Bounds on the attractor dimension for Low-Rm MHD turbulence between walls

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Some questions about Turbulence

- Number of degrees of freedom N in the flow ?

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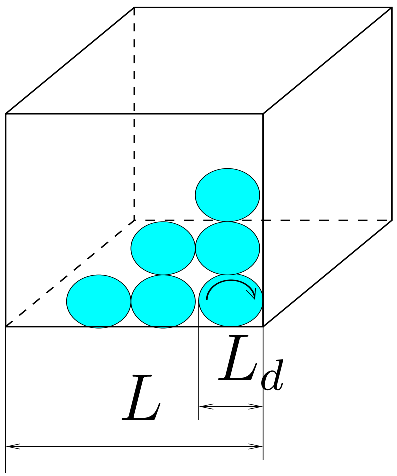

$$N \sim \left(\frac{L}{L_d}\right)^3$$

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- Number of degrees of freedom N in the flow ?
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$$\mathbf{v} = \sum_{m=1}^{\infty} c_m \mathbf{v}_m, \qquad \mathbf{v}(t=0) = \mathbf{v}_0$$

$$\mathbf{u} = \sum_{m=1}^{\infty} d_m \mathbf{u}_m, \qquad \mathbf{v}(t=0) = \mathbf{u}_0$$

$$\forall m \leq M \lim_{t\to 0} |c_m - d_m| = 0 \qquad \Rightarrow \lim_{t\to 0} \|\mathbf{u} - \mathbf{v}\| = 0$$

$M \lesssim N$ (Constantin et al. 85)

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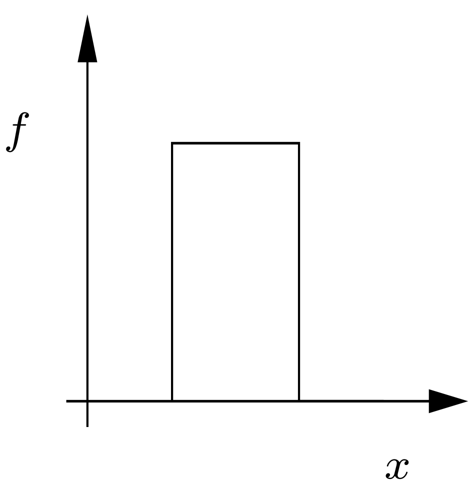
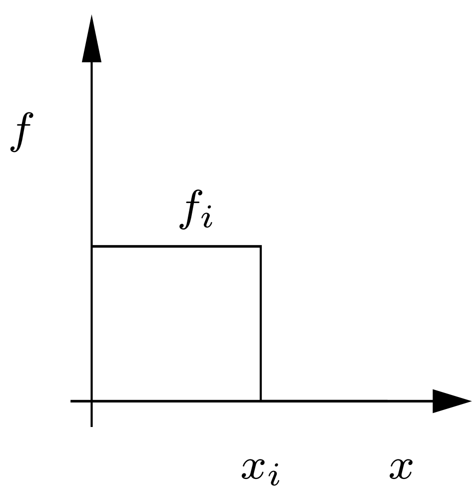
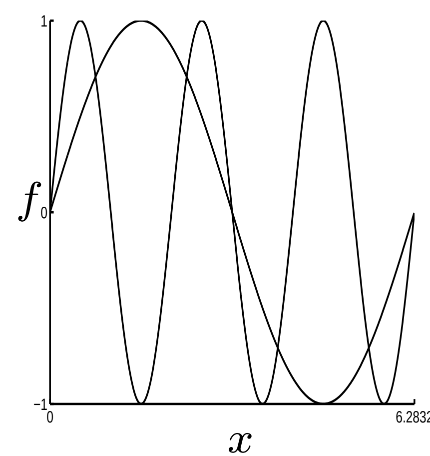


function to expand



2 modes needed



loads needed !

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Plan

- Small scales and determining modes in homogeneous turbulence
- Bounds on the attractor dimension in homogeneous turbulence
- Heuristics in MHD turbulence
- Bounds on the attractor dimension and least dissipative modes in MHD turbulence

The Heuristics of Kolmogorov (K41)

$$\partial_t \mathbf{u} = -\mathbf{u} \cdot \nabla \mathbf{u} - \nabla p + \frac{1}{Re} \nabla^2 \mathbf{u} + \mathbf{f}$$

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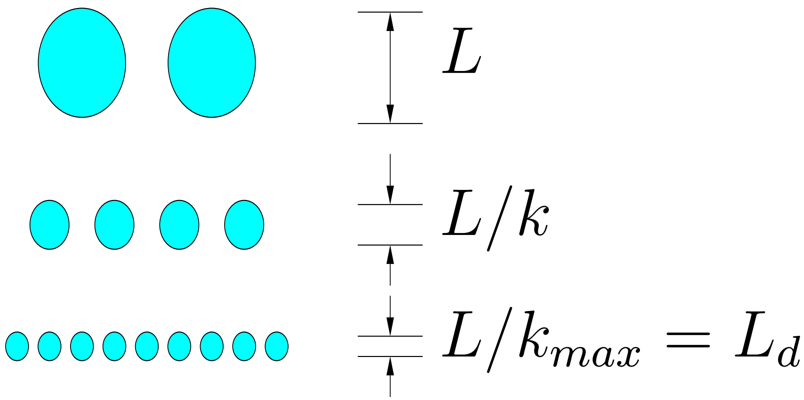

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$\rightarrow$ II. Inertial range

$\rightarrow$ III. Dissipative scales

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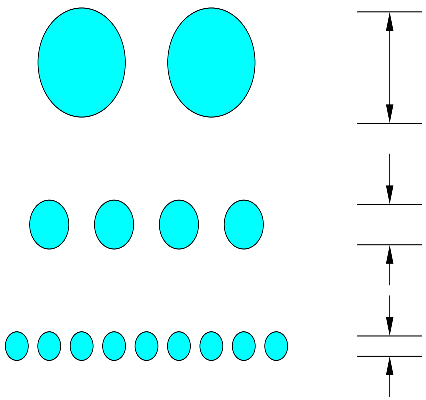

