, \alpha_1 P_1 + \alpha_2 P_2)$ is a little gem of a problem.

The extension of graded rings $R(C, A^{(d)}) \subset R(\widetilde{C}, \widetilde{A})$ is a normalisation, separating two transverse sheets along the u-axis. The affine cone over the nonnormal curve C is obtained by glueing the u_1 and u_2-axes together (different choices of glueing differ by a factor in $\mathbb{C}^*$, and lead to isomorphic rings). The functions compatible with this glueing are those that take the same value on u_1 and u_2-axes. Thus $R(C, A^{(d)}) \subset R(\widetilde{C}, \widetilde{A})$ is the subring generated as above, but with only one generator $u = u_1 - u_2$ in degree 5 instead of two. This proves the statement on generators of $R(S^{(d)}, A^{(d)})$ for $d = 5$. The cases $d \le 4$ are similar.

In case $d = 6$, the orbifold divisor on $\widetilde{C} = \mathbb{P}^1$ is

$$\widetilde{A} = n^*(A^{(d)}) = \frac{3}{5}P_1 + \frac{4}{5}P_2 - Q.$$

An identical calculation shows that $R(\widetilde{C}, \widetilde{A})$ is generated by

$$\begin{array}{rll} y & \text{in degree 2} & \text{with } \operatorname{div} y = \frac{1}{5}P_1 + \frac{3}{5}P_2, \\ z & \text{in degree 3} & \text{with } \operatorname{div} z = \frac{4}{5}P_1 + \frac{2}{5}P_2, \\ t & \text{in degree 4} & \text{with } \operatorname{div} t = P_1 + \frac{2}{5}P_1 + \frac{1}{5}P_2, \\ u_1, u_2 & \text{in degree 5} & \text{with } \operatorname{div}(u_1, u_2) = (2P_1, 2P_2). \end{array} \tag{2.5}$$

As before, the nonnormal subring $R(C, A^{(d)})$ is generated by y, z, t and $u = u_1 - u_2$, and one sees that the relations are

$$yt = z^2, \quad zu = t^2 - y^4.$$

That is, C is the complete intersection $C_{6,8} \subset \mathbb{P}(2,3,4,5)$, as required.

This proves the assertion of Theorem 2.1 on the generation of the rings $R(T^{(d)}, A^{(d)})$. This proof uses that $A^{(d)}$ is nef and big, but not that it is ample.

We now prove that $A^{(d)}$ is ample. It is enough to show that the anticanonical morphism of $T^{(d)}$ does not contract any curve Γ of T, or equivalently, that $T^{(d)}$ does not contain any curve with $A^{(d)}\Gamma = 0$. Now because the generators of $R(T^{(d)}, A^{(d)})$ include elements y_2, t in (2.4) or y, t in (2.5) that give the orbinates at $P \in A$, the anticanonical morphism of $T^{(d)}$ is an isomorphism near P, and so Γ cannot pass through P. On the other hand, a curve with $A^{(d)}\Gamma = 0$ is necessarily a component of a divisor in the mobile linear system $|A^{(d)}|$ if $d \le 5$, or $|2A^{(d)}|$ if $d = 6$.

One sees that $T = T_6 \subset \mathbb{P}(1,1,3,5)$ has a free pencil $|B|$ defined by $(u_1 : u_2)$, with a reducible fibre $u_2 = 0$ that splits into two components $F_i : (u_2 = l_i = 0)$, where, as in (2.1), the equation of T is $u_2 w = l_1(v, u_1^3) l_2(v, u_1^3)$ (compare Figure 2.1). Every effective Weil divisor is linearly equivalent to a positive linear combination of F_1, F_2. These satisfy $F_1^2 = F_2^2 = -\frac{2}{5}$ and $F_1 F_2 = \frac{3}{5}$, so that $iF_1 + jF_2$ is nef if only if $\frac{2}{3}j < i < \frac{3}{2}j$. Moreover, $iF_1 + jF_2$ can only move away from P if it is Cartier there, which happens if and only if $5 \mid (i + j)$. Next $iF_1 + jF_2$ a component of $|A| = |4F_1 + 4F_2|$ (resp. $|2A|$) implies $i, j \le 4$ (resp. $i, j \le 8$).

