Dynamo action in flows with cat's eyes

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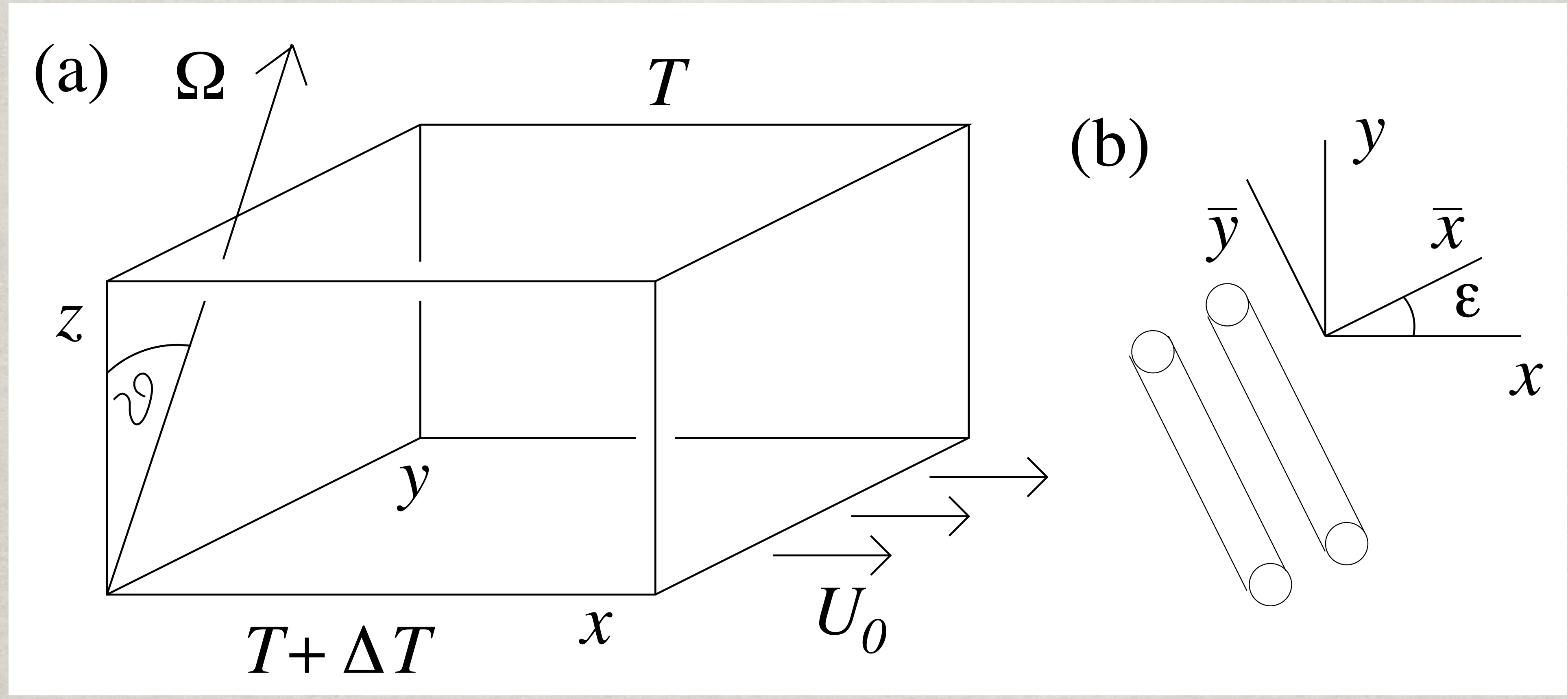
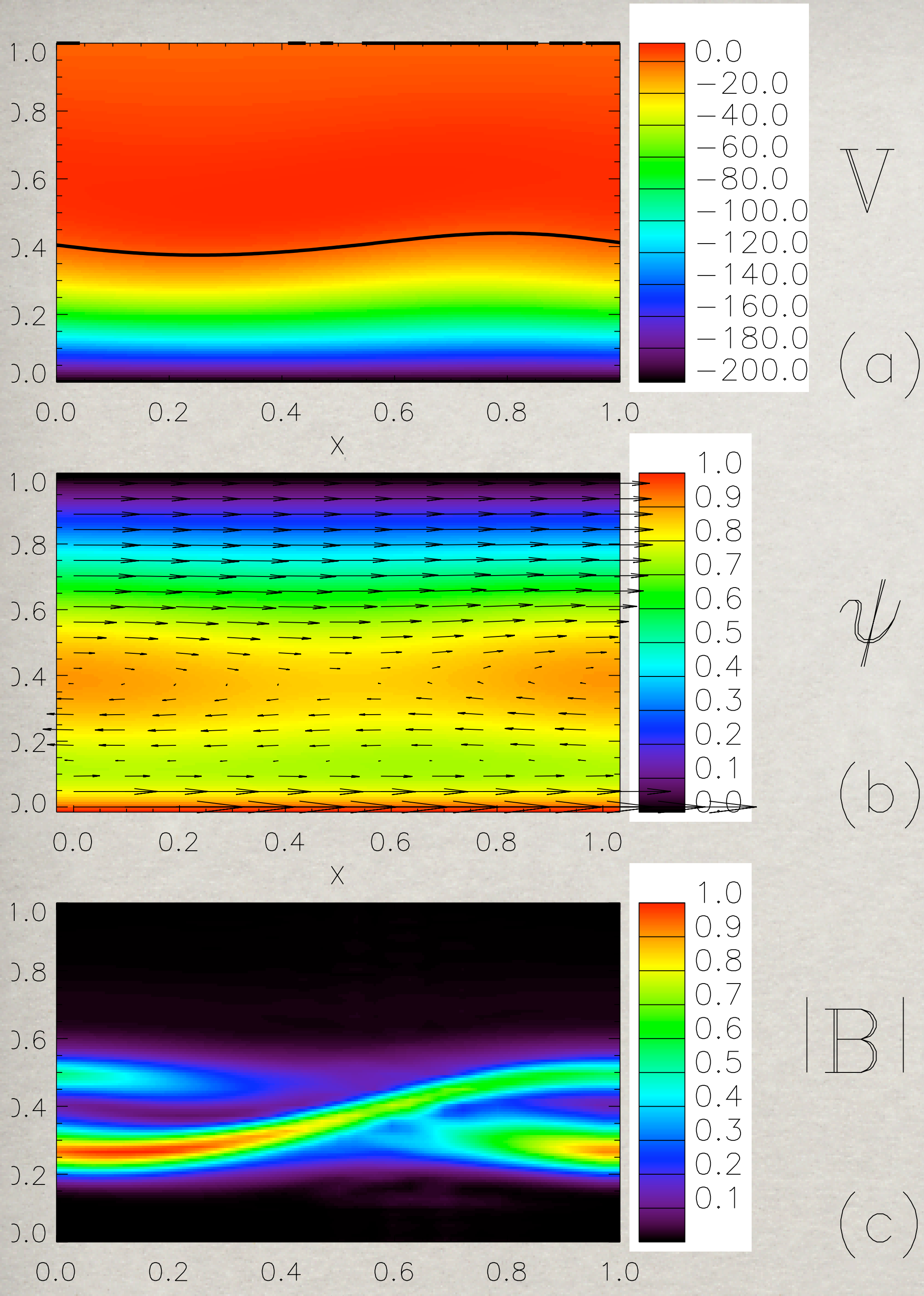
http://www.maths.ex.ac.uk/~adg