L → I. Large scales

L/k → II. Inertial range

$L/k_{max} = L_d$ → III. Dissipative scales

II. $E = E(\epsilon, L/k)$ $\qquad E \sim E(L)k^{-5/3}$

III. Inertia $\sim$ Dissipation $\qquad k_{max} \sim \left(\frac{U_L L}{\nu}\right)^{3/4}$

$$N \sim Re^{9/4}$$

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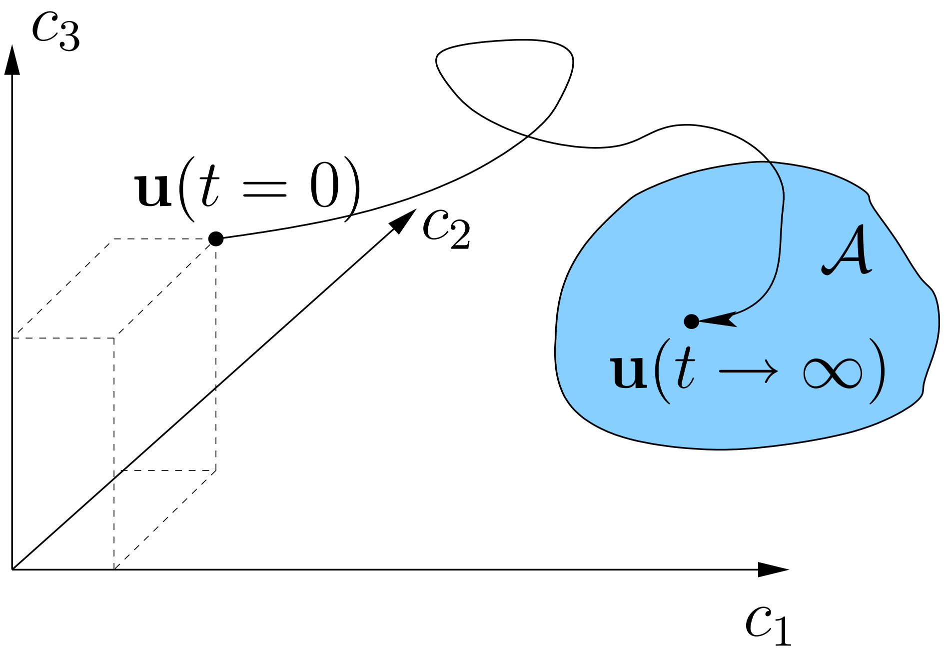
Global attractor for NS

$$\mathbf{u} = c_1(t)\mathbf{v}_1 + c_2(t)\mathbf{v}_2 + c_3(t)\mathbf{v}_3 + \ldots$$



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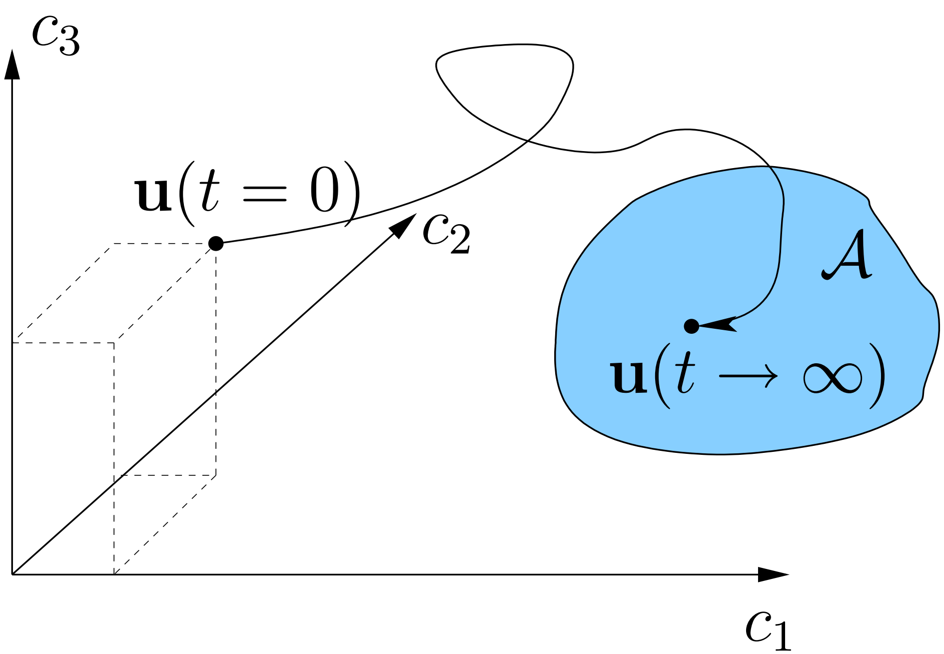

- $\forall t < \infty$, $\epsilon < \infty \Rightarrow$ there is a global attractor
  (Babin 78, Sermange 83)
- $\dim(\mathcal{A}) = cM$ (Constantin et al. 85)

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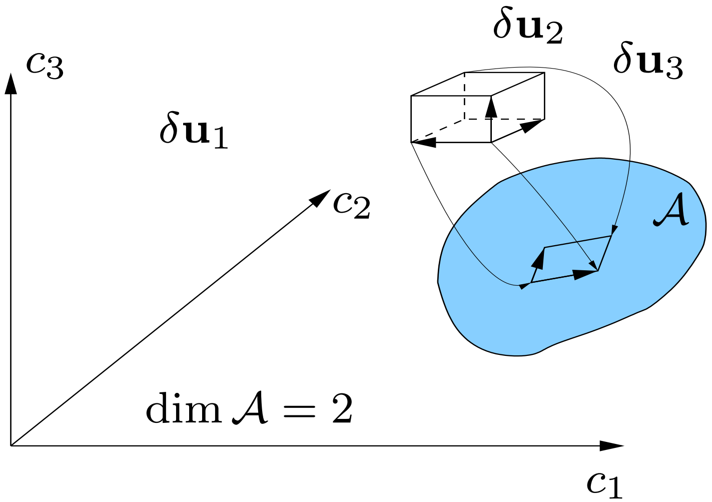
Estimate for d_M



$(\delta\mathbf{u}_i)_{i=1..n}$ independent disturbances in the vicinity of $\mathcal{A}$

$$n > \dim(\mathcal{A}) \Rightarrow V_n = \|\delta\mathbf{u}_1 \times \ldots \times \delta\mathbf{u}_n\| \rightarrow 0$$

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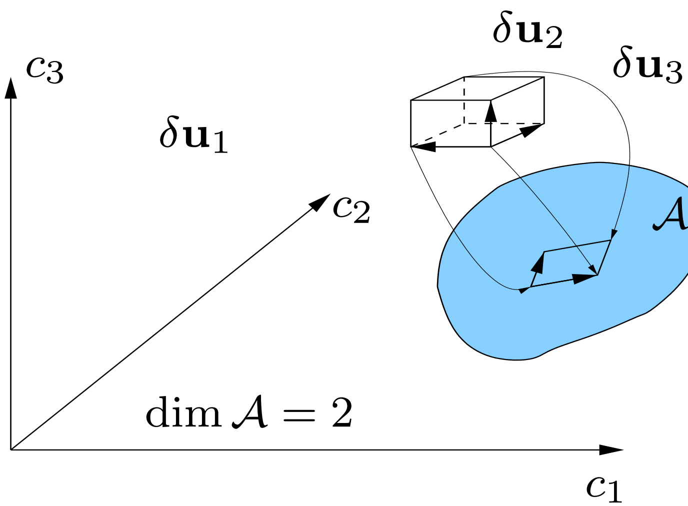

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$$\partial_t \delta\mathbf{u}_i = \mathcal{L}[\mathbf{u}]\delta\mathbf{u}_i + O(\|\delta\mathbf{u}_i\|^2)$$