Thus for $d \le 5$ we just have to handle $\Gamma \in |2F_1 + 3F_2|$ and $|3F_1 + 2F_2|$. Since $-K_T\Gamma = 4$ and $(\Gamma)^2 = 2$, RR gives $h^0(T, \Gamma) = 4$, and for general points, no 4 of $P_1, \dots, P_d$ are contained in Γ. This completes the proof if $d \le 5$. For $d = 6$ we also need to consider

$$4F_1 + 6F_2, \quad 5F_1 + 5F_2 \quad \text{and} \quad 7F_1 + 8F_2.$$

The proper transform of a curve $\Gamma \subset T$ will give $A^{(d)}\Gamma = 0$ if $\Gamma \in |A|$ passes through the P_i with multiplicity a_i, where

$$\sum a_i = -K_T\Gamma, \quad \sum a_i = (\Gamma)^2.$$

In the 3 cases above, the only solutions are

$$\begin{aligned} \Gamma = 4F_1 + 6F_2 :& \quad -K_T\Gamma = 8, \quad (\Gamma)^2 = 8, \quad \text{none;} \\ \Gamma = 5F_1 + 5F_2 :& \quad -K_T\Gamma = 8, \quad (\Gamma)^2 = 10, \quad (1,1,1,1,2,2); \\ \Gamma = 7F_1 + 8F_2 :& \quad -K_T\Gamma = 12, \quad (\Gamma)^2 = 24, \quad (2,2,2,2,2,2). \end{aligned}$$

The conclusion is that $A^{(d)}$ is ample if and only

$$\begin{array}{ll} (0) & \text{the } P_i \text{ are distinct and contained in an irreducible curve } C \in |A|; \\ (1) & \text{no 4 } P_i \text{ are contained in any } \Gamma \in |2F_1+3F_2| \text{ or } |3F_1+2F_2|; \\ (2) & |5F_1+5F_2-P_1-P_2-P_3-P_4-2P_5-2P_6| = \emptyset; \\ (3) & |7F_1+8F_2-2\sum P_i| = \emptyset \text{ and } |8F_1+7F_2-2\sum P_i| = \emptyset. \end{array} \tag{2.6}$$

Here conditions (2–3) are only required if $d = 6$.

These are open conditions on $P_1, \dots, P_d$, and they should fail in codimension 1. It remains to check that they are satisfied for general $P_1, \dots, P_6$. Write C for the unique curve of $|A|$ through $P_1, \dots, P_6$. Then any divisor Γ on T in Case (2) contains C: indeed,

$$\big(5F_1+5F_2-P_1-P_2-P_3-P_4-2P_5-2P_6\big)_{|C}$$

has degree 0, but is not linear equivalent to 0 on C for general $P_1, \dots, P_6$ (recall that C is a nodal cubic, so that its nonsingular points correspond to different points of the algebraic group $\operatorname{Pic} C = \mathbb{C}^*$). Thus $\Gamma = C + B$, where $|B| = |F_1+F_2|$ is the pencil of T. Clearly, the element of $|B|$ through P_5 does not in general pass through P_6. The argument in Case (3) is similar: a divisor Γ in Case (3) must be of the form $C+D$, where $D \in |3F_1+4F_2-\sum P_i|$. But

$$h^0(T, 3F_1+4F_2) < h^0(T, 4F_1+4F_2) = h^0(T, -K_T) = 7,$$

(see (2.2)) so that $|3F_1+4F_2|$ does not contain a curve through 6 general points of T. QED

Exercise 2.3 State and prove the analog of Theorem 2.1 for the cascade of Section 1. In other words, prove that the d point blowup of $\overline{\mathbb{F}}_3$ for $d \le 8$ has the properties asserted (without proof!) throughout Section 1.

Remark 2.4 The monomials in (2.2) map to some of the local generators of the sheaf of algebras $\bigoplus_{i=0}^4 \mathcal{O}_{T,P}(i)$ at the $\frac{1}{5}(2,4)$ singularity. Indeed, write ξ, η for ε^2 and ε^4 eigencoordinates on $\mathbb{C}^2$; then $\mathcal{O}_T$ is the sheaf of invariant functions, locally generated by $\xi^5, \xi^3\eta, \xi\eta^2, \eta^5$, whereas the eigensheaves $\mathcal{O}_{T,P}(i)$ are modules over $\mathcal{O}_{T,P}$, and are locally generated by

$$\begin{aligned} \mathcal{O}_{T,P}(1) &\ni \xi^3, \xi\eta, \eta^4 \\ \mathcal{O}_{T,P}(2) &\ni \xi, \eta^3 \\ \mathcal{O}_{T,P}(3) &\ni \xi^4, \xi^2\eta, \eta^2 \\ \mathcal{O}_{T,P}(4) &\ni \xi^2, \eta \\ \mathcal{O}_{T,P}(5) &\ni 1. \end{aligned} \tag{2.7}$$