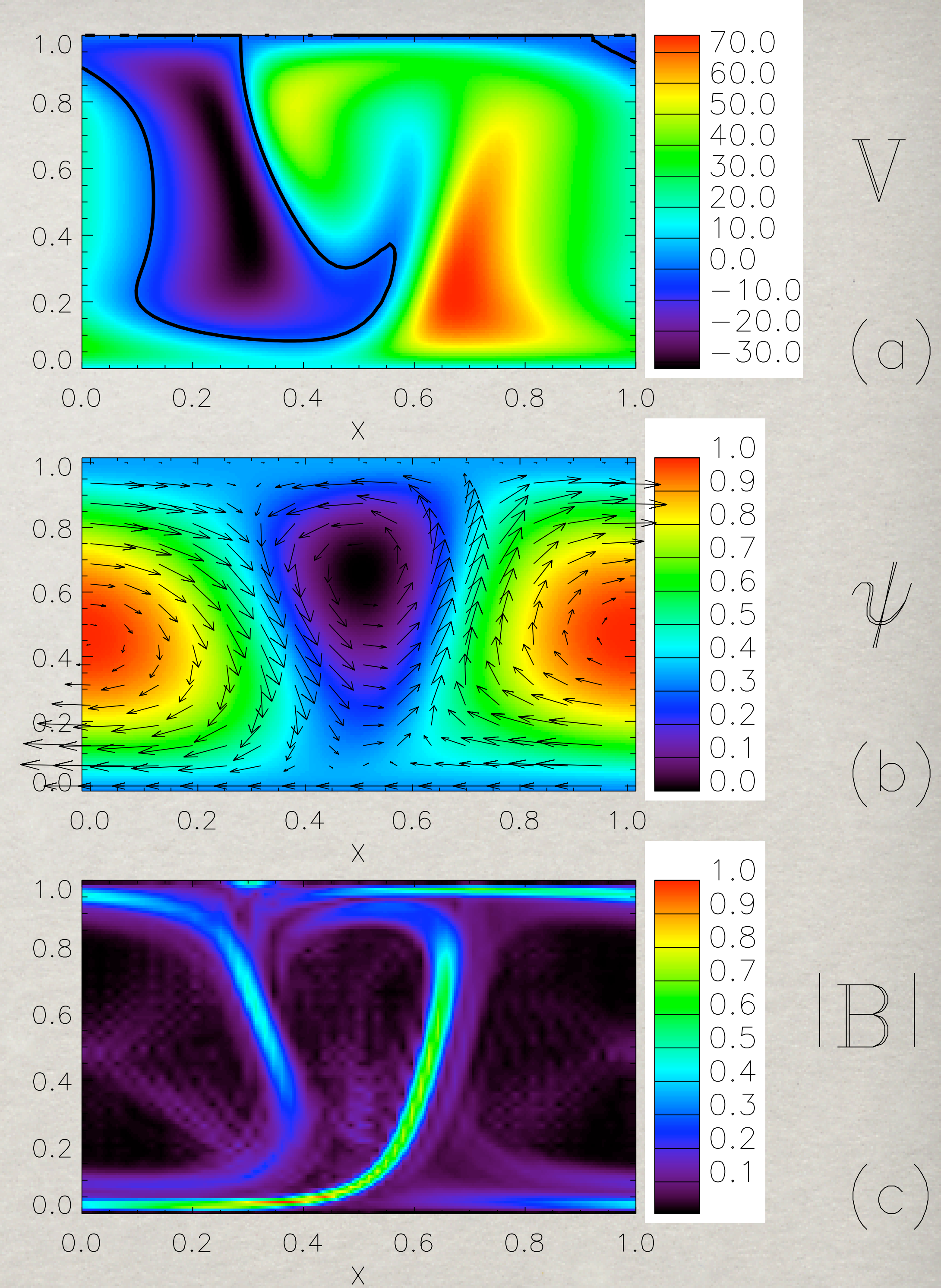
Dynamos in flows generated by Ekman instability



Flow driven by shear or convection in a rotating frame. Ekman layer can become unstable giving cat's eye rolls (Ponty, Gilbert, Soward 2001).

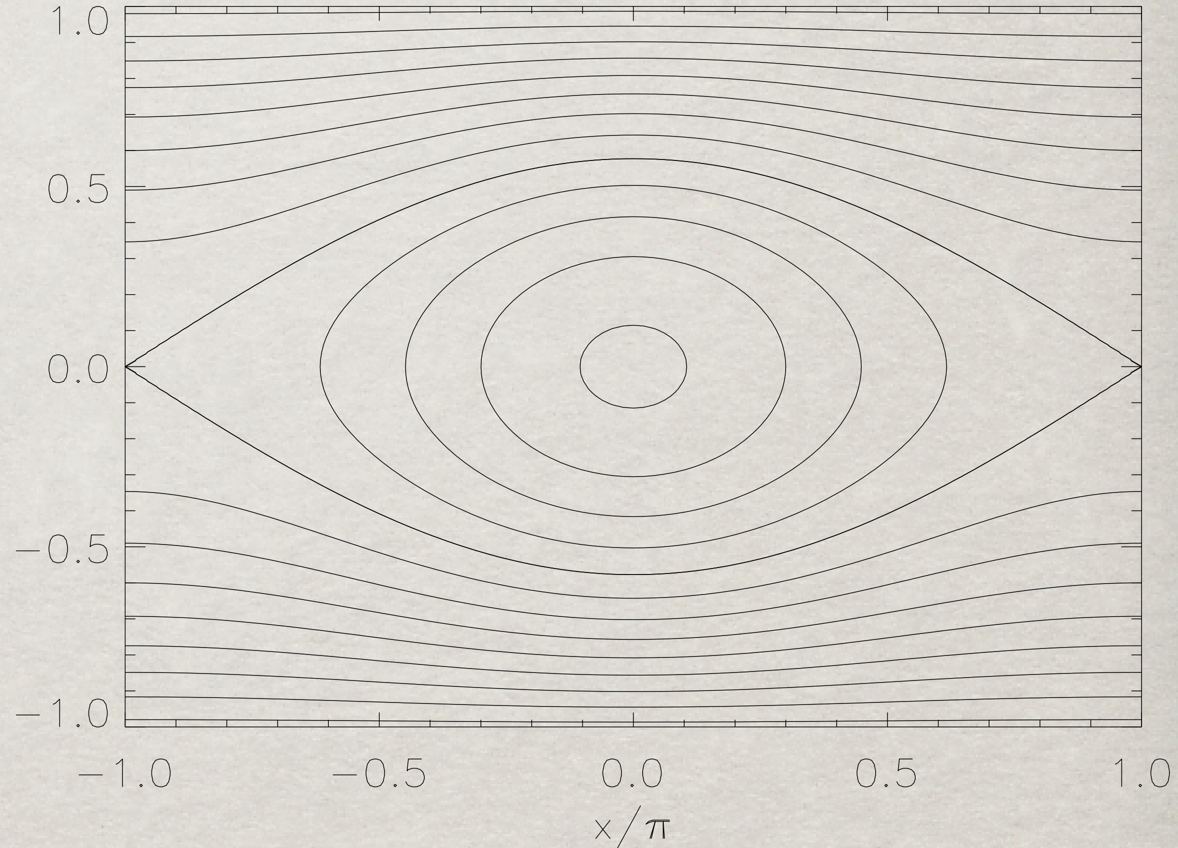


Ekman instability



Convective instability

Simplified plane layer flow



$$\boldsymbol{u} = (\psi_y, -\psi_x, w), \quad \psi = \psi(x, y), \quad w = w(x, y).$$

$$\psi = -\frac{ay^2 + b\cos x}{a + b\cos x} + \frac{b}{b - a}, \quad w = c\psi, \quad a = 5, \quad b = -1, \quad c = 1.$$

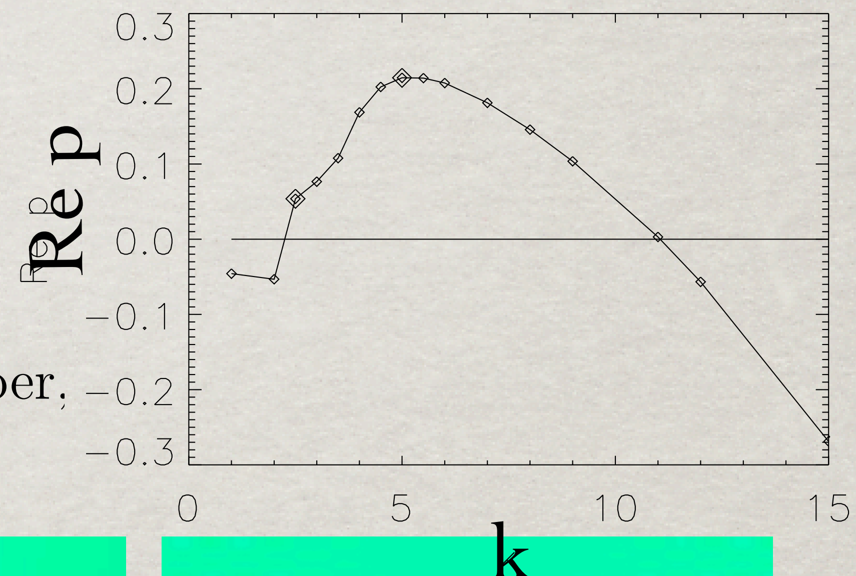
Ponomarenko and separatrix modes

$$\partial_t \boldsymbol{B} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) + \varepsilon \nabla^2 \boldsymbol{B}, \quad \nabla \cdot \boldsymbol{B} = 0,$$

$$\boldsymbol{B} = \boldsymbol{b}(x, y)\mathrm{e}^{pt + \mathrm{i}kz} + \text{complex conjugate.}$$

$R = \varepsilon^{-1}$ is the magnetic Reynolds number.

