Regularity and singularity in the 3d Navier-Stokes equations

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September 2005

The incompressible Navier-Stokes equations

$$\frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = 0$$

and

$$\nabla \cdot \mathbf{u} = 0$$

with initial condition $\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x})$.

> "...it is interesting to consider another boundary conidtion which has no physical meaning" (Temam, 1985):

Periodic boundary conditions on $Q = [0, L]^3$:

$$\mathbf{u}(\mathbf{x} + L\mathbf{e}_j, t) = \mathbf{u}(\mathbf{x}, t).$$

Zero total momentum:

$$\int_Q \mathbf{u} \, \mathrm{d}\mathbf{x} = 0.$$

Kinetic energy:

$$\|\mathbf{u}\|^2 := \int_Q |\mathbf{u}(\mathbf{x})|^2 \, \mathrm{d}\mathbf{x}.$$

$$H = \Big\{ \mathbf{u} : \ \nabla \cdot \mathbf{u} = 0 \quad \text{and} \quad \|\mathbf{u}\|^2 < +\infty \Big\}.$$

$H \subset [L^2(Q)]^3$.

Enstrophy:

$$\int_Q |\mathrm{curl}\, \mathbf{u}(\mathbf{x})|^2 \, \mathrm{d}\mathbf{x} = \|D\mathbf{u}\|^2.$$

$$V = \Big\{ \mathbf{u} : \ \nabla \cdot \mathbf{u} = 0, \quad \text{and} \quad \|D\mathbf{u}\|^2 < +\infty \Big\}.$$

$V \subset [H^1(Q)]^3$.

We will also use

$$H^2 = \{ \mathbf{u} : \ \|\Delta \mathbf{u}\|^2 < +\infty \}.$$

Fourier series:

for $\mathbf{u}$:

$$\mathbf{u}(\mathbf{x},t) = \sum_{\mathbf{k}\in\dot{\mathbb{Z}}^3} \hat{\mathbf{u}}(\mathbf{k},t)\mathrm{e}^{2\pi \mathrm{i}\mathbf{k}\cdot\mathbf{x}/L}$$

$(\dot{\mathbb{Z}}^3 = \mathbb{Z}^3 \setminus (0,0,0))$ with

$$\hat{\mathbf{u}}(\mathbf{k},t) = \overline{\hat{\mathbf{u}}(-\mathbf{k},t)}.$$

We have the Poincaré inequality

$$\|\mathbf{u}\|^2 \leq \lambda_1^{-1}\|D\mathbf{u}\|^2$$

and

$$\|D\mathbf{u}\|^2 \leq \lambda_1^{-1}\|\Delta\mathbf{u}\|^2$$

with $\lambda_1 = (2\pi/L)^2$.

for the pressure:

$$p(\mathbf{x},t) = \sum_{\mathbf{k}\in\dot{\mathbb{Z}}^3} \widehat{p}(\mathbf{k},t)\mathrm{e}^{2\pi \mathrm{i}\mathbf{k}\cdot\mathbf{x}/L}$$

The NSE in Fourier form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{u}}(\mathbf{k},t) + \nu|\mathbf{k}|^2\hat{\mathbf{u}}(\mathbf{k},t) + \mathrm{i}\mathbf{k}\widehat{p}(\mathbf{k},t) + \mathrm{i}\sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}}[\hat{\mathbf{u}}(\mathbf{k}',t)\cdot\mathbf{k}'']\hat{\mathbf{u}}(\mathbf{k}'',t) = 0$$

and

$$\mathbf{k}\cdot\hat{\mathbf{u}}(\mathbf{k},t) = 0 \quad \text{and} \quad \hat{\mathbf{u}}(\mathbf{k},0) = \hat{\mathbf{u}}_0(\mathbf{k}).$$

Eliminating the pressure gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{u}}(\mathbf{k},t) + \nu|\mathbf{k}|^2\hat{\mathbf{u}}(\mathbf{k},t) + \left(\mathbf{I} - \frac{\mathbf{k}\mathbf{k}^T}{|\mathbf{k}|^2}\right)\mathrm{i}\sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}}[\hat{\mathbf{u}}(\mathbf{k}',t)\cdot\mathbf{k}'']\hat{\mathbf{u}}(\mathbf{k}'',t) = 0$$

or

$$\frac{\partial \mathbf{u}}{\partial t} - \nu\boldsymbol{\Delta}\mathbf{u} + \Pi[(\mathbf{u}\cdot\nabla)\mathbf{u}] = 0.$$

where $(\Pi\phi)\hat{}(\mathbf{k}) = (\mathbf{I} - (\mathbf{k}\mathbf{k}^T/|\mathbf{k}|^2))\widehat{\phi}(\mathbf{k})$.