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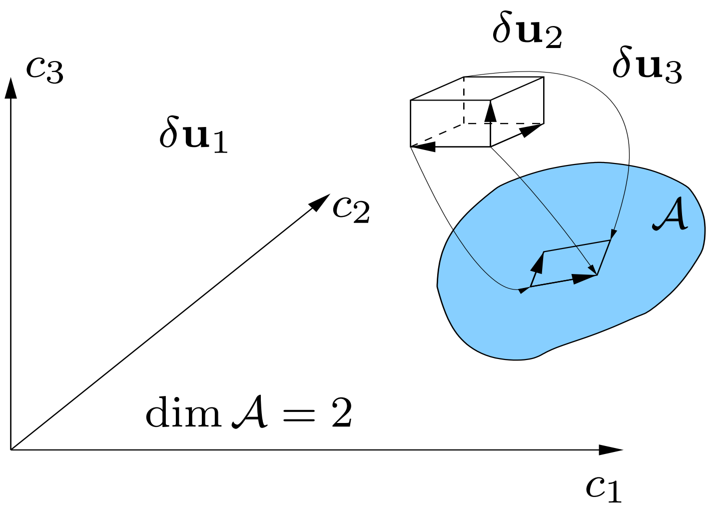

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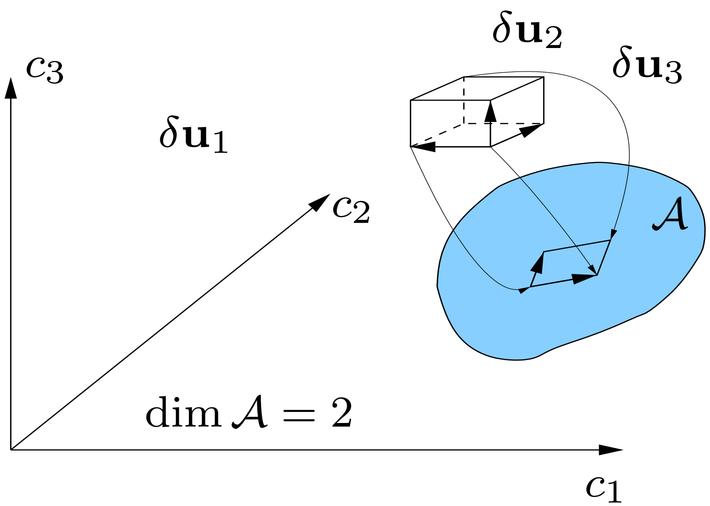

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$$d_M \leq \min n \in \{\max_{\mathcal{P}_n} \mathrm{Tr}(\mathcal{L}[\mathbf{u}]\mathcal{P}_n) < 0\}$$

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(Kaplan-Yorke)

Bounds d_M for Navier-Stokes with periodic b. c.

- Evolution equation for a perturbation $\delta\mathbf{u}_i$

$$\begin{aligned} \partial_t \delta\mathbf{u}_i &= -\delta\mathbf{u}_i \cdot \nabla\mathbf{u} - \mathbf{u} \cdot \nabla\delta\mathbf{u}_i + \frac{1}{Re}\nabla^2\delta\mathbf{u}_i \\ \mathcal{L}[\mathbf{u}] &= \mathcal{B}(\cdot, \mathbf{u}) + \mathcal{D} \end{aligned}$$

  - $\mathcal{B}(\cdot, \mathbf{u})$ produces modes $\rightarrow$ Tr$(\mathcal{B}(\cdot, \mathbf{u})\mathcal{P}_n) \geq 0$
  - $\mathcal{D}$ dissipative $\rightarrow$ Tr$(\mathcal{D}\mathcal{P}_n) \leq 0$

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- 3d periodic case

$$\mathrm{Tr}\mathcal{B}(\cdot,\mathbf{u})\mathcal{P}_n \leq nRe^2 \qquad \Rightarrow d_M \leq cRe^3$$

$$\mathrm{Tr}\mathcal{D}\mathcal{P}_n \leq -cn^{\frac{5}{3}}$$

(Constantin et. al. 85) $\qquad N \sim Re^{\frac{9}{4}}$!!

$\longrightarrow d_M$

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- 2d periodic case
  $d_M \leq \mathcal{G}^{\frac{1}{3}}(1+\ln\mathcal{G})^{\frac{1}{2}}$ (Constantin et al. 88) Log-optimal ! (0hkitani 89)

$\rightarrow d_M$

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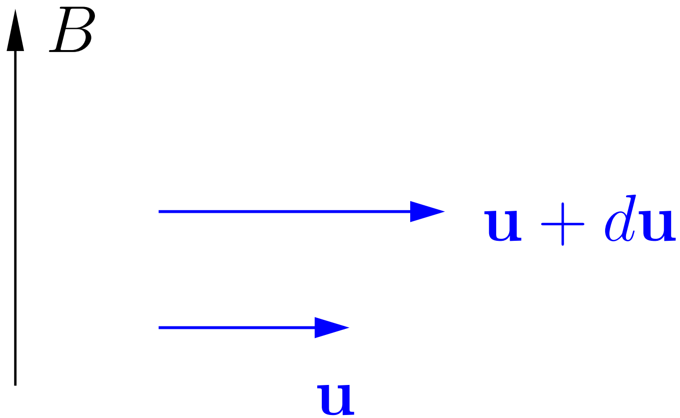
Heuristics on MHD turbulence

Lorentz Forces



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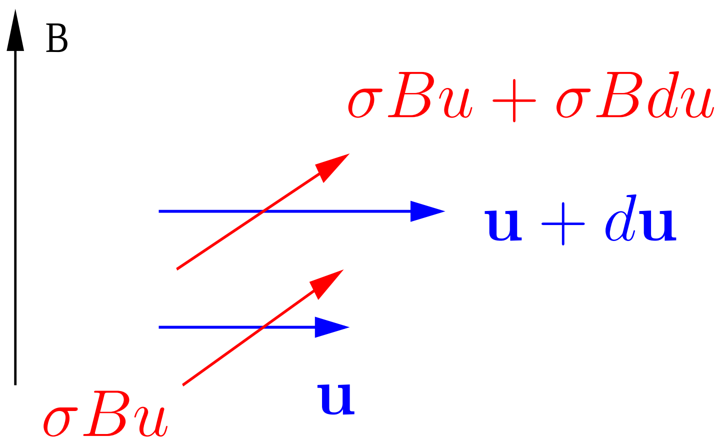

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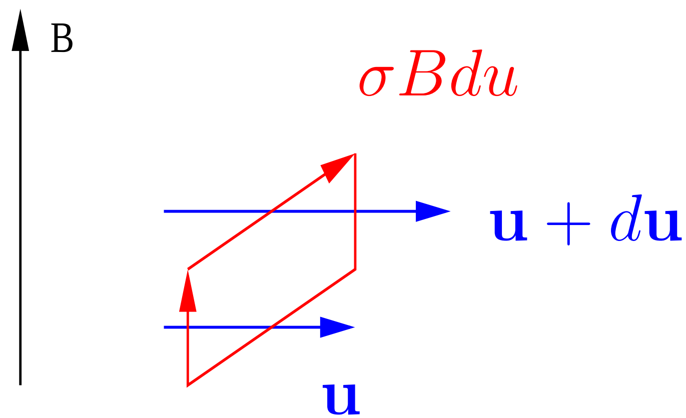