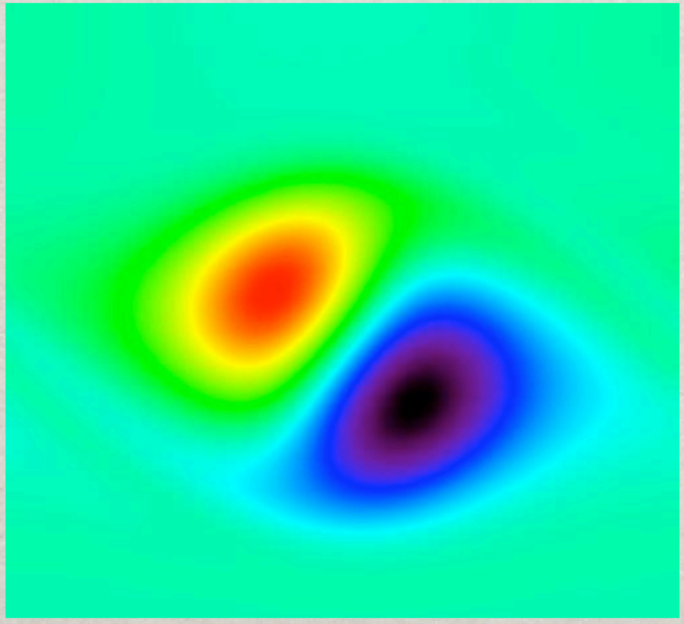
B_z is plotted in the (x, y)-plane

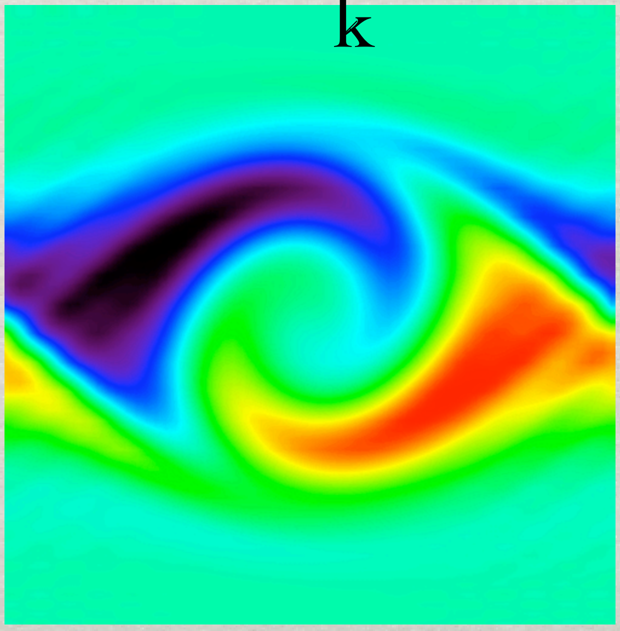


$R = 500.$

$k = 2.5,$

$k = 5.$

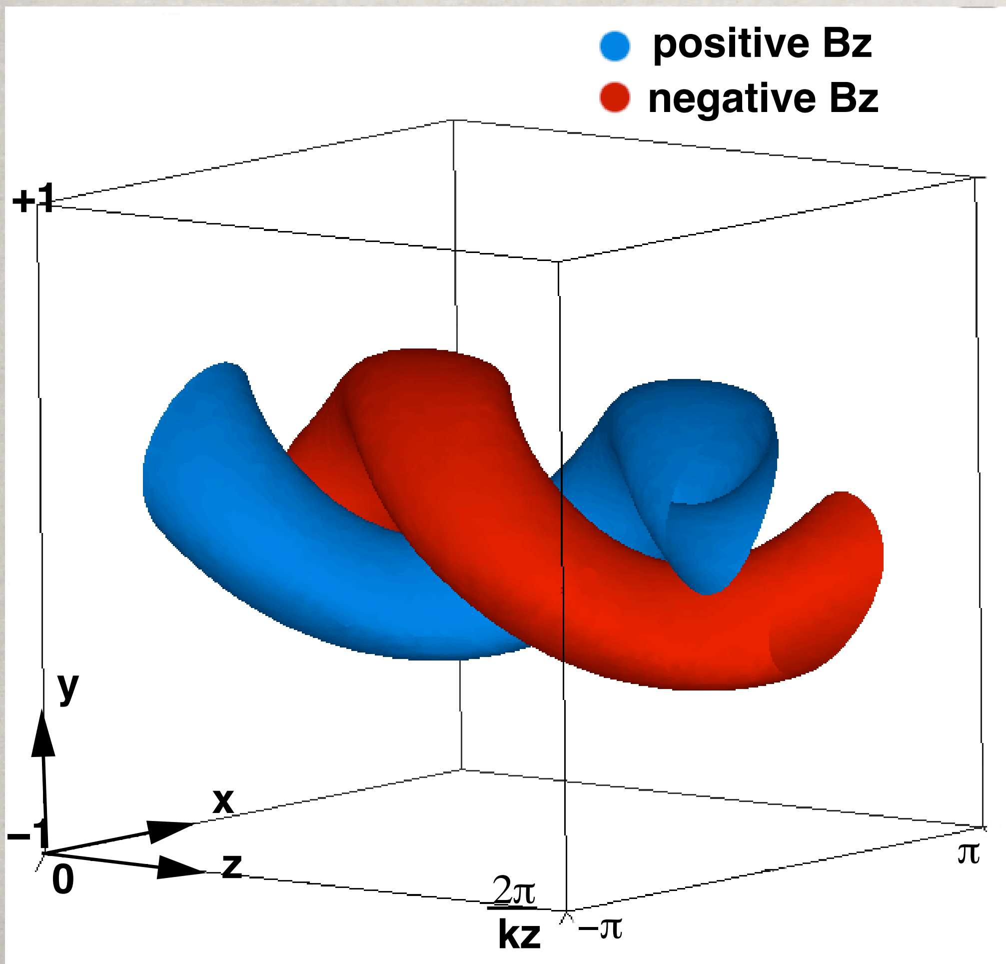


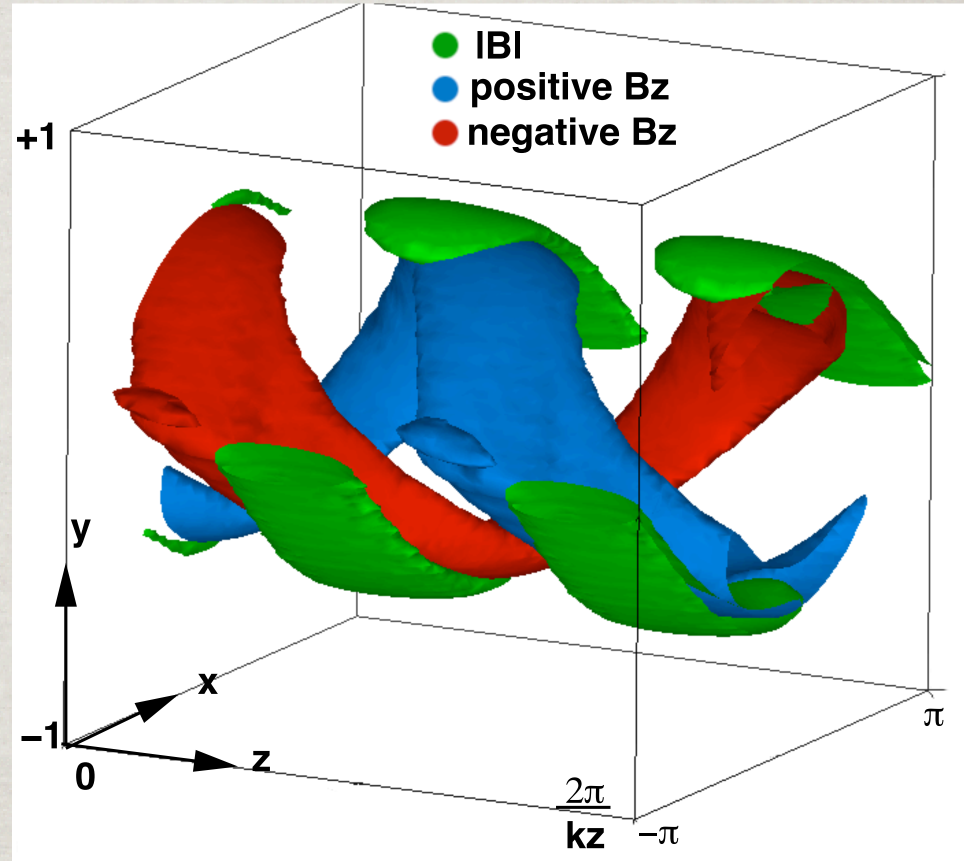


Field structure

$$R = 500.$$

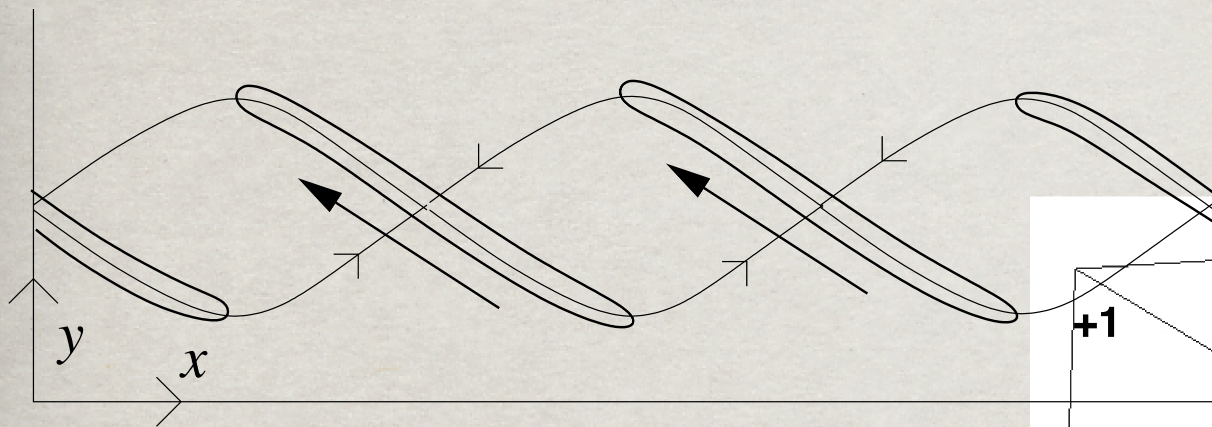
$k = 2.5,$

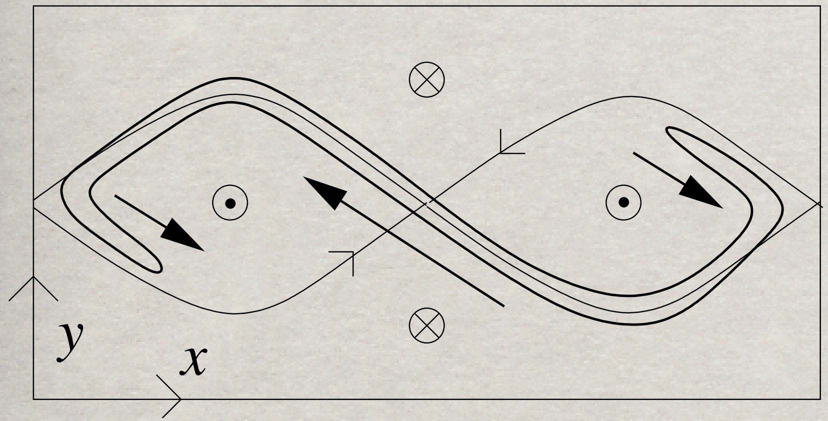


