Vanishing enstrophy dissipation in 2D NS turbulence in the inviscid limit

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Outline

- The fundamental assumption of Batchelor's theory of freely evolving 2D NS turbulence: $\nu\langle|\nabla\omega|^2\rangle \longrightarrow \chi > 0$ as $\nu \longrightarrow 0$, i.e. enstrophy dissipation remains finite in the inviscid limit.

- A *rigorous* bound for χ (Tran, Phys. Fluids 2005):

$$\nu\langle|\nabla\omega|^2\rangle \le \|\omega(0)\|_\infty \langle\omega(0)^2\rangle.$$

- But sharper (*semi-analytic*) bounds imply

$$\nu\langle|\nabla\omega|^2\rangle \longrightarrow 0 \text{ as } \nu \longrightarrow 0.$$

- Concluding remarks

Batchelor's theory
Analogy between 3D enstrophy & 2D palinstrophy production

$$\omega_t + (u \cdot \nabla)\omega - (\omega \cdot \nabla)u = \nu \Delta \omega$$

<u>3D</u> : $$\frac{1}{2}\frac{d}{dt}\langle|\omega|^2\rangle = \langle \omega \cdot (\omega \cdot \nabla)u \rangle - \nu \langle |\nabla \omega|^2 \rangle$$

$\nu \searrow \langle|\omega|^2\rangle \nearrow$ such that $\nu\langle|\omega|^2\rangle \longrightarrow \chi > 0$ as $\nu \longrightarrow 0$

<u>2D</u> : $$\frac{1}{2}\frac{d}{dt}\langle|\nabla\omega|^2\rangle = \langle \Delta\omega (u \cdot \nabla)\omega \rangle - \nu \langle |\Delta \omega|^2 \rangle$$

$\nu \searrow \langle|\nabla\omega|^2\rangle \nearrow$ such that $\nu\langle|\nabla\omega|^2\rangle \longrightarrow \chi > 0$ as $\nu \longrightarrow 0$

An inertial range $Z(k) = c\chi^{2/3}k^{-1}$ extending to $k_\nu = \chi^{1/6}/\nu^{1/2}$ becomes established.

A bad analogy

The production of 3D enstrophy & that of 2D palinstrophy by advective transfer are only *phenomenologically* analogous. There is a huge mathematical difference.

- 3D: $\langle \omega \cdot (\omega \cdot \nabla) u \rangle \longrightarrow \langle |u||\omega||\nabla\omega| \rangle$

  good$\nearrow$ *bad* $\nwarrow$*ugly*

- 2D: $\langle \Delta\omega (u \cdot \nabla)\omega \rangle \longrightarrow \langle |\nabla u||\omega||\Delta\omega| \rangle$

  good$\nearrow$ *good* $\nwarrow$*ugly*

An upper bound for the enstrophy dissipation

Vorticity equation $\omega_t + J(\psi, \omega) = \nu \Delta \omega$

Palinstrophy evolution $\langle \Delta\omega \cdots \rangle$

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\langle |\nabla\omega|^2 \rangle &= \langle \Delta\omega J(\psi,\omega) \rangle - \nu \langle |\Delta\omega|^2 \rangle \\
&= \langle \omega J(\psi_x, \omega_x) \rangle + \langle \omega J(\psi_y, \omega_y) \rangle - \nu \langle |\Delta\omega|^2 \rangle \\
&\leq \langle |\omega| \left(|\nabla\psi_x||\nabla\omega_x| + |\nabla\psi_y||\nabla\omega_y| \right) \rangle - \nu \langle |\Delta\omega|^2 \rangle \\
&\leq \|\omega\|_\infty \langle \omega^2 \rangle^{1/2} \langle |\Delta\omega|^2 \rangle^{1/2} - \nu \langle |\Delta\omega|^2 \rangle \\
&= \langle |\Delta\omega|^2 \rangle^{1/2} \left(\|\omega\|_\infty \langle \omega^2 \rangle^{1/2} - \nu \langle |\Delta\omega|^2 \rangle^{1/2} \right) \\
&\leq \frac{\langle |\Delta\omega|^2 \rangle^{1/2}}{\langle \omega^2 \rangle^{1/2}} \left(\|\omega\|_\infty \langle \omega^2 \rangle - \nu \langle |\nabla\omega|^2 \rangle \right),
\end{aligned}
$$

where $\langle |\Delta\omega|^2 \rangle^{1/2} \geq \langle |\nabla\omega|^2 \rangle / \langle \omega^2 \rangle^{1/2}$ has been used. It follows that $\nu \langle |\nabla\omega|^2 \rangle \leq \|\omega(0)\|_\infty \langle \omega(0)^2 \rangle$ for $t > 0$ if it holds for $t = 0$.

