The Magnetorotational Instability

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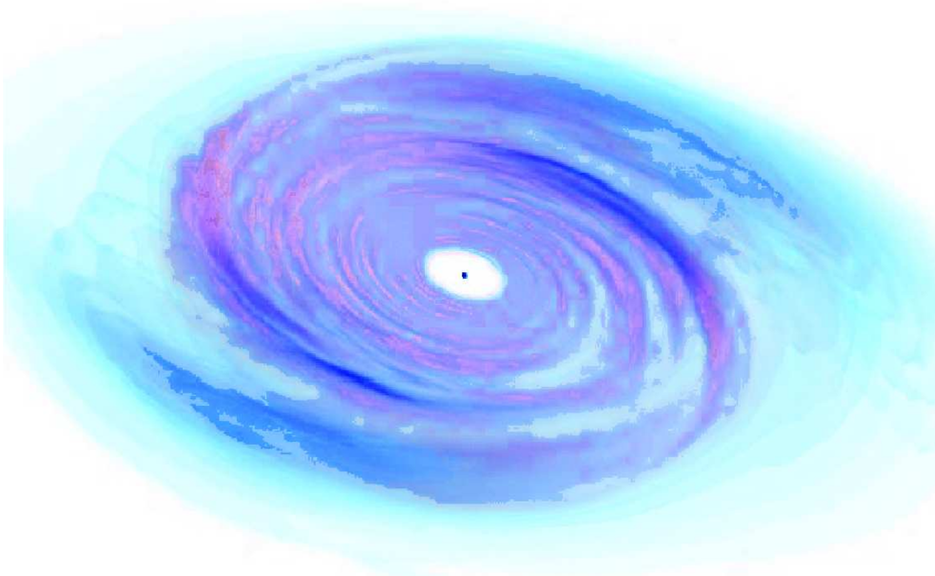
Consider an astrophysical accretion disk



Angular velocity $\Omega \sim r^{-3/2}$ decreases outward
Angular momentum $\Omega r^2 \sim r^{1/2}$ increases outward

For material to spiral inward, it must lose angular momentum, that is, transfer it to material further out.

How to accomplish this?

Molecular viscosity orders of magnitude too small.

Invoke turbulent viscosities... How to generate turbulence?

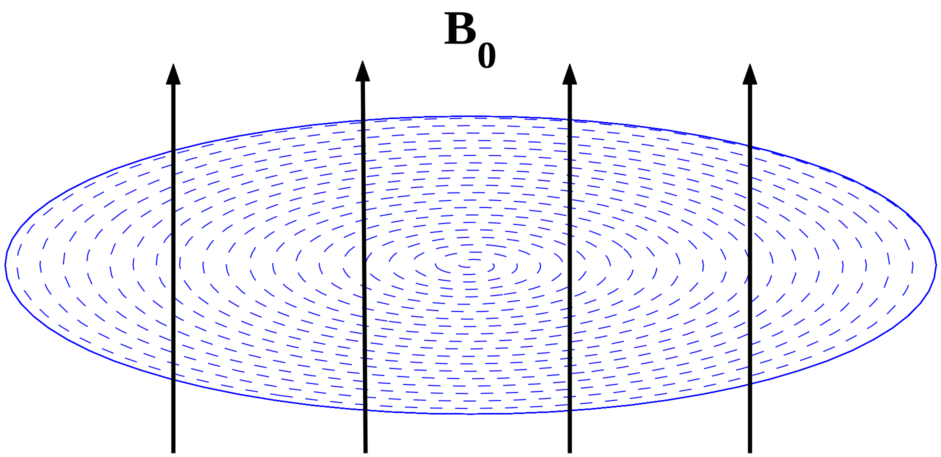
The Rayleigh Criterion: if the angular momentum increases outward, the flow is hydrodynamically stable.

The Magnetorotational Instability

A flow is *magneto*hydrodynamically stable only if the angular velocity itself increases outward, not just the angular momentum.