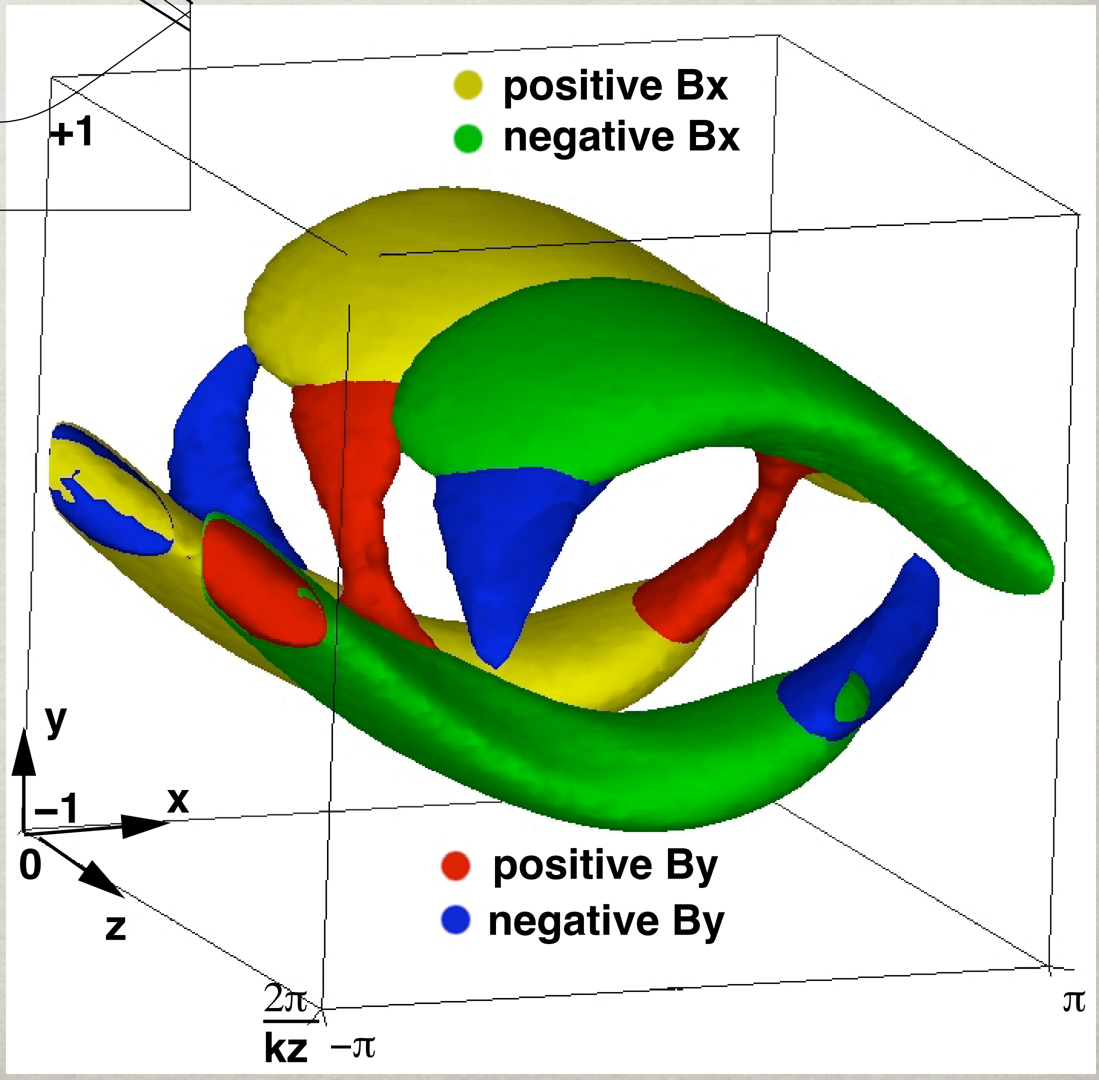


$k = 5.$

Dynamo mechanism

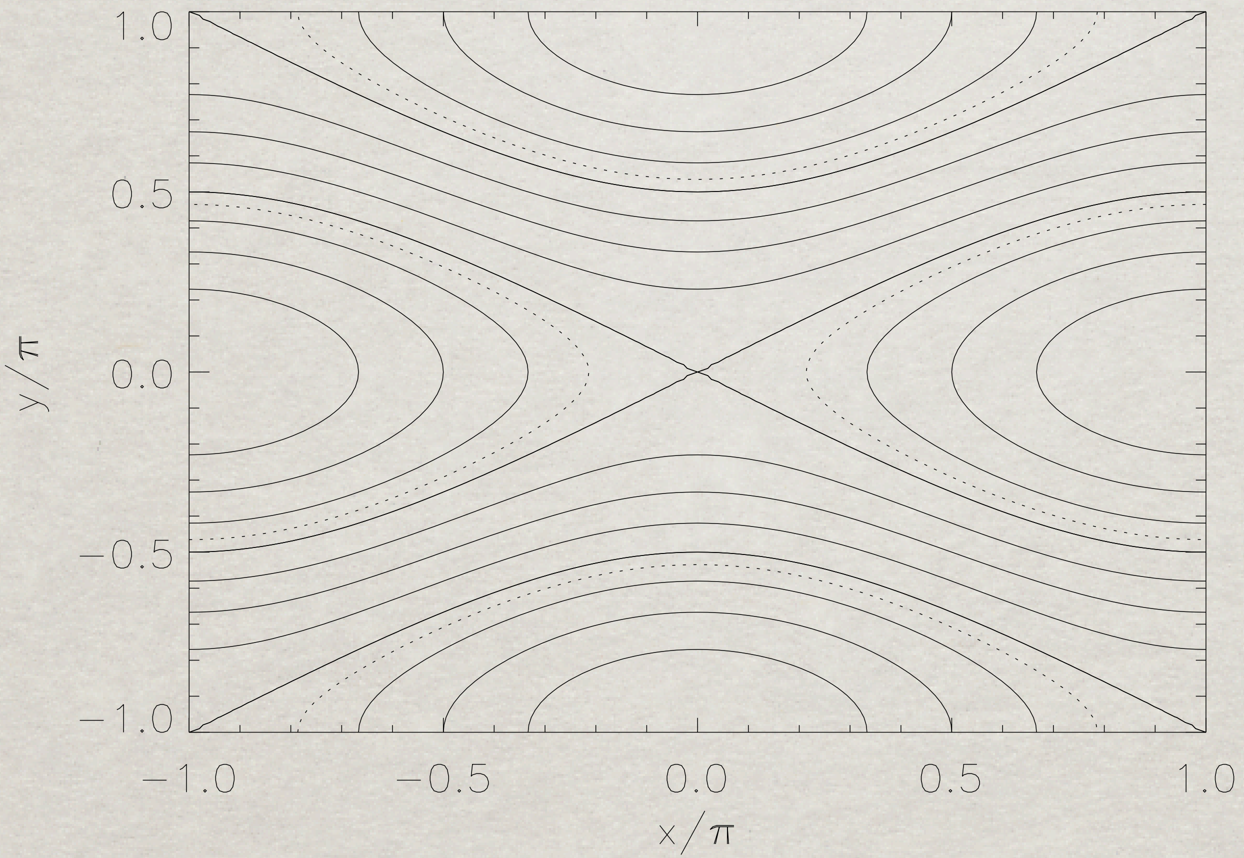






Stretch-fold-shear: Roberts, Soward, Bayly, Childress.

ABC flow (C=0)

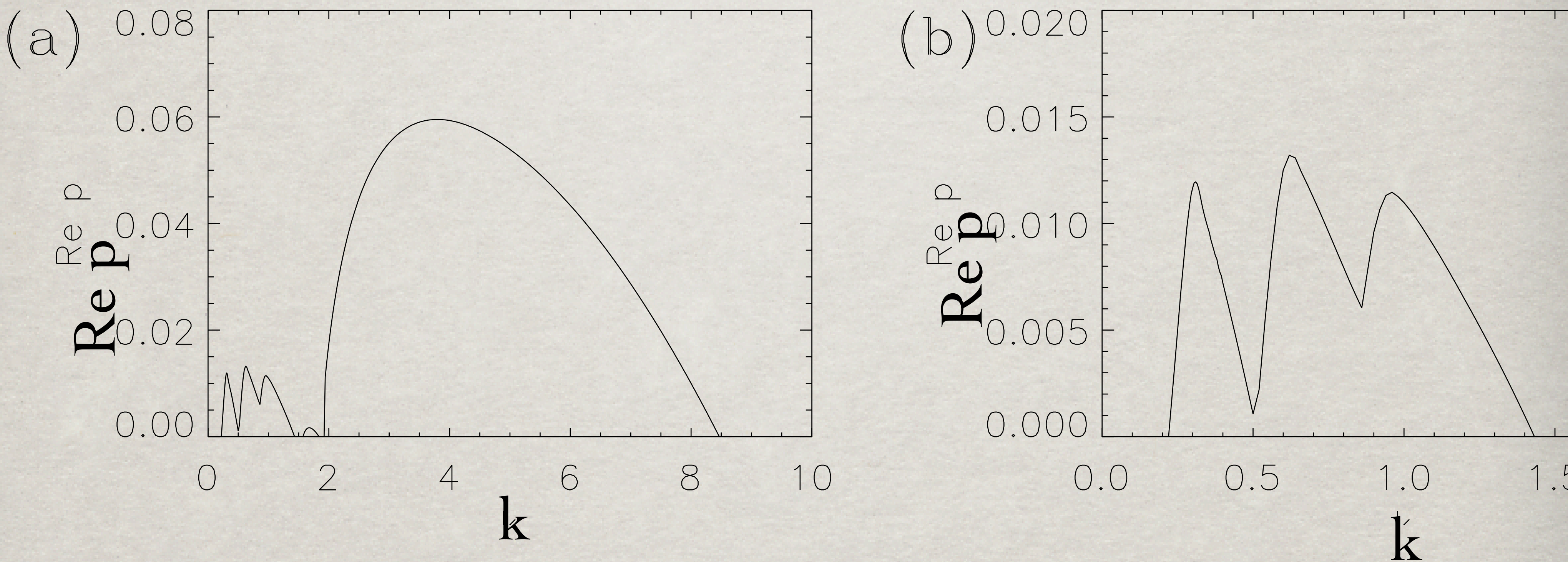


$$\psi = A\cos x - B\cos y, \quad w = \psi,$$

$$\boldsymbol{u} = A(0, \sin x, \cos x) + B(\sin y, 0, -\cos y).$$