Then the homogeneous to inhomogeneous correspondence at P (setting $\sqrt[5]{w} = 1$) has the effect

$$u_1 \mapsto \eta \quad \text{and} \quad v \mapsto \xi,$$

so that the generators of $R(T, -K_T)$ map to local generators of $\mathcal{O}_{T,P}(i)$ by

$$\begin{aligned}
&\text{deg 1:} \quad x_1 = u_1^4 \mapsto \eta^4, \quad x_6 = u_1 v \mapsto \xi\eta, \quad \emptyset \mapsto \xi^3;\\
&\text{deg 2:} \quad y_1 = u_1^3 w \mapsto \eta^3, \quad y_2 = vw \mapsto \xi;\\
&\text{deg 3:} \quad z = u_1^2 w^2 \mapsto \eta^2, \quad x_6 y_2 = u_1 v^2 w \mapsto \xi^2\eta, \quad \emptyset \mapsto \xi^4;\\
&\text{deg 4:} \quad t = u_1 w^3 \mapsto \eta, \quad y_2^2 = v^2 w^2 \mapsto \xi^2;\\
&\text{deg 5:} \quad u = w^4 \mapsto 1.
\end{aligned} \tag{2.8}$$

The remaining generators in (2.7) are hit by monomials in these generators: for example, $\xi^3 \in \mathcal{O}_T(1)$ is first hit by y_2^3 in degree 6. Thus $\mathcal{O}_T(i)$ is not always generated by its H^0, and not just because the $H^0(\mathcal{O}_X(i))$ are too small. However, by ampleness, $R(T, -K_T)$ maps surjectively to local generators of $\bigoplus_{i=0}^4 \mathcal{O}_T(i)$, so that, for example, the orbinates ξ and η must be hit by some generators of $R(T, -K_T)$.

Remark 2.5 (Detailed calculations of Type II projection) We hope eventually to use the two cascades of surfaces treated in Sections 1 and 2 as exercises in understanding Type II unprojection as in [Ki], Section 9, and in particular, solve the unfinished calculation in loc. cit., 9.12. The unprojection from $S_{10} \subset \mathbb{P}(1,2,3,5)$ to $S_{4,4} \subset \mathbb{P}(1,1,2,2,3)$ is covered by the equations of [Ki], 9.8. The only little surprise here is that, instead of increasing the codimension by 2, one of the entries in the 5×5 Pfaffian matrix is a unit, and one of the equations masks the variable of degree 5 as a combination of other variables.

On the other hand, the unprojection from $S_{6,8} \subset \mathbb{P}(1,2,3,4,5)$ leads to a codimension 4 ring, and the calculation is similar to the one unfinished in [Ki], 9.12. The image Γ of $\mathbb{P}^1 \hookrightarrow \mathbb{P}(1,2,3,4,5)$ cannot be projectively normal; indeed, if v_1, v_2 are coordinates on $\mathbb{P}^1$ and x, y, z, t, u coordinates on $\mathbb{P}(1,2,3,4,5)$, the two rings have monomials

	$\mathbb{P}(1,2,3,4,5)$	$\mathbb{P}^1$
in degree 1	x	v_1, v_2
in degree 2	x^2, y	v_1^2, v_1v_2, v_2^2
in degree 3	x^3, xy, z	$v_1^3, v_1^2v_2, v_1v_2^2, v_2^3$
in degree 4	x^4, x^2y, y^2, xz, t	$S^4(v_1, v_2)$

and the restriction map from $\mathbb{P}(1,2,3,4,5)$ to Γ clearly misses at least one monomial in each degree $1, 2, 3$. Choose $S_{6,8}$ containing Γ. Unprojecting it adds one linear generator, and one generator in each degree 2, 3 and 4 corresponding to these missing monomials (this will be explained better in [qG]). The old variables of degree 3 and 4 are masked by equations, and this gives rise to a codimension 4 surface $S' \subset \mathbb{P}(1,1,2,2,3,4,5)$. We still do not know how to complete this calculation directly.

Remark 2.6 In the projection from $\overline{\mathbb{F}}_3$ of Section 1, we always assumed that the blown up points were in general position. In the classic epic of del Pezzo surfaces, there are lots of interesting degenerations, most simply if 3 points in $\mathbb{P}^2$ become collinear. The simplest way that blowups of $\overline{\mathbb{F}}_3$ degenerate is that two points come to lie on a fibre l of the ruling of $\mathbb{F}_3$. If we project from two points on l, the birational transform of the fibre l becomes a -2-curve, and contracting it together with the negative section of $\mathbb{F}_3$ gives a $\frac{1}{5}(1,2)$ singularity. Thus all the surfaces in Section 2 are degenerate projections of those in Section 1. For example, $T_6 \subset \mathbb{P}(1,1,3,5)$ is a projection of $\overline{\mathbb{F}}_3$ from 2 points in a fibre (see Figure 2.1). This gives a top down elimination argument as on page 231 that might allows us to complete the tricky Type 2 unprojection calculation just discussed.