Velikhov 1959, Chandrasekhar 1960 (but not 1961!)

Balbus & Hawley 1991 (cited over 800 times)



$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{U} = -\nabla \Phi + \mathbf{B} \cdot \nabla \mathbf{B} / \mu\rho$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{U} \times \mathbf{B})$$

$$\mathbf{U} = \Omega r \, \hat{\mathbf{e}}_\phi + \mathbf{u} \qquad \mathbf{B} = B_0 \, \hat{\mathbf{e}}_z + \mathbf{b}$$

WKB Analysis: Look for modes $\mathbf{u}, \mathbf{b} \sim \exp(ikz + i\omega t)$

$$i\omega\, u_\phi + r^{-1}(\Omega r^2)'\, u_r = ikB_0\, b_\phi/\mu\rho$$

$$i\omega\, u_r - 2\Omega\, u_\phi = ikB_0\, b_r/\mu\rho$$

$$i\omega\, b_\phi = ikB_0\, u_\phi + r\Omega'\, b_r$$

$$i\omega\, b_r = ikB_0\, u_r$$

Nonmagnetic:

$$i\omega\, u_r - 2\Omega\, u_\phi = 0, \qquad i\omega\, u_\phi + r^{-1}(\Omega r^2)'\, u_r = 0$$

$$\omega^2 = 2\Omega\, r^{-1}(\Omega r^2)', \quad \text{so stable if} \quad (\Omega r^2)' > 0$$

<u>The Rayleigh Criterion</u>

Magnetic:

$$\omega^4 - \omega^2\left[2k^2 v_A^2 + 2\Omega r^{-1}(\Omega r^2)'\right] + k^2 v_A^2\left[k^2 v_A^2 + r(\Omega^2)'\right] = 0$$

$$v_A = B_0 \big/ \sqrt{\mu\rho}$$

If we now set $v_A = 0$, do we recover the previous nonmagnetic result? Not quite!

$$\omega^4 = \omega^2\left[2\Omega r^{-1}(\Omega r^2)'\right]$$

$$\omega^4 - \omega^2 \left[2k^2 v_A^2 + 2\Omega r^{-1} (\Omega r^2)'\right] + k^2 v_A^2 \left[k^2 v_A^2 + r(\Omega^2)'\right] = 0$$

$$\omega^4 - b\omega^2 + c = 0 \quad \Longrightarrow \quad \omega_\pm^2 = \frac{1}{2}\left[b \pm \sqrt{b^2 - 4c}\right]$$

Stability requires

- $b^2 - 4c > 0$, otherwise $\omega_\pm^2$ would be complex
- $4c > 0$, otherwise ω_-^2 would be negative

Stability therefore requires that $k^2 v_A^2 + r(\Omega^2)' > 0$

For $v_A \to 0$, this reduces to $\Omega' > 0$, **not** $(\Omega r^2)' > 0$

Even for $v_A \neq 0$, we can always let $k \to 0$,

so the stability criterion is always $\Omega' > 0$.

The Magnetorotational Instability

Any flow in which Ω decreases outward is unstable,

regardless of how weak or strong the field $\mathbf{B}_0$ *is.*

$$(i\omega)_{\max} = r|\Omega'|/2, \quad \text{for} \quad k^2 v_A^2 = -r\Omega'(\Omega + r\Omega'/4)$$

$$\text{For } \Omega \sim r^{-3/2}, \;\; (i\omega)_{\max} = \tfrac{3}{4}\,\Omega, \quad \text{for} \;\; kv_A = \tfrac{\sqrt{15}}{4}\Omega$$