The idea

- The bound $\nu\langle|\nabla\omega|^2\rangle \le \|\omega(0)\|_\infty \langle\omega(0)^2\rangle$ is excessive, a direct consequence of the highly generous inequality
$$1 \le \frac{\langle|\Delta\omega|^2\rangle^{1/2}\langle\omega^2\rangle^{1/2}}{\langle|\nabla\omega|^2\rangle}(\longrightarrow \infty \text{ as } \nu \longrightarrow 0).$$

- The employment of this inequality severely undermines the palinstrophy dissipation term, effectively giving an edge to the production term.

- By sacrificing a reasonable degree of rigour, we can derive sharper estimates for $\langle|\Delta\omega|^2\rangle$, allowing us to deduce that
$$\nu\langle|\nabla\omega|^2\rangle \longrightarrow 0 \text{ as } \nu \longrightarrow 0.$$

The main argument

$$\frac{1}{2}\frac{d}{dt}\langle|\nabla\omega|^2\rangle \le \langle|\Delta\omega|^2\rangle^{1/2}\left(\|\omega\|_\infty\langle\omega^2\rangle^{1/2} - \frac{\langle|\Delta\omega|^2\rangle^{1/2}}{\langle|\nabla\omega|^2\rangle}\nu\langle|\nabla\omega|^2\rangle\right)$$

- $\nu \to 0$: (i) If $\nu\langle|\nabla\omega|^2\rangle \longrightarrow \chi > 0$, then $\langle|\Delta\omega|^2\rangle^{1/2}/\langle|\nabla\omega|^2\rangle < \infty$.

  (ii) If $\langle|\Delta\omega|^2\rangle^{1/2}/\langle|\nabla\omega|^2\rangle \longrightarrow \infty$, then $\nu\langle|\nabla\omega|^2\rangle \longrightarrow 0$.

- We will show that for all spectra of interest,

$$\frac{\langle|\Delta\omega|^2\rangle^{1/2}}{\langle|\nabla\omega|^2\rangle} \longrightarrow \infty.$$

  It follows that the evolution of the turbulence along the assumed spectra necessarily requires $\nu\langle|\nabla\omega|^2\rangle \longrightarrow 0$.

Admissible vorticity spectra

We allow for mean-square vorticity spectra of the form

$$Z(k) = \begin{cases} Ck^{-\beta} & \text{if } k_0 \le k \le k_\nu, \\ D(k) & \text{if } k \ge k_\nu, \end{cases}$$

where $1 \le \beta \le 3$ and $k_\nu \longrightarrow \infty$ as $\nu \longrightarrow 0$ and

$$C \int_{k_0}^{k_\nu} k^{2-\beta} \, dk = c^{-1} \int_{k_\nu}^{\infty} k^2 D(k) \, dk.$$

Here c is a constant of order unity. This means that in the inviscid limit the inertial range becomes infinitely wide and that the enstrophy is most strongly dissipated around k_ν or beyond. For $c = 1$, this equation states that as much enstrophy dissipation is occurring for $k > k_\nu$ as for $k < k_\nu$. We could indeed *define* k_ν by the condition $c = 1$.

Why $1 \leq \beta \leq 3$?

- The case $\beta > 3$ means $\langle|\nabla\omega|^2\rangle < \infty$ since the spectrum of $\langle|\nabla\omega|^2\rangle$, $k^2 Z(k)$, is steeper than k^{-1}. It follows that $\nu\langle|\nabla\omega|^2\rangle \longrightarrow 0$.

- The case $\beta < 1$ is unphysical since this would correspond to $\langle|\omega|^2\rangle/\langle|\nabla\psi|^2\rangle \to \infty$. But the decay of a vorticity reservoir with $\langle|\omega|^2\rangle/\langle|\nabla\psi|^2\rangle = s^2$ is constrained by $\langle|\omega|^2\rangle/\langle|\nabla\psi|^2\rangle \leq s^2$ (Tran & Shepherd, Physica D 2002), which forbids such a divergence.

The case $1 < \beta < 3$

$$\langle|\omega|^2\rangle = \int Z(k)\,dk = \frac{C}{\beta-1}k_0^{1-\beta} \Rightarrow C = (\beta-1)k_0^{\beta-1}\langle|\omega|^2\rangle$$

$$\langle|\nabla\omega|^2\rangle = C\int_{k_0}^{k_\nu} k^{2-\beta}\,dk + \int_{k_\nu}^{\infty} k^2 D(k)\,dk = C\frac{1+c}{3-\beta}\,k_\nu^{3-\beta}$$

$$\begin{aligned}\langle|\Delta\omega|^2\rangle &= C\int_{k_0}^{k_\nu} k^{4-\beta}\,dk + \int_{k_\nu}^{\infty} k^4 D(k)\,dk \\ &\geq C\int_{k_0}^{k_\nu} k^{4-\beta}\,dk + k_\nu^2\int_{k_\nu}^{\infty} k^2 D(k)\,dk \\ &= \frac{C}{5-\beta}\,k_\nu^{5-\beta} + \frac{Cc}{3-\beta}\,k_\nu^{5-\beta}\end{aligned}$$

$$\Rightarrow \quad \frac{\langle|\Delta\omega|^2\rangle^{1/2}}{\langle|\nabla\omega|^2\rangle} \geq C^{-1/2}k_\nu^{(\beta-1)/2} \longrightarrow \infty$$