$$A = \sqrt{3/5}, \quad B = 2A,$$

Growth rates



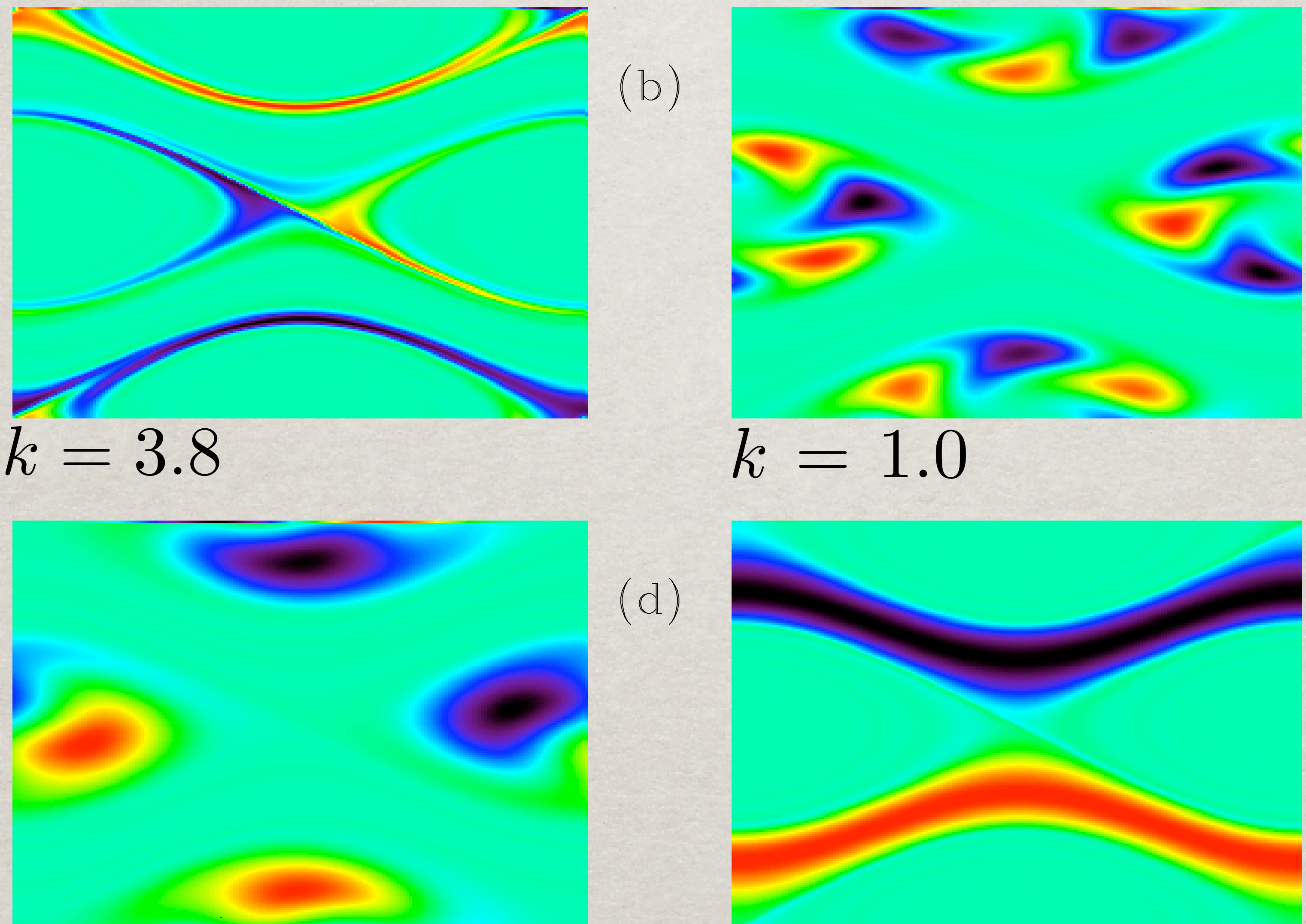
$$R = 1000$$

$$\partial_t \boldsymbol{B} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) + \varepsilon \nabla^2 \boldsymbol{B}, \quad \nabla \cdot \boldsymbol{B} = 0,$$

$$\boldsymbol{B} = \boldsymbol{b}(x, y) \mathrm{e}^{pt + \mathrm{i}kz} + \text{complex conjugate.}$$