This type of contraction between surfaces with log terminal singularities corresponds to the bad links of [CPR], 5.5. We do not make this too precise. The fact that we blow up a point, then unexpectedly contract the line l with negative discrepancy is analogous to Sarkisov links involving an antiflip. The regular kind of blowup of a nonsingular point in a del Pezzo cascade decreases K_S^2 by 1, and the Hilbert function $P_S(t)$ by $t/(1-t)^3 = t+3t^2+6t^3+10t^4+\cdots$; whereas the special blowup (of a point contained in a curve of degree 2/3 that is a component of a split fibre of the conic bundle structure) considered here only decreases K_S^2 by 14/15, and $P_S(t)$ by

$$\frac{t(1+2t+3t^2+2t^3+3t^4+2t^5+t^6)}{(1-t)(1-t^3)(1-t^5)} = t+3t^2+6t^3+9t^4+14t^5+20t^6+26t^7+\cdots$$

3 Final remarks

3.1 Why weighted projective varieties?

Nonsingular surfaces over a field k that are rational or ruled over $\overline{k}$ (that is, have $\kappa = -\infty$) are prominent objects of study in birational geometry and in Diophantine geometry. By a theorem of Castelnuovo (a distinguished precursor of Mori theory!), such a surface can be blown down (over k) to a minimal surface, which is a del Pezzo surface of rank 1, or a conic bundle over a curve with relative rank 1. In justifying the pre-eminent position of the cubic surfaces among del Pezzo surfaces, Peter Swinnerton-Dyer observes that del Pezzo surfaces of degree ≥ 4 are in most respects too simple to be interesting, whereas del Pezzo surfaces of degree 2 and 1 tend to be much too difficult. Whereas the cubic surface is associated with the root systems E_6, those of degree 2 and 1, the weighted hypersurfaces $S_4 \subset \mathbb{P}(1^3, 2)$ and $S_6 \subset \mathbb{P}(1^2, 2, 3)$, are associated with E_7 and E_8, and are much more complicated from essentially every point of view (Galois theory, biregular and birational geometry, Diophantine arithmetic, etc.). In Peter's words:

> "if your research adviser gives you a problem involving del Pezzo surfaces of degree 2 and 1, it means he really *hates* you."

In view of this, working with del Pezzo surfaces with cyclic singularities may seem perverse, since it leads to even more exotic weighted projective constructions. For example our model case is $S_{10} \subset \mathbb{P}(1, 2, 3, 5)$, the 8 point blowup of $\overline{\mathbb{F}}_3$. It makes sense to write down the equation of S_{10} over any field, and to ask for its solutions: does this lead to any interesting problems of birational geometry or Diophantine arithmetic? The Galois group of the configuration of eight -1-curves is clearly the symmetric groups S_8. In contrast to the minimal cubic surface, this is birational over $\overline{k}$ in an obvious way to the conic bundle $\mathbb{F}_3 \to \mathbb{P}^1$, with the marked section, and a set of 8 points defined over k. This suggests that our surfaces are actually simpler objects and do not involve especially difficult or interesting Diophantine issues. On a more positive note, log del Pezzo surfaces come in large infinite families, among which we can surely always find some really complicated case for the graduate student who deserves that special attention.

3.2 Log del Pezzo surfaces and Fano 3-folds of index 2

3.2.1 The fabulous half-elephant

Our main motivation was of course to use log del Pezzo surfaces to study Fano 3-folds of Fano index $f = 2$. The *Fano index* of a Fano 3-fold X in the Mori category is the maximum natural number f such that $-K_X = fA$ with A a Weil divisor of X. Our model is the general strategy of Altınok, Brown and Reid [ABR], that uses K3 surfaces as technical background and motivation in the study of Fano 3-folds of index 1. If X is a Fano with $-K_X = 2A$ twice an ample Weil divisor, a sufficiently good surface $S \in |A|$ is a del Pezzo surface (if it exists, see below); an element of $|-K_X|$ is called an elephant, so $S \in |A|$ is a *half-elephant*. In the two cascades of Sections 1 and 2, all the del Pezzo surfaces up to codimension 3 extend in an unobstructed way to Fano 3-folds. Thus for example, we have Fano 3-folds of index 2

$$X_{10} \subset \mathbb{P}(1^2, 2, 3, 5), \quad X_{4,4} \subset \mathbb{P}(1^3, 2^2, 3), \quad \text{and} \quad X_{\mathrm{Pf}} \subset \mathbb{P}(1^4, 2^2, 3)$$

extending the del Pezzo surfaces of Table 1.1. What happens in cases of codimension 4 is a computation based on the same projection cascade that we have not had time to finish; the basic question is to find all Pfaffian 3-folds $X_{\mathrm{Pf}} \subset \mathbb{P}(1^4, 2^2, 3)$ containing a linearly embedded $\mathbb{P}^2 \hookrightarrow \mathbb{P}(1^4, 2^2, 3)$. It seems likely that the single unprojection type for del Pezzo surfaces from codimension 3 to 4 splits into Tom and Jerry cases for Fano 3-folds that are essentially different (compare [Ki], Example 6.4 and 6.8 and [P1]–[P2]).