Truncate the Fourier series expansion of $\mathbf{u}$:

$$\mathbf{u}_N(\mathbf{x},t) = \sum_{|\mathbf{k}|\le N} \hat{\mathbf{u}}_N(\mathbf{k},t)\mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\cdot\mathbf{x}/L}$$

to obtain the <u>Galerkin approximation</u>:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}&\hat{\mathbf{u}}_N(\mathbf{k},t) + \nu|\mathbf{k}|^2\hat{\mathbf{u}}_N(\mathbf{k},t)\\ &+ \left(\mathbf{I} - \frac{\mathbf{k}\mathbf{k}^T}{|\mathbf{k}|^2}\right)\mathrm{i}\sum_{\substack{\mathbf{k}'+\mathbf{k}''=\mathbf{k}\\ |\mathbf{k}'|,|\mathbf{k}''|\le N}} [\hat{\mathbf{u}}_N(\mathbf{k}',t)\cdot\mathbf{k}'']\hat{\mathbf{u}}_N(\mathbf{k}'',t)\\ &\quad = 0.\end{aligned}$$

This finite-dimensional system of ODEs has a unique solution

$$\{\hat{\mathbf{u}}_N(\mathbf{k},t)\}_{|\mathbf{k}|\le|N|}$$

that exists for all $t \ge 0$, and

$$\|\mathbf{u}_N(t)\|^2 + \nu\int_0^t \|D\mathbf{u}_N(s)\|^2\,\mathrm{d}s \le \|\mathbf{u}(0)\|^2.$$

The Bolzano-Weierstrass Theorem

Given a bounded sequence of real numbers

$$|a_j| \leq M \qquad \text{for all} \qquad j = 1, 2, \ldots$$

one can find a subsequence $\alpha_n = a_{j_n}$

(j_n are integers such that $j_{n+1} > j_n$)

such that

$$\alpha_n \to a^*, \qquad \text{where} \qquad |a^*| \leq M.$$

Given a sequence of functions $\mathbf{u}_N$ with

$$\int_Q |\mathbf{u}_N(\mathbf{x})|^2 \, \mathrm{d}\mathbf{x} = \|\mathbf{u}_N\|^2 \le M^2$$

there exists a subsequence $\mathbf{u}_{N_j}$ such that every Fourier coefficient converges:

$$\mathbf{u}_{N_j}(\mathbf{k}) \to \mathbf{u}(\mathbf{k})$$

for every $\mathbf{k}$ as $j \to \infty$.

But for the sequence

$$\mathbf{u}_N(\mathbf{x}) = \mathrm{e}^{2\pi \mathrm{i}(N\mathbf{e}_1)\cdot\mathbf{x}/L} = \mathrm{e}^{2N\pi \mathrm{i}x/L}$$

we have

$$\mathbf{u}_N(\mathbf{k}) \to 0 \qquad \text{as} \qquad N \to \infty$$

for every $\mathbf{k}$.

Weak convergence of $\mathbf{u}_N$ to $\mathbf{u}$,

$$\mathbf{u}_N \rightharpoonup \mathbf{u} :$$

for every $\mathbf{v} \in H$,

$$\int_Q \mathbf{u}_N(x) \cdot \mathbf{v}(\mathbf{x})\, \mathrm{d}\mathbf{x} \to \int_Q \mathbf{u}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x})\, \mathrm{d}\mathbf{x}.$$

or

the Fourier coefficients of $\mathbf{u}_N$ converge to those of $\mathbf{u}$ and the kinetic energy of $\{\mathbf{u}_N\}$ is uniformly bounded.

Inequalities are preserved:

$$\mathbf{u}_n \rightharpoonup \mathbf{u} \qquad \Rightarrow \qquad \|\mathbf{u}\| \leq \liminf_{n\to\infty} \|\mathbf{u}_n\|.$$

but equalities are not (in general).