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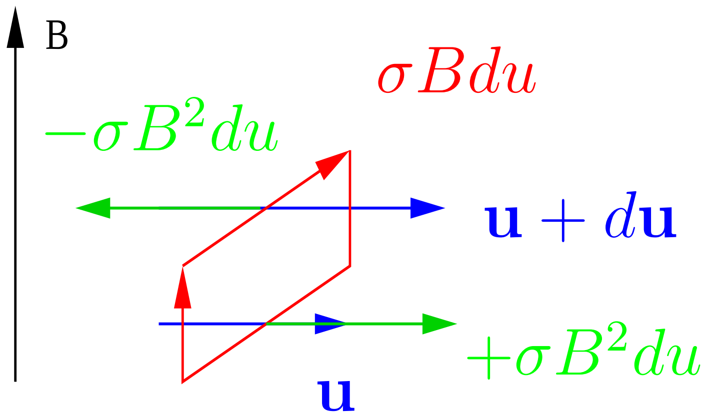

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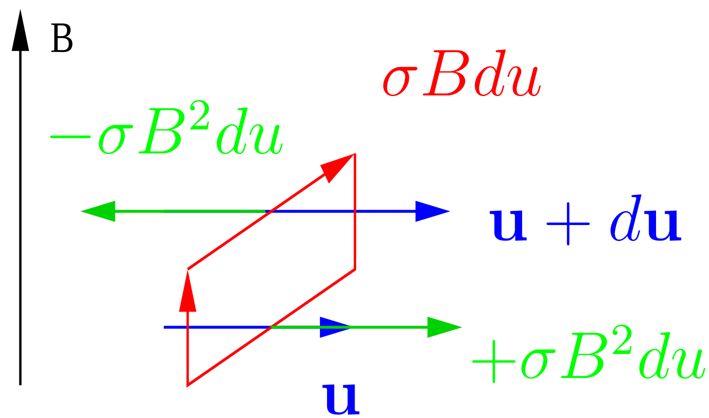
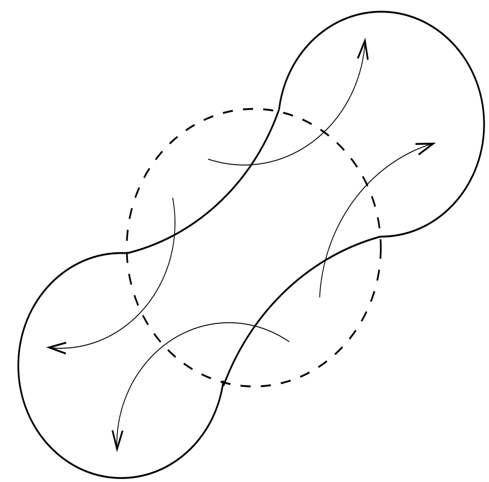


vs.

Inertia



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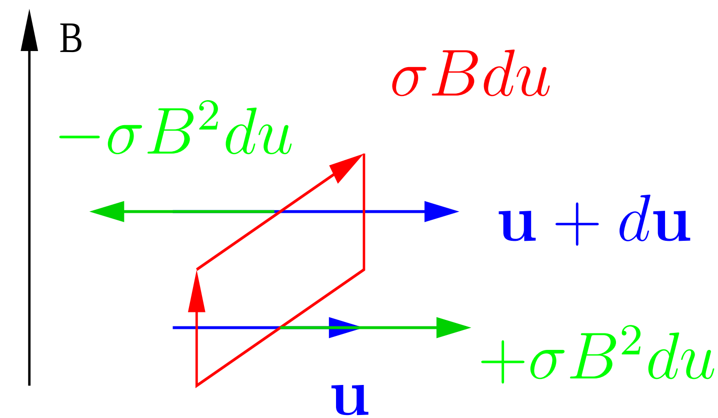

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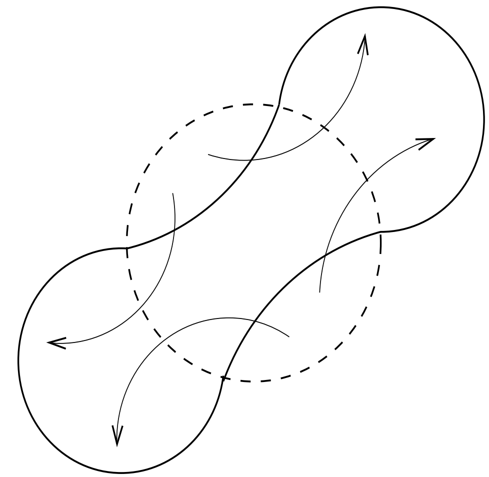
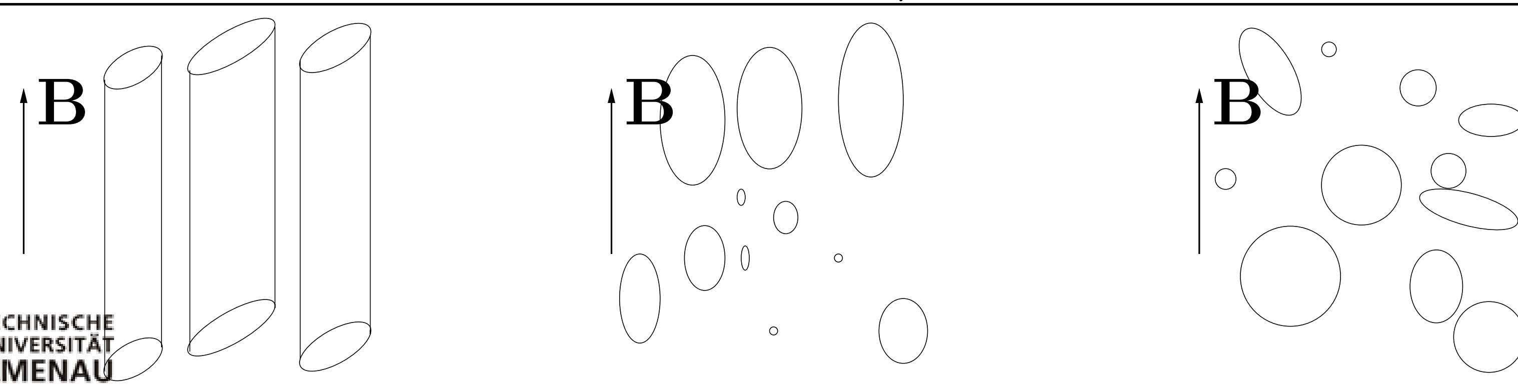