B_z in x-y plane

$R = 1000$



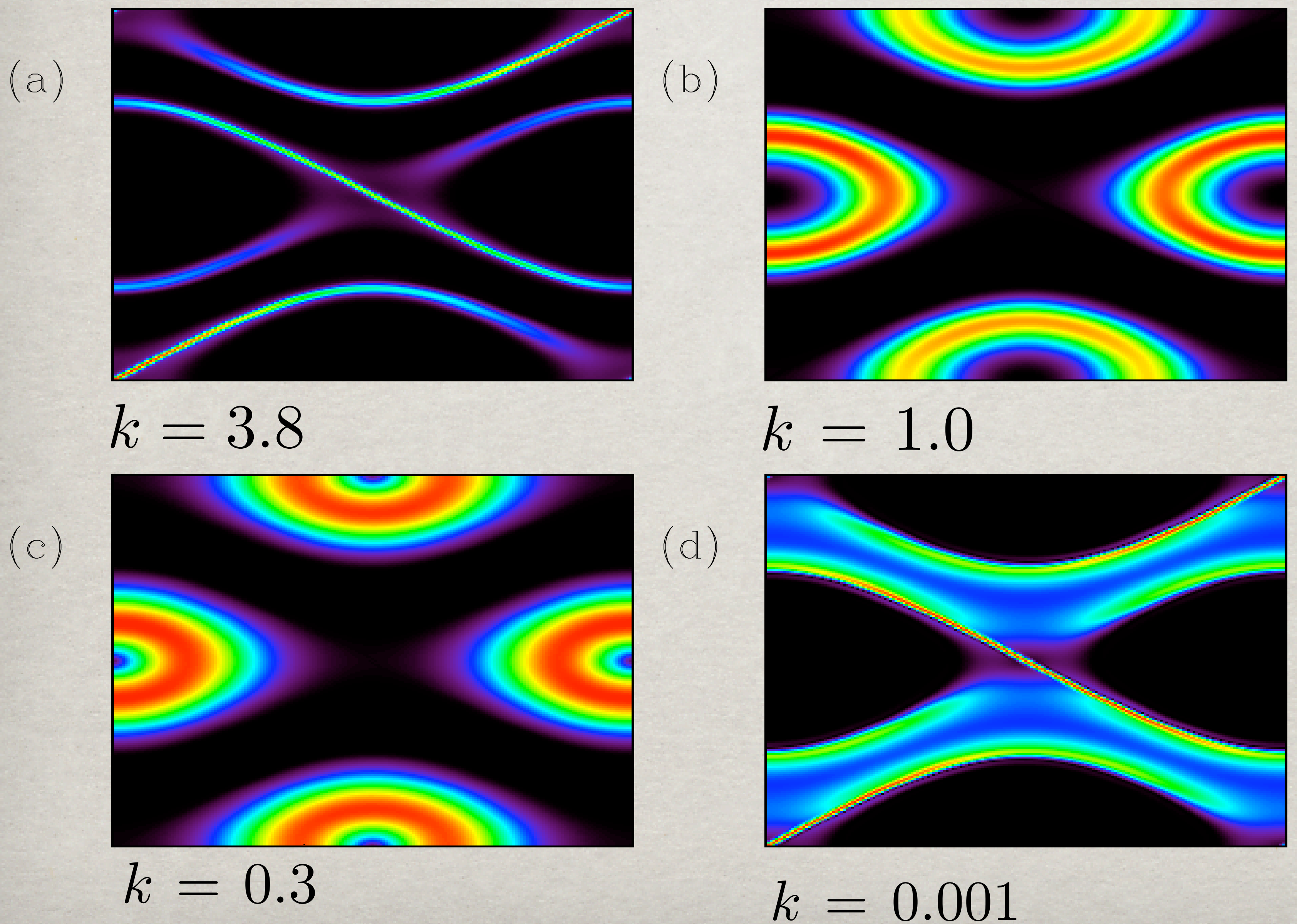
(a) $k = 3.8$

(b) $k = 1.0$

(c) $k = 0.3$

(d) $k = 0.001$

|B|^2 averaged over z, in x-y plane

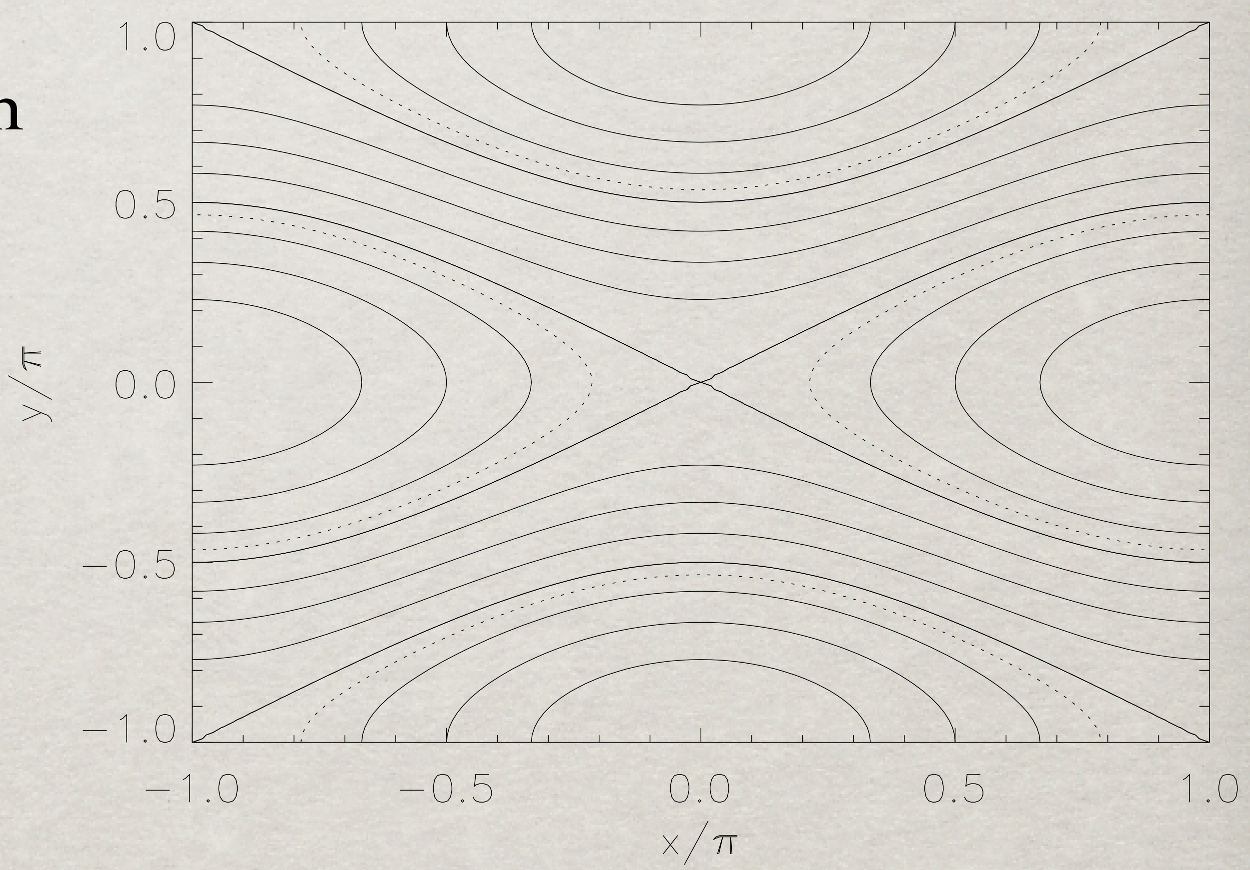


Criterion for Ponomarenko modes

These localise on stream surfaces satisfying:

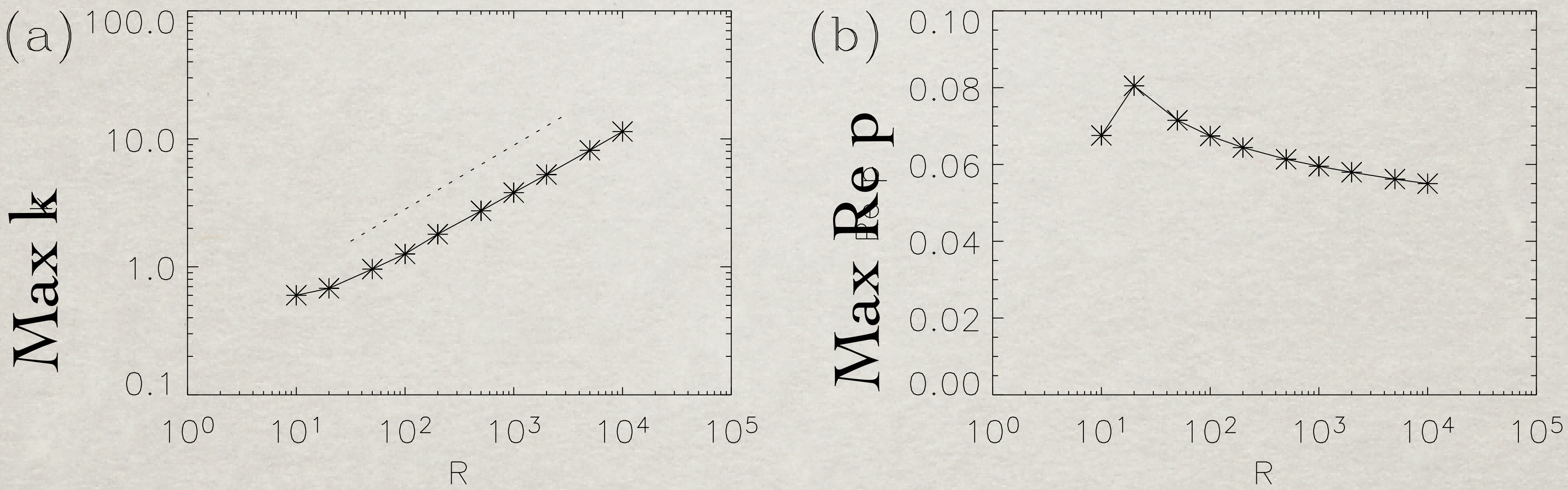
$$\left|\frac{\Omega''(\psi)}{\Omega'(\psi)} - \frac{w''(\psi)}{w'(\psi)}\right| < 4\,\frac{|\alpha_m|}{\gamma}$$

$$|\psi| > \psi_c \simeq 0.943.$$



Growing Ponomarenko modes can only occur if this purely geometrical condition is satisfied: generation overcomes enhanced diffusion (Soward 1990, Gilbert & Ponty 2000).

MAXIMUM GROWTH RATES



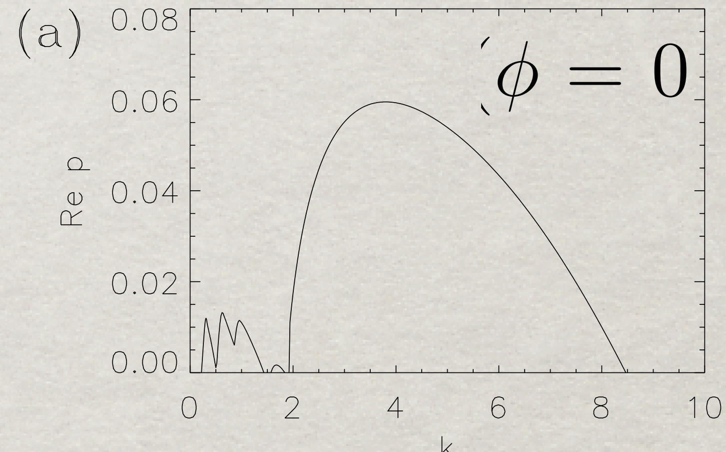
Fastest growing modes for each value of R: maximise Re p over k for given R.
Appear to have

$$k = o(R^{1/2}), \quad \operatorname{Re} p = o(1),$$

as far as we can tell numerically (cf. Soward 1997).

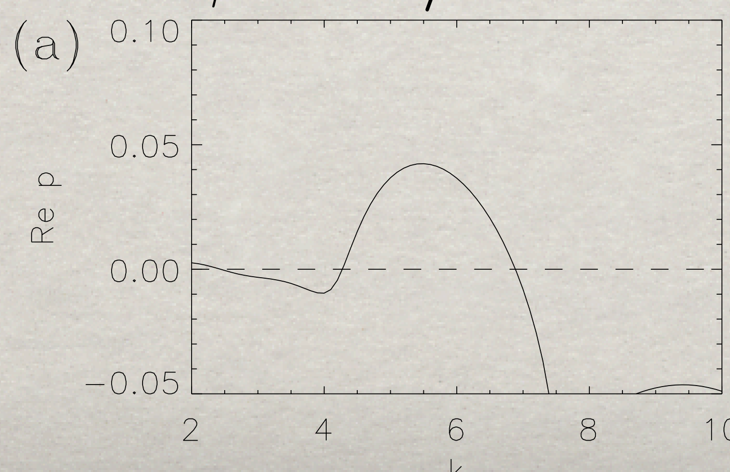
W NOT A FUNCTION OF PSI

$$\psi = A\cos x - B\cos y, \quad w = A\cos x - B\cos(y - \phi),$$

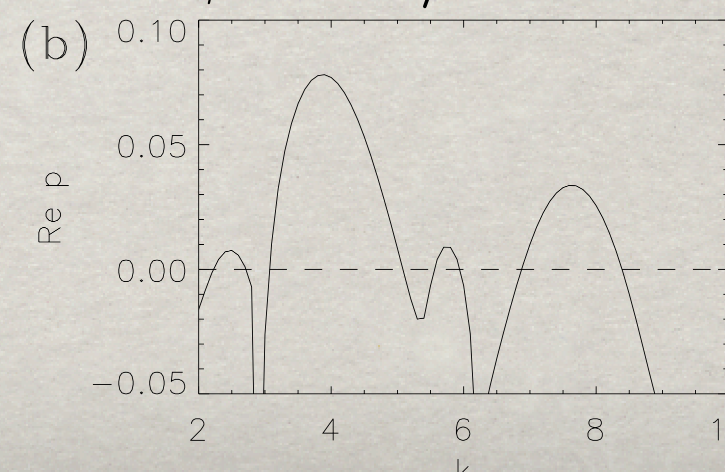


Earlier examples had w a function of psi, but this is not the case for the Ekman instability.

$\phi = \pi/36$



$\phi = \pi/12$



Sensitivity to value of k linked to z-motion

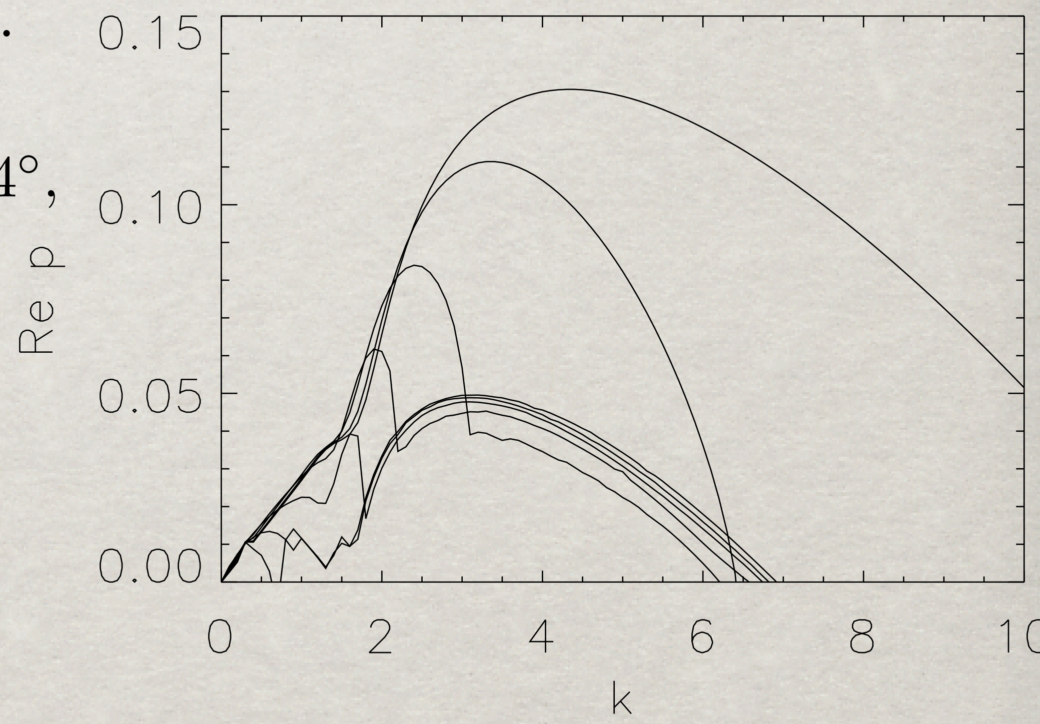
From Roberts' flow to cat's eyes

$$\boldsymbol{u} = A(0, \sin x, \cos x) + B(\sin y, 0, -\cos y).$$

$$A = \sqrt{3}\sin\xi, \quad B = \sqrt{3}\cos\xi.$$

$R = 1000$ and $\xi = 45°,\ 44.5°,\ 44°,$

$43.5°,\ 43°,\ 42.5°$ and $42°$



G.O. Roberts' flow modes (square cells) appear distinct from cat's eye modes. As the channels open up the Roberts' mode disappears and the cat's eye mode takes over