Estimates uniform in N:

$$\|\mathbf{u}_N(t)\|^2 + \nu \int_0^t \|D\mathbf{u}_N(s)\|^2 \, \mathrm{d}s \le \|\mathbf{u}(0)\|^2.$$

With

$$\mathbf{b}_N(\mathbf{k}) = \left(\mathrm{I} - \frac{\mathbf{k}\mathbf{k}^T}{|\mathbf{k}|^2}\right) \sum_{\substack{\mathbf{k}'+\mathbf{k}''=\mathbf{k} \\ |\mathbf{k}'|,|\mathbf{k}''|\le N}} [\hat{\mathbf{u}}_N(\mathbf{k}',t)\cdot\mathbf{k}'']\hat{\mathbf{u}}_N(\mathbf{k}'',t)$$

we have only

$$|\widehat{\mathbf{b}}_N(\mathbf{k})| \le \|\mathbf{u}_N\| \|D\mathbf{u}_N\|.$$

So integrate in time:

$$\hat{\mathbf{u}}(\mathbf{k},t) = \hat{\mathbf{u}}(\mathbf{k},t_0) - \nu \int_{t_0}^t |\mathbf{k}|^2 \hat{\mathbf{u}}(\mathbf{k},s)\, \mathrm{d}s$$
$$- \int_{t_0}^t \left(\mathbf{I} - \frac{\mathbf{k}\mathbf{k}^T}{|\mathbf{k}|^2}\right) \mathrm{i} \sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}} [\hat{\mathbf{u}}(\mathbf{k}',s)\cdot\mathbf{k}'']\hat{\mathbf{u}}(\mathbf{k}'',s)\, \mathrm{d}s.$$

Global existence of weak solutions:

A weak solution is a function $\mathbf{u}(\mathbf{x}, t)$, with

$$\mathbf{u} \in L^\infty(0, T; H) \cap L^2(0, T; V)$$

that satisfies the time integrated equations for each individual Fourier component.

Theorem. *There exists at least one weak solution such that the energy inequality*

$$\|\mathbf{u}(t)\|^2 + \nu \int_0^t \|D\mathbf{u}(s)\|^2 \, \mathrm{d}s \leq \|\mathbf{u}_0\|^2 \qquad (1)$$

holds. Furthermore $(\mathbf{u}(t), \phi)$ *is continuous for every* $\phi \in H$.

Given an initial condition with finite kinetic energy, there exists at least one solution whose kinetic energy remains finite, and its Fourier coefficients evolve continuously in time.

Local existence of strong solutions:

$$\left|\int [(\mathbf{u}\cdot\nabla)\mathbf{v}]\cdot\mathbf{w}\,\mathrm{d}\mathbf{x}\right| \le c_2\|D\mathbf{u}\|\|D\mathbf{v}\|^{1/2}\|\Delta\mathbf{v}\|^{1/2}\|\mathbf{w}\|.$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|D\mathbf{u}\|^2 + \nu\|\Delta\mathbf{u}\|^2 \le \frac{c_2}{\nu^3}\|D\mathbf{u}\|^6$$

and so

$$\|D\mathbf{u}(t)\|^2 \le \frac{\|D\mathbf{u}_0\|^2}{\sqrt{1-2kt\|D\mathbf{u}_0\|^4}}$$

where $k = c_2/\nu^3$.

Theorem. *Given* $\mathbf{u}_0$ *with* $\|D\mathbf{u}_0\| < \infty$ *there exists a time* $T > 0$ *such that*

$$\mathbf{u} \in L^\infty(0,T;V)\cap L^2(0,T;H^2)$$

and the solution is unique on $[0,T)$*.*

Given an initial condition with finite enstrophy, there exists a solution whose enstrophy remains finite for some (possible small) time interval, and while this is the case there are no other solutions.