On the other hand, the codimension 5 surface $S^{(4)} \subset \mathbb{P}(1^5, 2^2, 3)$ of (1.4) probably does not have any extension in degree 1 to a Fano 3-fold of index 2: we conjecture this because it seems hard to incorporate a new variable x_6 of degree 1 into the equations (1.4) in a nontrivial way to give a 3-fold having only terminal singularities.

3.2.2 A good half-elephant is an extremely rare beast

In contradiction to our initial hopes, most Fano 3-folds X of index 2 do not have a half-elephant, and most log del Pezzo surface S do not extend to a Fano 3-fold of index 2. An obvious necessary global condition is $P_1(X) \geq 1$, but there are also severe local restrictions on the basket of quotient singularities: each quotient singularity $\frac{1}{r}(1, a, r-a)$ in the basket of X must have $2a \cong \pm 1 \bmod r$ (so that when we rewrite the singularity as $\frac{1}{r}(2, 2a, r-2a)$, the equation of S in degree 1 can be one of the orbinates). In slightly different terms, as we saw in 1.2, a del Pezzo surface S with a singularity of type $\frac{1}{r}(a,b)$, polarised by $-K_S = A$, so that $a + b \cong 1 \bmod r$, can only extend to a Fano 3-fold of index 2 if $a+1$ or $b+1 \cong 0 \bmod r$ (compare Example 1.2), so that $\frac{1}{r}(1,a,b)$ is terminal.

These conditions restricts the several thousand baskets for index 2 Fanos to just a handful having a possible log del Pezzo surface as half-elephant. Table 3.1 is a preliminary list of a few $f = 2$ Fano 3-folds without any projections from smooth points (not complete, but possibly fairly typical). Apart from Nos. 1 and 2 that we already know from Sections 1–2, the only cases in this list having a good half-elephant are No. 12, $X_{10,12} \subset \mathbb{P}(1,2,3,5,6,7)$ and No. 14, $X_{8,10} \subset \mathbb{P}(1,2,3,4,5,5)$.

3.2.3 Fano 3-folds of index 2 and projections

Quite independently of del Pezzo surfaces, Fano 3-folds of index 2 usually have projections based on blowing up a nonsingular point, so often belong to projection cascades. Suppose that X is a Fano 3-fold in the Mori category (that is, with at worst terminal singularities) and $-K_X = 2A$ with A a Weil divisor. Consider the blowup $\sigma\colon X' \to X$ at a nonsingular point $P \in X$ with exceptional surface $E \cong \mathbb{P}^2$. Then by the adjunction formula for a blowup, $-K_{X'} = 2A'$, where $A' = \sigma^* A - E$. If $A^3 > 1$ and $P \in X$ is

1.	$X_{10} \subset \mathbb{P}(1,1,2,3,5)$	$\frac{1}{3}(2,2,1)$
2.	$X_{6,8} \subset \mathbb{P}(1,1,2,3,4,5)$	$\frac{1}{5}(1,2,4)$
3.	$X_{10,14} \subset \mathbb{P}(1,2,2,5,7,9)$	$\frac{1}{9}(2,2,7)$
4.	$X_{12,14} \subset \mathbb{P}(1,2,3,4,7,11)$	$\frac{1}{11}(2,4,7)$
5.	$X_{8,10} \subset \mathbb{P}(1,2,2,3,5,7)$	$\frac{1}{3}(2,2,1), \frac{1}{7}(2,2,5)$
6.	$X_{22} \subset \mathbb{P}(1,2,3,7,11)$	$\frac{1}{3}(2,2,1), \frac{1}{7}(2,3,4)$
7.	$X_{10,12} \subset \mathbb{P}(1,2,3,4,5,9)$	$\frac{1}{3}(2,2,1), \frac{1}{9}(2,4,5)$
8.	$X_{6,10} \subset \mathbb{P}(1,2,2,3,5,5)$	$2 \times \frac{1}{5}(2,2,3)$
9.	$X_{8,12} \subset \mathbb{P}(1,2,3,4,5,7)$	$\frac{1}{5}(1,3,4), \frac{1}{7}(2,3,4)$
10.	$X_{26} \subset \mathbb{P}(1,2,5,7,13)$	$\frac{1}{5}(2,2,3), \frac{1}{7}(1,2,6)$
11.	$X_{6,8} \subset \mathbb{P}(1,2,2,3,3,5)$	$\frac{1}{3}(2,2,1), \frac{1}{5}(2,2,3)$
12.	$X_{10,12} \subset \mathbb{P}(1,2,3,5,6,7)$	$2 \times \frac{1}{3}(2,2,1), \frac{1}{7}(1,2,6)$
13.	$X_{14,18} \subset \mathbb{P}(2,2,3,7,9,11)$	$2 \times \frac{1}{3}(2,2,1), \frac{1}{11}(2,2,9)$
14.	$X_{8,10} \subset \mathbb{P}(1,2,3,4,5,5)$	$\frac{1}{3}(2,2,1), 2 \times \frac{1}{5}(1,2,4)$
15.	$X_{12,14} \subset \mathbb{P}(2,2,3,5,7,9)$	$\frac{1}{3}(2,2,1), \frac{1}{5}(2,2,3), \frac{1}{9}(2,2,7)$
16.	$X_{10,14} \subset \mathbb{P}(2,2,3,5,7,7)$	$\frac{1}{3}(2,2,1), 2 \times \frac{1}{7}(2,2,5)$
17.	$X_{10,12} \subset \mathbb{P}(2,2,3,5,5,7)$	$2 \times \frac{1}{5}(2,2,3), \frac{1}{7}(2,2,5)$
18.	$X_{10,12} \subset \mathbb{P}(2,3,3,4,5,7)$	$4 \times \frac{1}{3}(2,2,1), \frac{1}{7}(2,3,4)$
19.	$X_{6,6} \subset \mathbb{P}(1,1,2,2,3,5)$	$\frac{1}{5}(2,2,3)$