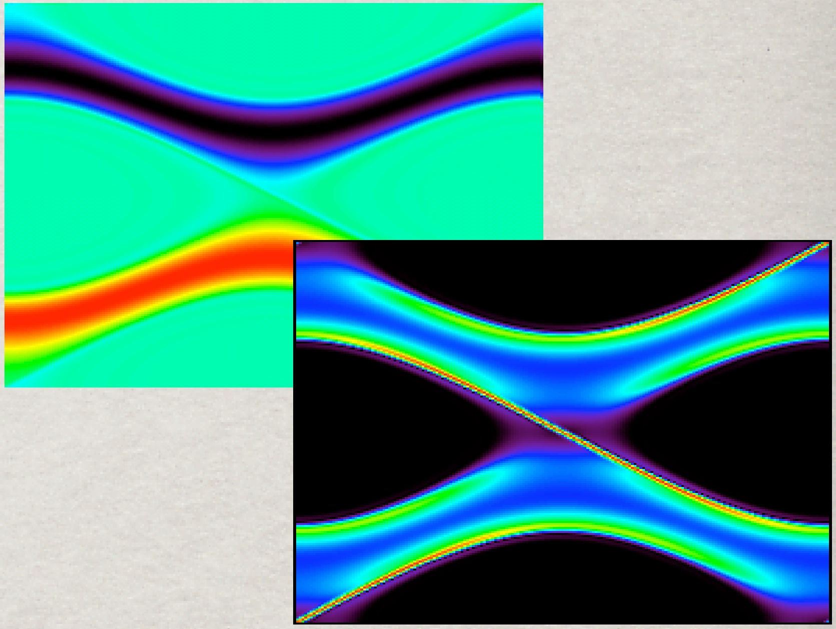
Low k: alpha effect

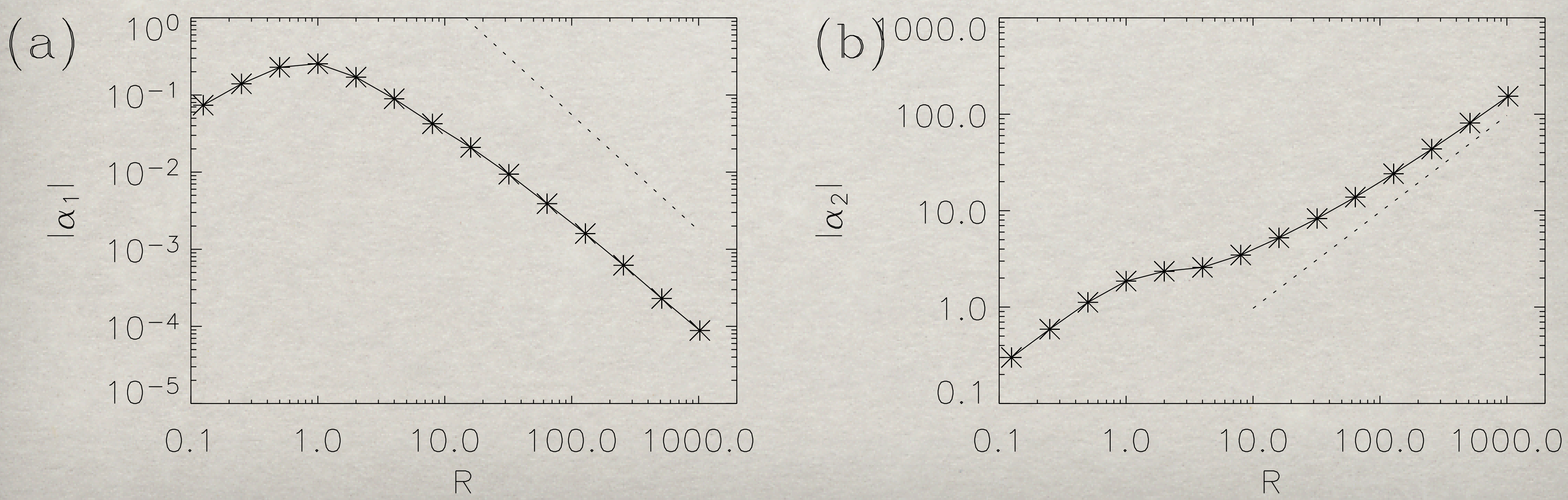
$$\partial_t \overline{B}_1 = -\partial_z(\alpha_2 \overline{B}_2), \quad \partial_t \overline{B}_2 = \partial_z(\alpha_1 \overline{B}_1),$$

$$p \simeq \pm k\sqrt{\alpha_1 \alpha_2}. \qquad 0 < k \ll 1,$$

$$\alpha_1 = O(R^{-3/2}), \quad \alpha_2 = C_1 R.$$

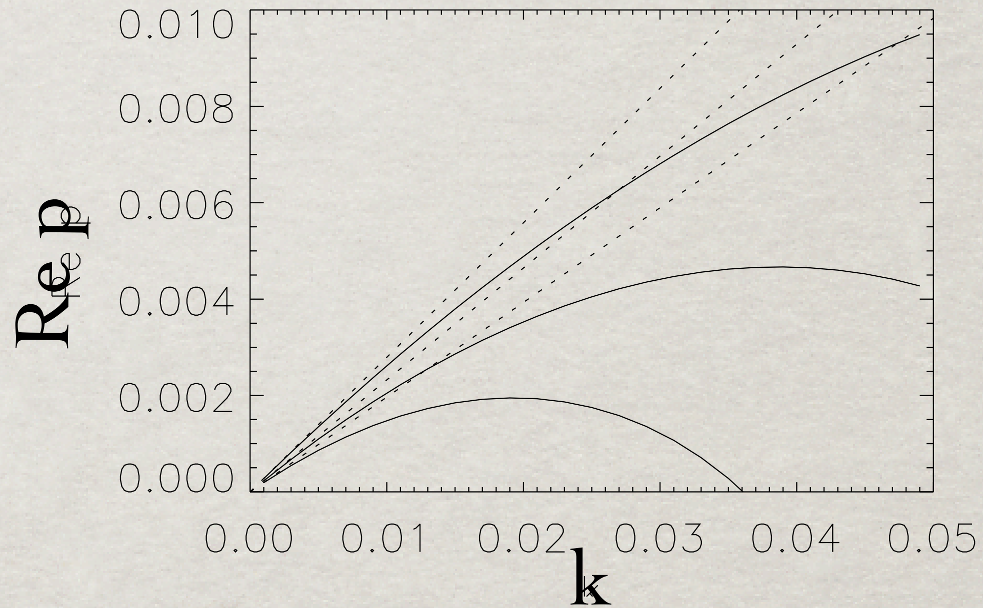
Childress & Soward 1989





Window of low k modes

R=32, 64, 128
Solid: numerical,
dotted: alpha-effect



Alpha effect only functions in a narrow window

$$0 < k \leq O(R^{-1})$$

that vanishes with increasing R (Childress & Soward 1989).

Comparison

$$\delta = (B - A)/(B + A).$$

β	R'	$R'^{1/2}\alpha^{(+)}$: sim	bl	asy	$R'^{1/2}\alpha^{(-)}$: sim	bl	asy
0.5	25	0.2214			1.3792		
0.5	64	0.2227			1.3805		
0.5	100	0.2232			1.3812		
0.5	∞		0.2246	0.2248		1.383	1.320
1.0	64	0.1172			3.3582		
1.0	100	0.1177			3.3646		
1.0	144	0.1179			3.3680		
1.0	∞		0.1186	0.1250		3.377	3.455

Table 1
Comparison with table 1 of Childress and Soward (1989). Here $\beta = \delta R'^{1/2}$. Columns 3–5 give results for $R'^{1/2}\alpha^{(+)}$ and columns 6–8 give those for $R'^{1/2}\alpha^{(-)}$. The labels are 'sim' for results computed numerically using a two-dimensional simulation for the value of R' given, 'bl' for the numerical solution of the boundary layer equations (valid in the limit $R' \to \infty$) taken from Childress and Soward (1989), and 'asy' for their asymptotic approximation valid for large R' and large β.

Conclusions

- Global alpha-effect modes: small window for small k; require double periodicity.
- Cat's eye modes: require single periodicity; fastest growing modes.
- Ponomarenko modes: require only a single overturning cell; slow growing modes.
- Dynamics: modes for Ekman instability equilibrate with low-level, irregular magnetic fields (Pu Zhang).