$Ha^2/Re >> 1$ $\quad$ $Ha^2/Re = \frac{\sigma B^2 L}{\rho U} \sim 1$ $\quad$ $Ha^2/Re << 1$



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Kolmogorov-like heuristics in MHD

$$\partial_t \mathbf{u} = -\mathbf{u} \cdot \nabla \mathbf{u} - \nabla p + \frac{1}{Re} \left(\nabla^2 \mathbf{u} - Ha^2 \partial_{z^2}^2 \nabla^{-2} \mathbf{u} \right) + \mathbf{f}$$

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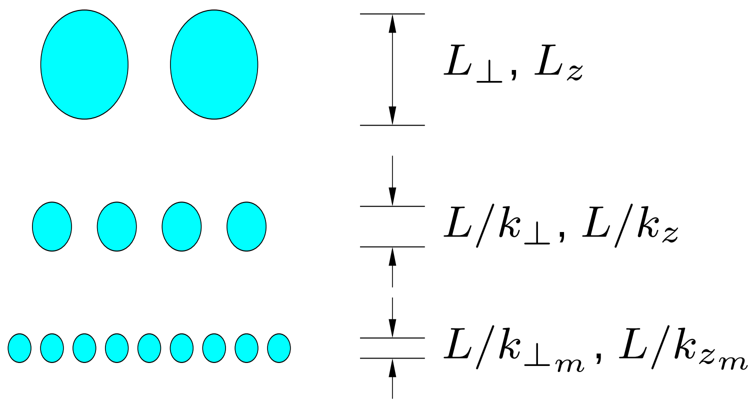

→ I. Large scales

→ II. Inertial range (Joule !)

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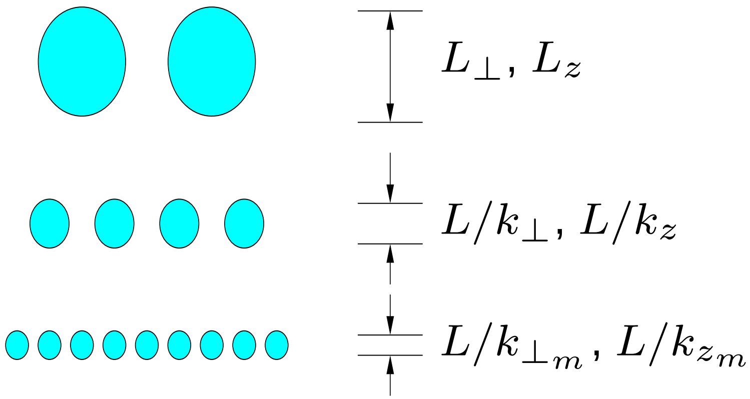

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II. Inertia $\sim$ Lorentz and $\frac{k_\perp}{k_z} = cst$ $\qquad$ $E \sim E(L) k_\perp^{-3}$ (Alemany et al. 79)

III. Inertia $\sim$ Dissipation $\qquad$ $k_{\perp_m} \sim Re^{1/2}$ and $k_{z_m} \sim Re/Ha$

$$N \sim \frac{Re^2}{Ha}$$

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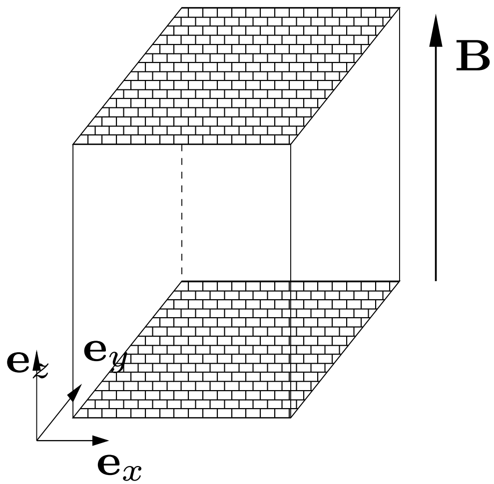
Bounds for d_M in MHD

- The problem...



- incompressible, conducting fluid
- $2\pi L$-x, y periodic box
- $\mathbf{u} = 0$ and $\mathbf{j} \cdot \mathbf{n} = 0$ at $z = -1, 1$
- forced flow

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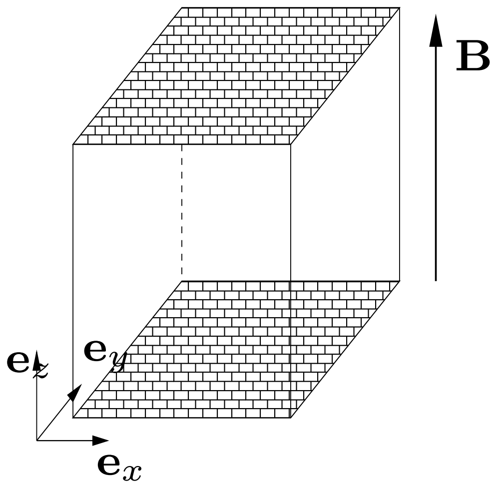

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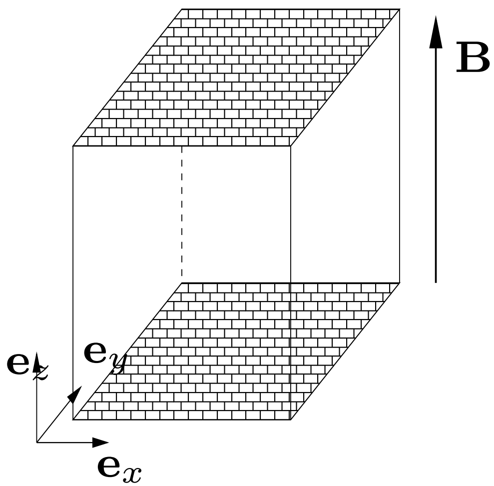

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\mathcal{L}[\mathbf{u}] &= \mathcal{B}(\cdot, \mathbf{u}) + \mathcal{D}_{Ha}
\end{aligned}$$

- Bound on d_M achieved for the set of n least dissipative eigenmodes of $\mathcal{D}_{Ha}$