Table 3.1: Some index 2 Fano 3-folds

general then A' is nef and big, and defines a birational contraction $X' \to \overline{X}$, where $\overline{X}$ is again a (singular) Fano 3-fold of index 2 containing a copy of $E \cong \mathbb{P}^2$ with $\overline{A}_{|E} \cong \mathcal{O}_{\mathbb{P}^2}(1)$; in general, $\overline{X}$ will have finitely many nodes on E, corresponding to the lines on X through P. The inclusion $R(\overline{X}, \overline{A}) \subset R(X, A)$ is the quasi-Gorenstein unprojection of E (in the sense of [PR] and [qG]). This means that Fano 3-folds of index 2 could in principle be constructed by starting from a variety such as one of Table 3.1, force it to contain an embedded plane $E \cong \mathbb{P}^2$ of degree 1, which can then be contracted to a nonsingular point by an unprojection. This calculation has a number of entertaining features, not the least the question of how to describe embeddings (say) $\mathbb{P}^2 \hookrightarrow \mathbb{P}(1,2,2,5,6,9)$ and codimension 2 complete intersections $X_{10,14}$ containing the image.

The nonsingular case is well known: for example, a Fano 3-fold $X \subset \mathbb{P}^7$ of index 2 and degree 6 has a projection $X \dashrightarrow \overline{X}$, that coincides with the linear projection from a point, whose image is a linear section of the Grassmannian $\mathrm{Grass}(2,5)$ containing a linearly embedded plane $\mathbb{P}^2 \subset \overline{X} \hookrightarrow \mathrm{Grass}(2,5)$. There are two different ways of embedding a plane $\mathbb{P}^2 \hookrightarrow \mathrm{Grass}(2,5)$ related to Schubert conditions, and these give rise to the two families of unprojection called Tom and Jerry, corresponding to the linear section of the Segre embedding of the hyperplane section of $\mathbb{P}^2 \times \mathbb{P}^2$, and $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$. See [P1]–[P2] for details.

3.2.4 Alternative birational treatments

Whereas Table 3.1 (or a suitable completion), together with unprojection of planes to nonsingular points, could thus provide a basis for a detailed classification of Fano 3-folds of index 2 (or at least for their numerical invariants), it is possible that many of these varieties could be studied more easily by birational methods: in this paper we have mainly concentrated on projections from nonsingular points, but each projection can presumably be completed to a Sarkisov link (Corti [Co]), giving rise to a birational description.

There are alternative birational methods, for example, based on projections from quotient singularities; these may take us outside the Mori category, as with the "Takeuchi program" used by Takagi in his study of Fano 3-folds with singular index 2 (see [T]). Most of the del Pezzo surfaces and Fano 3-folds we treat here in fact have projections of Type I. For example, $X_{6,8} \subset \mathbb{P}(1,1,2,3,4,5)_{x_1,x_2,y,z,t,u}$ has equations

$$ux_1 = A_6(x_2,y,z,t) \quad \text{and} \quad uz = B_8(x_2,y,z,t),$$

so that eliminating u gives a birational map from $X_{6,8}$ to the hypersurface

$$X_9 : (Bx - Az) \subset \mathbb{P}(1,1,2,3,4).$$

Algebraically this is a Type I projection, in fact of the simplest $Bx - Ay$ type (see [Ki], Section 2). However, from the point of view of the Sarkisov program, it is quite different: introducing the weighted ratio $x_2 : y : t$ makes the $(1,2,4)$ blowup at P, not the Kawamata blowup – it is the blowup $X_1 \to X$ with exceptional surface E of discrepancy $2/5$, so that $-K_{X_1} = 2(A - 1/5E)$. This preserves the index 2 condition, but introduces a line of A_1 singularities along the y,t axis on X_9, taking us out of the Mori category. Compare also Example 3.1.