Uniqueness of strong solutions:

Let $\mathbf{w} = \mathbf{u} - \mathbf{v}$, so that

$$\frac{\partial \mathbf{w}}{\partial t} - \nu \Delta \mathbf{w} + \Pi(\mathbf{w} \cdot \nabla)\mathbf{u} + \Pi(\mathbf{v} \cdot \nabla)\mathbf{w} = 0.$$

Take the inner product with $\mathbf{w}$, and integrate:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{w}\|^2 + \nu\|D\mathbf{w}\|^2 = -\int [(\mathbf{w} \cdot \nabla)\mathbf{u}] \cdot \mathbf{w}\,\mathrm{d}\mathbf{x}$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{w}\|^2 + \nu\|D\mathbf{w}\|^2 \leq \frac{k^2}{\nu}\|D\mathbf{u}\|\|\Delta\mathbf{u}\|\|\mathbf{w}\|^2.$$

Ignoring the $\nu\|D\mathbf{w}\|^2$ term gives

$$\|\mathbf{w}(t)\|^2 \leq \exp\left(\frac{k^2}{\nu}\int_0^t \|D\mathbf{u}(s)\|\|\Delta\mathbf{u}(s)\|\,\mathrm{d}s\right)\|\mathbf{w}(0)\|^2$$

and

$$\int_0^t \|D\mathbf{u}(s)\|\|\Delta\mathbf{u}(s)\|\,\mathrm{d}s$$
$$\leq \left(\int_0^t \|D\mathbf{u}(s)\|^2\right)^{1/2}\left(\int_0^t \|\Delta\mathbf{u}(s)\|^2\right)^{1/2}.$$

Integral bounds and continuity

$$\|D^s\mathbf{u}\|^2 = \sum_{\mathbf{k}} |\mathbf{k}|^{2s}|\hat{\mathbf{u}}(\mathbf{k})|^2 < \infty$$

$$\Downarrow$$

$$\mathbf{u}(\mathbf{x}) = \sum_{\mathbf{k}} \hat{\mathbf{u}}(\mathbf{k})\mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\cdot\mathbf{x}/L}$$

is continuous, provided that $s > 3/2$.

Similarly if

$$\|D^s\mathbf{u}\|^2 = \sum_{\mathbf{k}} |\mathbf{k}|^{2s}|\hat{\mathbf{u}}(\mathbf{k})|^2 < \infty$$

for $s > k + 3/2$ then all derivatives of $\mathbf{u}$ up to order k are continuous.

If $\|D^s\mathbf{u}\|^2 < +\infty$ for every s then $\mathbf{u}$ is smooth.

If $\|D^s\mathbf{u}\|^2 \leq Ms!b^{-s}$ then $\mathbf{u}$ is analytic.

If $\mathbf{u}$ is a strong solution then $\mathbf{u}$ is analytic for all $t > 0$ (Foias & Temam, 1989).

Write $\|\mathbf{u}\|_\tau = \|\mathrm{e}^{\tau|\nabla|}\mathbf{u}\|$ where

$$\mathrm{e}^{\tau|\nabla|}\mathbf{u} = \sum_{\mathbf{k}} \mathrm{e}^{\tau|\mathbf{k}|}\hat{\mathbf{u}}(\mathbf{k})\mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\cdot\mathbf{x}/L}.$$

$$\|\mathbf{u}\|_\tau^2 \leq M \quad \Rightarrow \quad \|D^s\mathbf{u}\|^2 \leq \frac{Ms!}{(2\tau)^s}.$$

$$\left|\int [(\mathbf{u}\cdot\nabla]\mathbf{v})\cdot\mathrm{e}^{2\tau|\nabla|}\boldsymbol{\Delta}\mathbf{w}\right| \leq c\|D\mathbf{u}\|_\tau^{1/2}\|\boldsymbol{\Delta}\mathbf{u}\|_\tau^{1/2}\|D\mathbf{v}\|_\tau\|\boldsymbol{\Delta}\mathbf{w}\|_\tau.$$

Multiplying the equation by $-\mathrm{e}^{2t|\nabla|}\boldsymbol{\Delta}\mathbf{u}$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|D\mathbf{u}\|_t^2 + \nu\|\boldsymbol{\Delta}\mathbf{u}\|_t^2 \leq \frac{2}{\nu}\|D\mathbf{u}\|_t^2 + \frac{c}{\nu^3}\|D\mathbf{u}\|_t^6.$$

Which yields

$$\|\mathrm{e}^{t|\nabla|}D\mathbf{u}\|^2 \leq \frac{\|D\mathbf{u}(0)\|}{\sqrt{1 - ct\|D\mathbf{u}(0)\|^4/\nu}}.$$

From

$$\frac{\mathrm{d}}{\mathrm{d}t}\|D\mathbf{u}\|^2 + \nu\|\Delta\mathbf{u}\|^2 \leq \frac{c_2}{\nu^3}\|D\mathbf{u}\|^6$$

and

$$\|D\mathbf{u}(t)\|^2 \leq \frac{\|D\mathbf{u}_0\|^2}{\sqrt{1 - 2kt\|D\mathbf{u}_0\|^4}}$$

we have...