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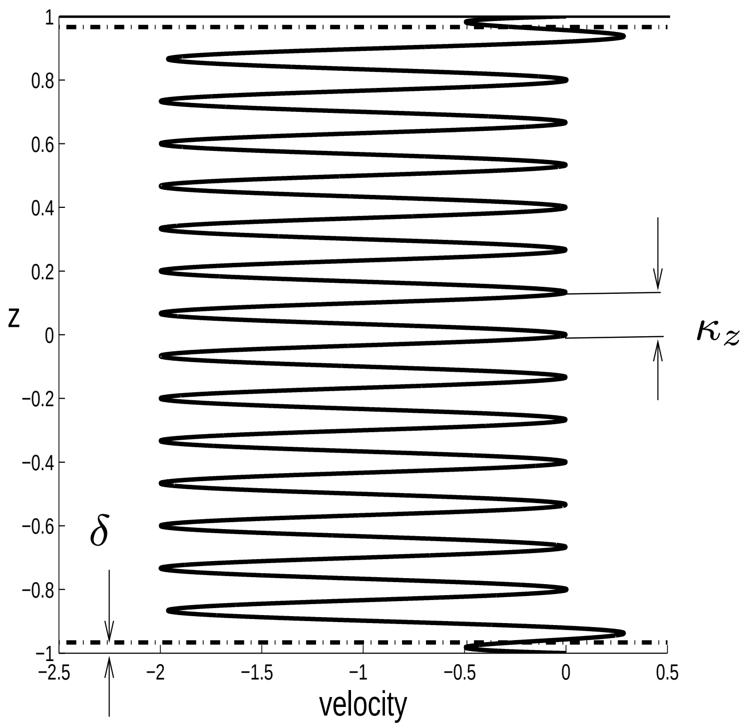
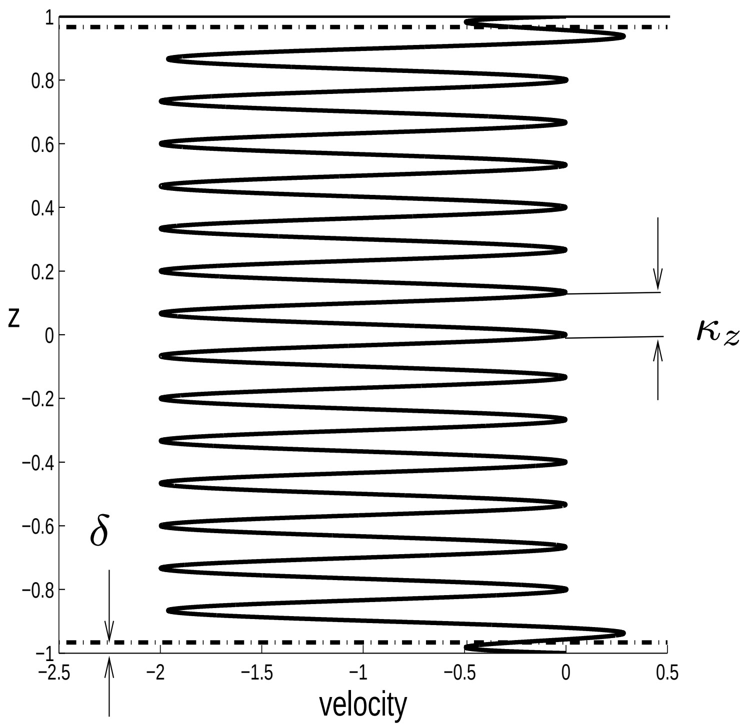
Eigenmodes of $\mathcal{D}_{Ha}$



- Fourier modes in the (x, y) plane $k_\perp^2 = k_x^2 + k_y^2$
- Dispersion relation $\lambda = -k_\perp^2 - \mu_z^2 + \frac{1}{Ha^2} \frac{\mu_z^2}{k_\perp^2 + \mu_z^2}$
- $\mu = \pm\delta$ or $\mu = \pm i\kappa_z$

Eigenmodes of $\mathcal{D}_{Ha}$



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- One eigenmode defined by (k_x, k_y, κ_z) or (k_x, k_y, δ)
  κ_z spans a discrete real spectrum
- We have to minimise $\sum \lambda(k_x, k_y, \kappa_z)$

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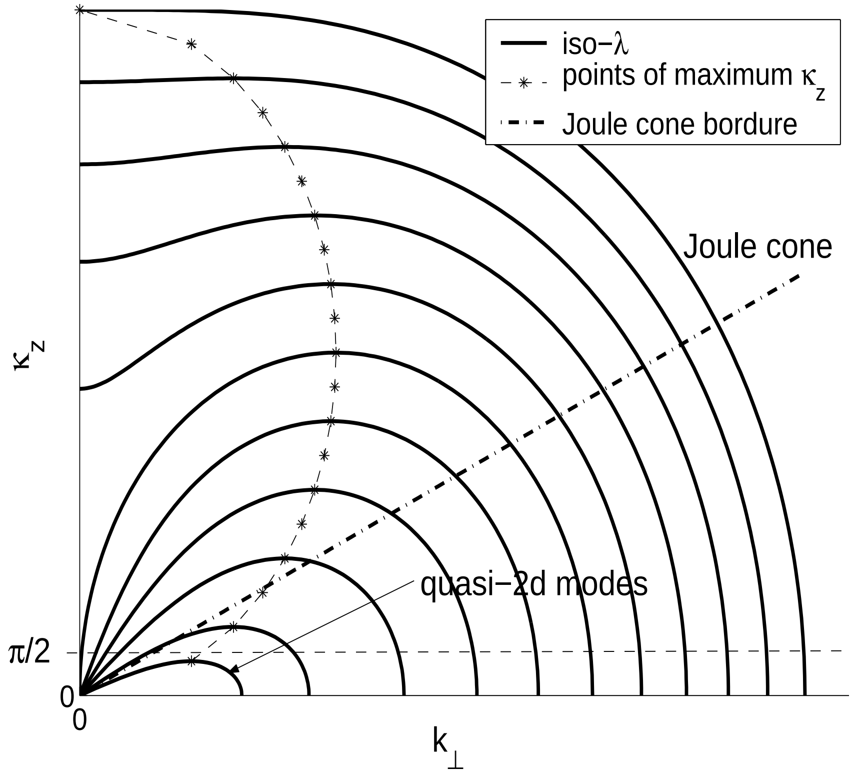
Modes classification according to κ_z