3.2.5 How many Fano 3-folds of index ≥ 3 are there?

Fano 3-folds of index $f \geq 3$ do not form projection cascades – a blowup $X' \to X$ changes the index. Another way of seeing this is to note that for $f \geq 3$, orbifold RR applied to $\chi(-A) = 0$ gives a formula for A^3 in terms of the basket of singularities $\mathcal{B} = \{\frac{1}{r}(1, a, r-a)\}$, in much the same what that $\frac{Ac_2}{12}$ is determined by the classic orbifold RR formula for $\chi(\mathcal{O}_X)$:

$$\frac{(-K_X)c_2}{24} = 1 - \sum_{\mathcal{B}} \frac{r^2-1}{12r},$$

(see [YPG], Corollary 10.3).

The numerical invariants of a Fano 3-fold are the data going into the orbifold RR formula, giving the Hilbert series; compare [ABR], Section 4. It consists of A^3, $\frac{Ac_2}{12}$ and the basket of singularities $\mathcal{B}$; for $f \geq 3$, the first two rational numbers are determined by $\mathcal{B}$.

Suzuki's Univ. of Tokyo thesis [Su], [Su1] (based in part on Magma programming by Gavin Brown [GRD]) contains lists of the possible numerical invariants of Fano 3-folds of index $f \geq 2$. She proves in particular that $f \leq 19$, with $f = 19$ if and only if X has the same Hilbert series as weighted projective space $\mathbb{P}(3,4,5,7)$ (we conjecture of course that then $X \cong \mathbb{P}(3,4,5,7)$.) For $f = 3, \ldots, 19$, the number of possible numerical types is bounded as follows:

f	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
n_f	12	9	7	1	5	3	2	0	3	0	1	0	0	0	1	0	1
N_f	20	24	14	5	11	5	2	1									

Here n_f is a lower bound, and N_f a rough upper bound: n_f refers to the number of established cases in codimension ≤ 2, that is, weighted projective spaces, hypersurfaces or codimension 2 complete intersections. N_f is the number of candidate baskets, that includes cases in codimension 4 and 5 that we expect to be able to justify with more work, together with many less reputable candidates.[1] For $f \geq 9$ the number n_f is correct, except for an annoying (and thoroughly disreputable) candidate with $f = 10$.

Rather remarkably, there are no codimension 3 Pfaffians except for the case $S^{(6)}$ of Section 1 (see (1.1)) with $f = 2$; so far we are unable to determine which candidate cases in codimension ≥ 4 really occur (which accounts for the uncertainties in the list). By analogy with Mukai's results for nonsingular Fanos, one may speculate that Fano 3-folds in higher codimension should often be quasilinear sections of certain "key varieties", such as the weighted Grassmannians treated in Corti and Reid [CR], and there may be some convincing reason why there are few codimension ≥ 3 cases.

3.2.6 How many interesting cascades are there?

For present purposes, for a cascade to be of interest, at least one of the graded rings at the bottom must be explicitly computable; for us to get some

[1] There are currently some problems with the upper bound N_f; the rigorous bound is much larger than given here. For details, see Suzuki's thesis [Su1].

benefit, it should realistically have codimension ≤ 3. Also, we must be able to identify the surface at the top of the cascade, for example, because it has higher Fano index, so is a simpler object in a Veronese embedding. The cascades of Sections 1–2 illustrate how these conditions work in ideal settings. These conditions are restrictive, and probably only allow a small number of numerical cases. Thus, whereas each of $\overline{\mathbb{F}}_k$ for $k = 7, 9, \dots$ is the head of a tall cascade, involving $k + 4$ blowups, a moment's thought along the lines of Exercise 1.2 shows that essentially none of the surfaces in it has anticanonical ring of small codimension. They do not extend to Fano 3-folds of index 2 for the reason given in Exercise 1.2 and 3.2.2.

As another example, consider the Fano 3-fold $X_{10,12} \subset \mathbb{P}(1, 2, 3, 5, 6, 7)$ of Table 3.1, No. 12 and its half-elephant $S_{10,12} \subset \mathbb{P}(2, 3, 5, 6, 7)$. This is a surface with quotient singularities $2 \times \frac{1}{3}(2, 2)$ and $\frac{1}{7}(2, 6)$ and $K^2 = \frac{2}{21}$. Its minimal resolution $\widetilde{S} \to S$ is a surface with $K^2_{\widetilde{S}} = -1$, so is a scroll $\mathbb{F}_n$ blown up 9 times, containing two disjoint -3-curves and a disjoint $-3, -2, -2$ chain of curves arising from the $\frac{1}{7}(2, 6)$ singularity. $\widetilde{S}$ can be constructed by blowing up $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ 9 times, with 3 of the centres on each of 2 sections, and 3 other centres infinitely near points along a nonsingular arc. It seems likely that if these blowups are chosen generically, this surface contains no -1-curves not passing through the singularities. Thus there seem to be more complicated cases in which there is no cascade at all. Now, in what way is $S_{10,12} \subset \mathbb{P}(2, 3, 5, 6, 7)$ so different from $T_{6,8} \subset \mathbb{P}(1, 2, 3, 4, 5)$ of Section 2?