<u>Rate of blowup:</u>

If $\|D\mathbf{u}(t)\| \to \infty$ as $t \to T$ then

$$\|D\mathbf{u}(t)\|^2 \geq \frac{1}{\sqrt{2k(T-t)}}.$$

<u>Global existence for small initial data:</u>

Theorem. *If* $\mathbf{u}_0$ *satisfies*

$$\|D\mathbf{u}_0\|^2 \leq c_2^{-1/2}\nu^2\lambda_1^{1/2}$$

then there is a unique strong solution which exists for all time.

The set of singular times I

(Leray, 1934)

Define

$$\Sigma = \{t \in \mathbb{R}^+ : \ \|\mathsf{D}\mathbf{u}(t)\| = \infty\}.$$

Then

$$\nu \int_0^\infty \|\mathsf{D}\mathbf{u}(t)\|^2 \, \mathsf{d}t \le \|\mathbf{u}_0\|^2$$

implies that $\mu(\Sigma) = 0$.

Furthermore

$$\mu\left(t \in \mathbb{R}^+ : \ \nu\|\mathsf{D}\mathbf{u}(s)\|^2 \ge \alpha\|\mathbf{u}_0\|^2\right) \le \alpha^{-1};$$

in particular

$$\begin{aligned}\mu(t \in \mathbb{R}^+ : \ \|D\mathbf{u}(s)\|^2 > c_2^{-1/2}\nu^2\lambda_1^{1/2}) \\ \le \frac{c_2^{1/2}}{\nu^3\lambda_1^{1/2}}\|\mathbf{u}_0\|^2 := T^*.\end{aligned}$$

Every weak solution is eventually strong.

Hausdorff measure & Hausdorff dimension

Define

$$\mu(X, d, \epsilon) \\ = \text{inf}\left\{\sum_i r_i^d : r_i \leq \epsilon \text{ and } X \subseteq \cup_i B(x_i, r_i)\right\},$$

where the $B(x_i, r_i)$ are balls with radius r_i.

The d-dimensional Hausdorff measure of X, $\mathscr{H}^d(X)$, is given by

$$\mathscr{H}^d(X) = \lim_{\epsilon \to 0} \mu(X, d, \epsilon).$$

The set of singular times II

(Leray, 1934; Scheffer, 1976)

We can write

$$[0,\infty) \setminus \Sigma = \bigcup_{q=1}^{\infty} J_q \cup (T,\infty),$$

where $J_q = (l_q, r_q)$ and $T \leq T^*$.

For $t \in J_q$ we have (with $a = \nu^3/2c_2$)

$$\|D\mathbf{u}(t)\|^2 \geq a(r_q - t)^{-1/2}.$$

Then

$$\sum_q |J_q|^{1/2} = \sum_q (r_q - l_q)^{1/2} > +\infty.$$

A simple argument gives

$$\mathscr{H}^{1/2}(\Sigma) = 0.$$

The set of space-time singularities

(Cafarelli, Kohn, & Nirenberg, 1982)

$(\mathbf{x}, t)$ is *regular* if $\mathbf{u}(\mathbf{x}, t)$ is essentially bounded in a neighbourhood of $(\mathbf{x}, t)$. Define

$Q_r(\mathbf{x}, t) = \{(\mathbf{y}, s) : \ |\mathbf{y}-\mathbf{x}| < r, \ t-r^2 < s < t+r^2\}$

Theorem. *There exists an $\epsilon > 0$ such that if $(\mathbf{u}, p)$ is a suitable weak solution and*

$$\limsup_{r\to 0} \frac{1}{r} \int_{Q_r(\mathbf{x},t)} |\nabla \mathbf{u}(\mathbf{x}, t)|^2 \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \leq \epsilon$$

then $(\mathbf{x}, t)$ is a regular point.

It follows that if

$$S = \{(\mathbf{x}, t) : \ (\mathbf{x}, t) \text{ is not regular}\}.$$

then

$$\mathscr{H}^1(S) = 0.$$