Eigenmodes repartition



$\rightarrow \mathcal{B}$ estimate

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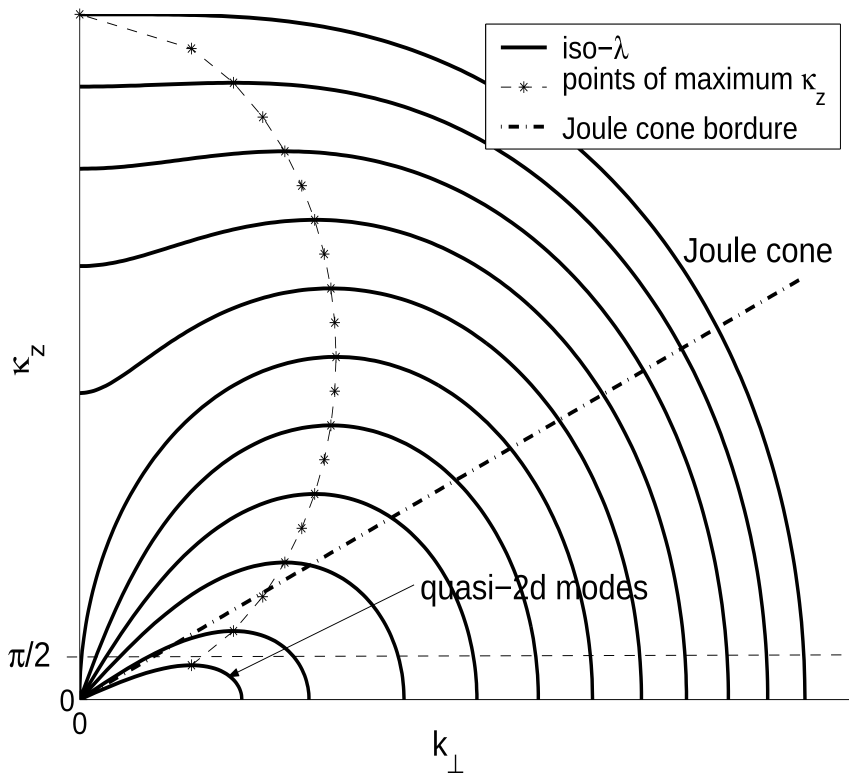
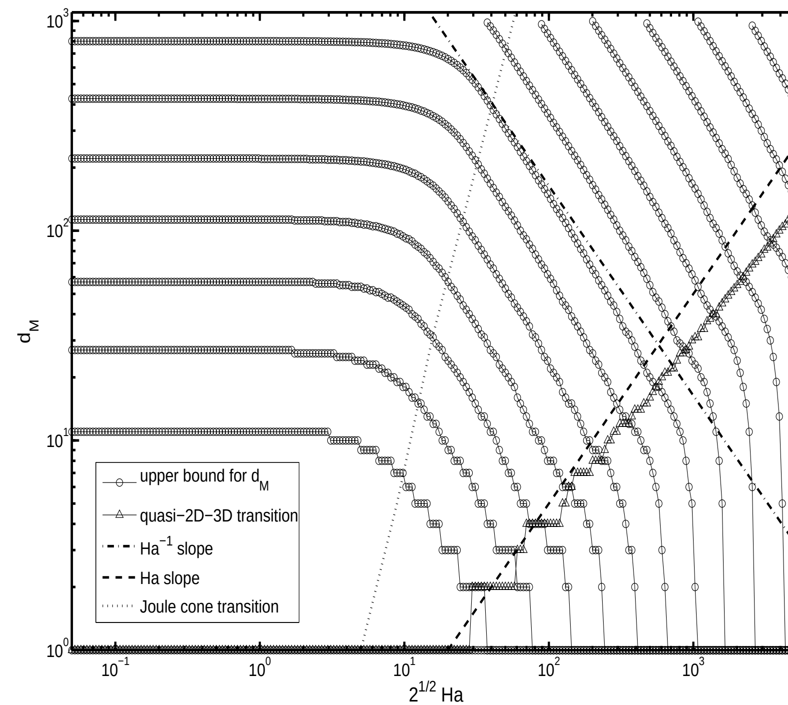


Attractor dimension



→ $\mathcal{B}$ estimate

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Analytical estimates in the case $\frac{Ha^2}{Re} \sim 1$

Estimates	Heuristics
$d_M \leq \frac{9\pi^5}{256\sqrt{2}} \frac{Re^4}{Ha}$	$N \sim \frac{Re^2}{Ha}$
$k_{\perp_m} \leq \left(\frac{3}{2\pi^2}\right)^{\frac{1}{4}} Re$	$k_{\perp_m} \sim Re^{\frac{1}{2}}$
$\kappa_z \leq \left(\frac{3}{2\pi^2}\right)^{\frac{1}{2}} \frac{Re^2}{Ha}$	$k_z \sim \frac{Re}{Ha}$
$\sin\theta \leq c\frac{Re}{Ha}$	$\sin\theta \sim \left(\frac{Re}{Ha^2}\right)^{\frac{1}{2}}$ (Sommeria & Moreau 82)

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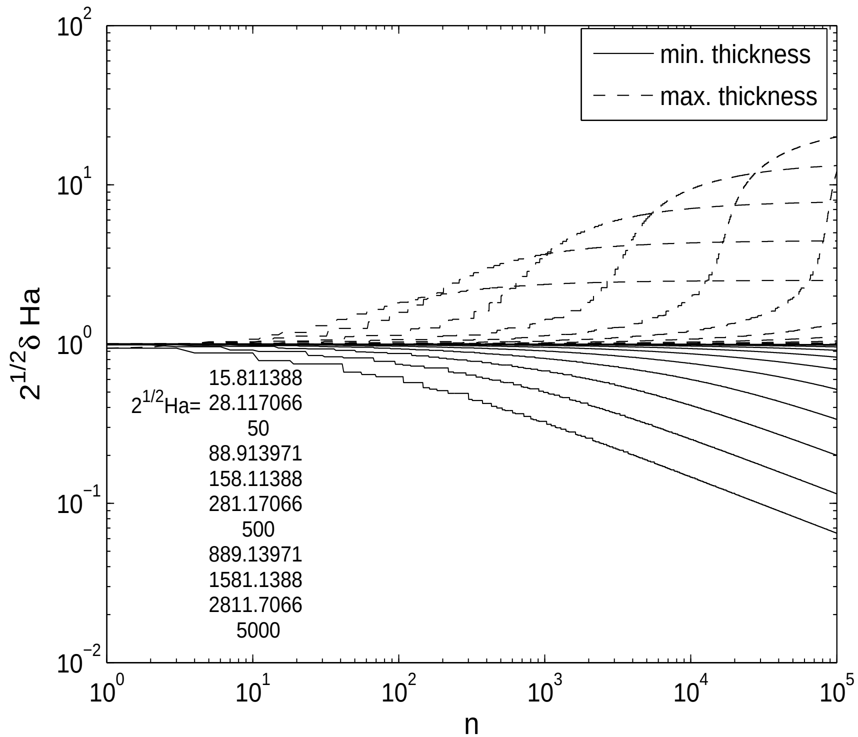
$d_M \leq \frac{9\pi^5}{256\sqrt{2}} \frac{Re^4}{Ha}$	$N \sim \frac{Re^2}{Ha}$
$k_{\perp_m} \leq \left(\frac{3}{2\pi^2}\right)^{\frac{1}{4}} Re$	$k_{\perp_m} \sim Re^{\frac{1}{2}}$
$\kappa_z \leq \left(\frac{3}{2\pi^2}\right)^{\frac{1}{2}} \frac{Re^2}{Ha}$	$k_z \sim \frac{Re}{Ha}$
$\sin\theta \leq c\frac{Re}{Ha}$	$\sin\theta \sim \left(\frac{Re}{Ha^2}\right)^{\frac{1}{2}}$ (Sommeria & Moreau 82)