3.3 Mirages

Mirages have been a common phenomenon in the study of weighted projective varieties since Fletcher's thesis. The question is to construct a graded ring and a plausible candidate for a variety in weighted projective space having a given Hilbert series. It happens frequently that we can find a graded ring, but it does not correspond to a good variety, for example, because one of the variables cannot appear in any relations for reasons of degree, so that the candidate variety is a weighted cone. See p. 229 and Example 3.1 below for typical cases.

A *mirage* is an unexpected component of a Hilbert scheme, that does not consist of the varieties that we want, but of some degenerate cases, e.g., cones, varieties with index bigger than specified, or varieties condemned to have some extra singularities. The Hilbert scheme of a family of Fano 3-folds may have other components, e.g., consisting of varieties with the same numerical data, but different divisor class group. For example, the second Veronese embedding of our index 2 Fanos $X_{10} \subset \mathbb{P}(1, 1, 2, 3, 5)$ gives an extra component of the family of Fano 3-folds of index 1 with $(-K)^3 = 2 + \frac{2}{3}$.

More generally, it is an interesting open problem to understand what these mirages really are, and to find formal criteria to deal with them systematically in computer generated lists. One clue is to consider how global sections of $\mathcal{O}_X(i)$ correspond to local sections of the sheaf of algebras $\bigoplus \mathcal{O}_{X,P}(i)$ as indicated in Remark 2.4.

Example 3.1 We work out one final legend that illustrates several points. Looking for a Fano 3-fold X of Fano index $f = 2$ with a $\frac{1}{11}(2,3,8)$ terminal quotient singularity $P \in X$ by our Hilbert series methods gives (we omit a couple of lines of Magma)

$$P_X(t) = \frac{(1-t^6)(1-t^9)(1-t^{10})}{\prod(1-t^{a_i}) : i \in [1,2,2,3,3,5,11]}\,.$$

That is, the Hilbert series of the c.i. $X_{6,9,10} \subset \mathbb{P}(1,2,2,3,3,5,11)$. As with the examples on p. 229, this candidate is a mirage for two reasons: the equations cannot involve the variable of degree 11, and there is no variable of degree 8 to act as orbinate at the singularity (this kind of thing seems to happens fairly often with candidate models). Adding a generator of degree 8 to the ring gives a codimension 4 model $X \subset \mathbb{P}(1,2,2,3,3,5,8,11)$. We expect that this model works: we can eliminate the variable of degree 11 by a Type I projection $X \dashrightarrow X'$ corresponding to the $(2,3,8)$ blowup, as described in 3.2.4. This weighted blowup subtracts

$$\frac{t^{11}}{(1-t^2)(1-t^3)(1-t^8)(1-t^{11})}$$

from $P(T)$, and a little calculation

$$P_X(t) - \frac{t^{11}}{(1-t^2)(1-t^3)(1-t^8)(1-t^{11})} = \frac{1-t^6-t^8-t^9-t^{10}+t^{12}+t^{13}+t^{14}+t^{16}-t^{22}}{(1-t)(1-t^2)^2(1-t^3)^2(1-t^5)(1-t^8)}$$

gives the model for the projected variety X' as the Pfaffian with weights

$$\begin{pmatrix} 1 & 2 & 3 & 5 \\ & 3 & 4 & 6 \\ & & 5 & 7 \\ & & & 8 \end{pmatrix} \quad \text{in} \quad \mathbb{P}(1,2,2,3,3,5,8).$$

Here X' is supposed to contain $\Pi = \mathbb{P}(2,3,8) : (x = y_1 = z_1 = t = 0)$. The two ways of achieving this are: take

$$\left.\begin{array}{rl} \text{Tom:} & \text{the first } 4\times 4 \text{ block} \\ \text{or Jerry:} & \text{the first 2 rows} \end{array}\right\} \quad \text{in the ideal} \quad I_\Pi = (x, y_1, z_1, t),$$

that is, something like

$$\begin{pmatrix} x & y_1 & z_1 & a_5 \\ & z_1 & y_1^2 & b_6 \\ & & t & c_7 \\ & & & d_8 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} x & y_1 & z_1 & x^5 \\ & z_1 & y_1^2 & y_1^3 + z_1^2 \\ & & b_6' & c_7' \\ & & & d_8' \end{pmatrix},$$

so that X can be constructed either as a Tom or a Jerry unprojection (see [PR], [P1]–[P2]). As in 3.2.4, the projected variety has a line of A_1 singularities along the y_2, z_2 axis.

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