The classical case $\beta = 1$

$$\langle|\omega|^2\rangle = \int Z(k)\,dk = C\,\ln(k_\nu/k_0) \Rightarrow C = \langle|\omega|^2\rangle / \ln(k_\nu/k_0)$$

$$\langle|\nabla\omega|^2\rangle = \frac{(1+c)\langle|\omega|^2\rangle}{2\ln(k_\nu/k_0)}\,k_\nu^2$$

$$\langle|\Delta\omega|^2\rangle \geq \frac{(1+2c)^{1/2}\langle|\omega|^2\rangle^{1/2}}{2[\ln(k_\nu/k_0)]^{1/2}}\,k_\nu^2$$

$$\Rightarrow \quad \frac{\langle|\Delta\omega|^2\rangle^{1/2}}{\langle|\nabla\omega|^2\rangle} \geq \frac{[\ln(k_\nu/k_0)]^{1/2}}{\langle|\omega|^2\rangle^{1/2}} \longrightarrow \infty$$

An improved bound for the enstrophy dissipation ($\beta = 1$)

$$\frac{1}{2}\frac{d}{dt}\langle|\nabla\omega|^2\rangle \le \langle|\Delta\omega|^2\rangle^{1/2}\left(\|\omega\|_\infty\langle\omega^2\rangle^{1/2} - \frac{\langle|\Delta\omega|^2\rangle^{1/2}}{\langle|\nabla\omega|^2\rangle}\nu\langle|\nabla\omega|^2\rangle\right)$$

$$\le \left(\frac{\langle|\Delta\omega|^2\rangle \ln(k_\nu/k_0)}{\langle|\omega|^2\rangle}\right)^{1/2}\left(\frac{\|\omega\|_\infty\langle\omega^2\rangle}{[\ln(k_\nu/k_0)]^{1/2}} - \nu\langle|\nabla\omega|^2\rangle\right)$$

$$\Rightarrow \nu\langle|\nabla\omega|^2\rangle \le \frac{\|\omega\|_\infty\langle\omega^2\rangle}{[\ln(k_\nu/k_0)]^{1/2}} \approx \frac{\|\omega\|_\infty\langle\omega^2\rangle}{(\ln Re)^{1/2}}$$

$$Re = \frac{\|\omega\|_\infty}{\nu s^2} \approx \frac{k_\nu^2}{k_0^2} \quad \text{and} \quad s^2 = \langle\omega^2\rangle/\langle|\nabla\psi|^2\rangle$$

Concluding remarks

- The palinstrophy production rate is $\propto \langle|\Delta\omega|^2\rangle^{1/2}$ (relatively weak) while the palinstrophy dissipation rate is $\nu\langle|\Delta\omega|^2\rangle$.

- The balance between palinstrophy production and viscous dissipation is delicate. In the limit $\nu \longrightarrow 0$, the dissipation remains sufficiently strong to keep the growth of palinstrophy in check.

- As $Re \longrightarrow \infty$, $\langle|\nabla\omega|^2\rangle \longrightarrow \infty$ but not sufficiently rapidly to maintain a finite $\nu\langle|\nabla\omega|^2\rangle$. For Batchelor's spectrum, $\propto k^{-1}$, $\nu\langle|\nabla\omega|^2\rangle \leq \|\omega\|_\infty \langle\omega^2\rangle(\ln Re)^{-1/2}$. But for any steeper spectrum the dissipation vanishes much more rapidly.

Further remarks

In the inviscid limit, equation

$$\frac{1}{2}\frac{d}{dt}\langle|\nabla\omega|^2\rangle \leq \langle|\Delta\omega|^2\rangle^{1/2}\left(\|\omega\|_\infty\langle\omega^2\rangle^{1/2} - \nu\langle|\Delta\omega|^2\rangle^{1/2}\right)$$

implies $\nu\langle|\Delta\omega|^2\rangle^{1/2}\langle\omega^2\rangle^{1/2} \leq \|\omega\|_\infty\langle\omega^2\rangle$. This quantity has the dimension of enstrophy dissipation and may remain finite. The dimensionless quantity

$$\nu\langle|\Delta\omega|^2\rangle^{1/2}/\langle\omega^2\rangle$$

could be used in dimensional analyses.

The present result would contradict Batchelor's assumption but might not invalidate some aspects of his theory. In fact, his dimensional arguments would go through if the limiting value χ is replaced by the achievable maximum enstrophy dissipation.