$\longrightarrow$ Match on Ha, but Re^2 instead of Re

Modes classification according to δ



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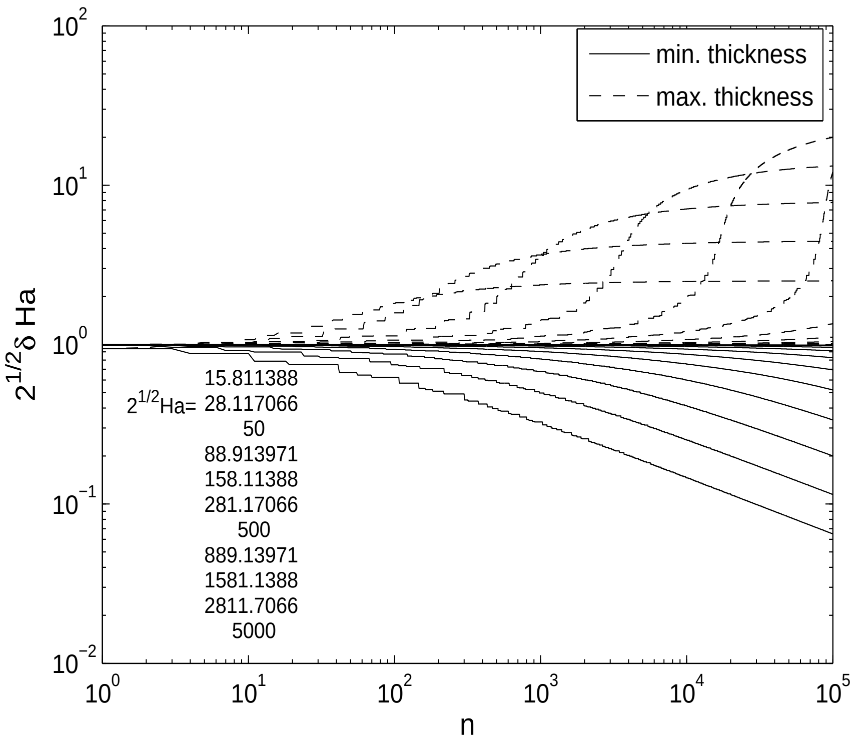
Modes classification according to δ



- Estimates

$$\delta_{min} = \frac{1}{Ha} \frac{1}{\sqrt{2}\sqrt{1+\frac{Re^2}{2Ha^2}}}$$

$$\delta_{max} = \frac{1}{Ha} \frac{1}{\sqrt{2}\sqrt{1-\frac{Re^2}{2Ha^2}}}$$

- Heuristics
  - $\delta_{laminar} \sim \frac{1}{Ha}$
  - double deck turbulent Hartmann layer

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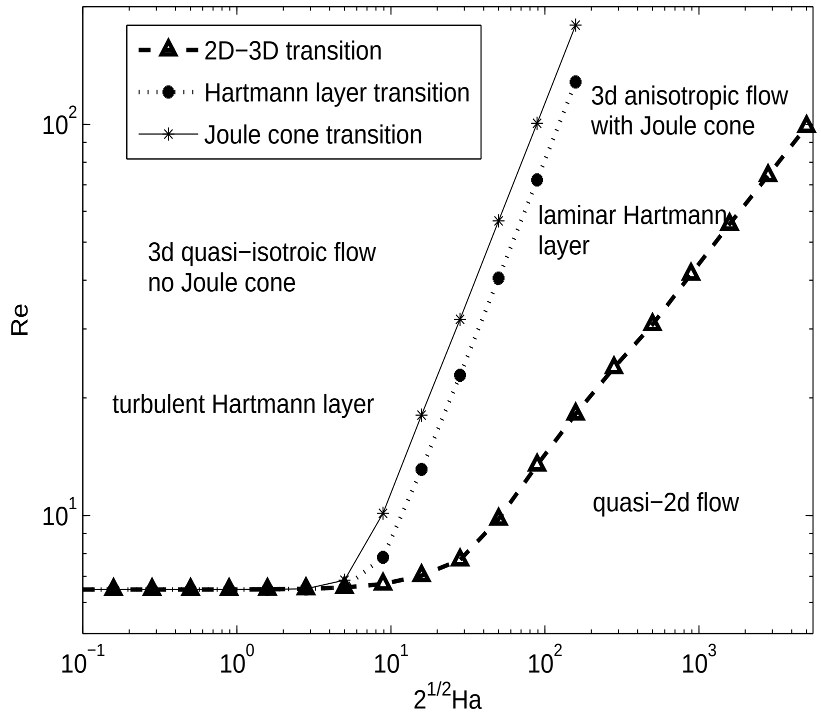
Modes classification according to δ



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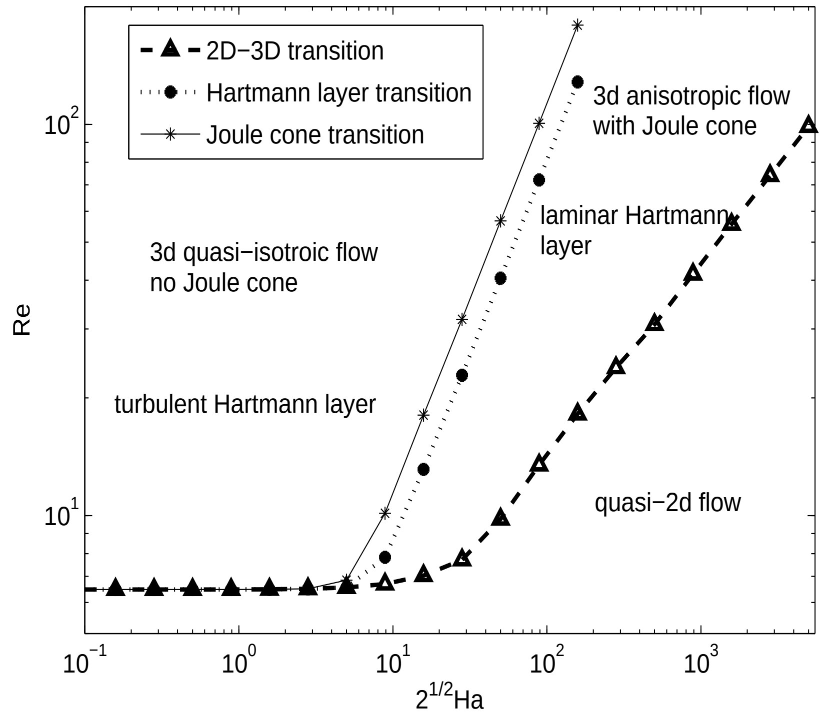
Modes classification according to δ



	Estimates	Heuristics
boundary separating quasi- 2D and 3D sets:	$Re^2 = cHa$	$Re \sim Ha$
b. s. sets with and without Joule cone:	$Re = \frac{\sqrt{3}}{5^{1/4}} Ha^2$	$Re \sim Ha$
b. s. sets with single and multiple layers:	$Re = cHa^2$	?

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Concluding remarks

- Strict bounds for the attractor dimension between 2 walls, with good agreement on the exponent of Ha
- A particular forcing can take the real flow very far from the bounds
- Bounds could be improved with a better estimate from the inertial terms
- Hierarchy of least dissipative modes mimic the flow in great details
- LDM don't carry any Energy information but could be a good basis of functions for DNS