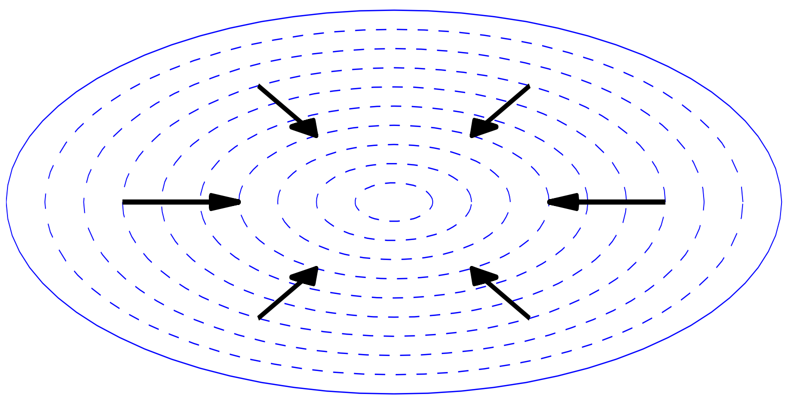
$$\text{Wavelength } \lambda = \tfrac{2\pi}{k} = \tfrac{8\pi}{\sqrt{15}}\,\Omega^{-1}\,v_A \sim B_0$$

- If B_0 is too weak, λ is too small, and diffusive effects kill off the MRI after all.

- If B_0 is too strong, λ is bigger than the entire disk, again killing off the MRI.

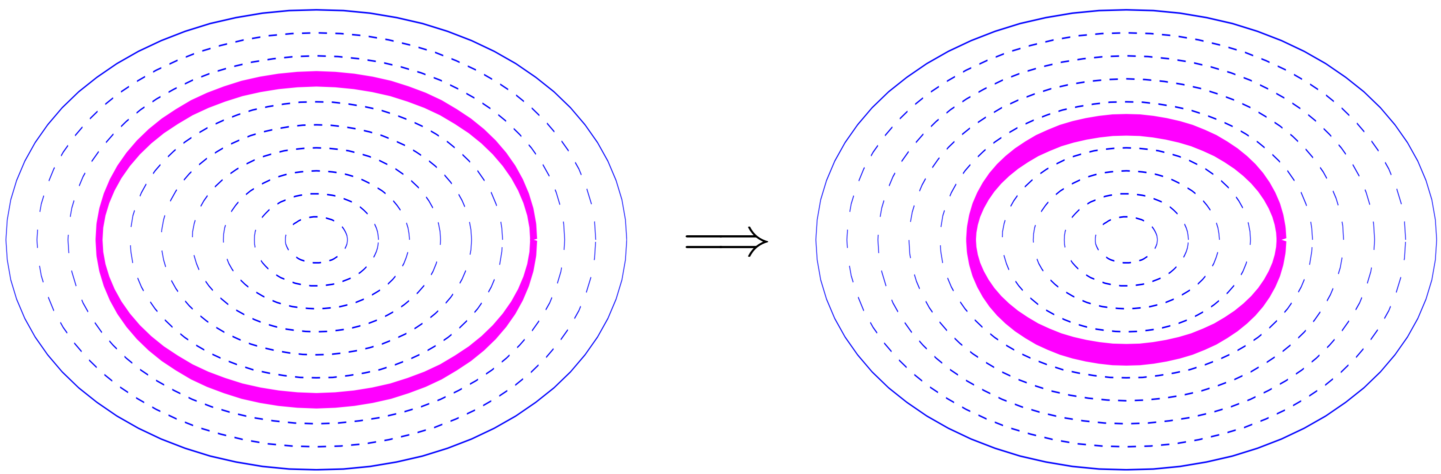
The Rayleigh Criterion: A Physical Interpretation

$$\underline{-\nabla\Phi = -\,\Omega^2 r\,\hat{\mathbf{e}}_r}$$



Suppose the angular momentum increases outward.

Now imagine a disturbance that moves a ring of fluid inward:



Since the force field $-\nabla\Phi$ is purely radial, the ring will conserve its A.M. Its A.M. (and hence velocity) is then greater than that of the surrounding fluid at the new radius. It is therefore rotating too fast for the force field at this r, so it will move outward again. The flow is therefore stable.

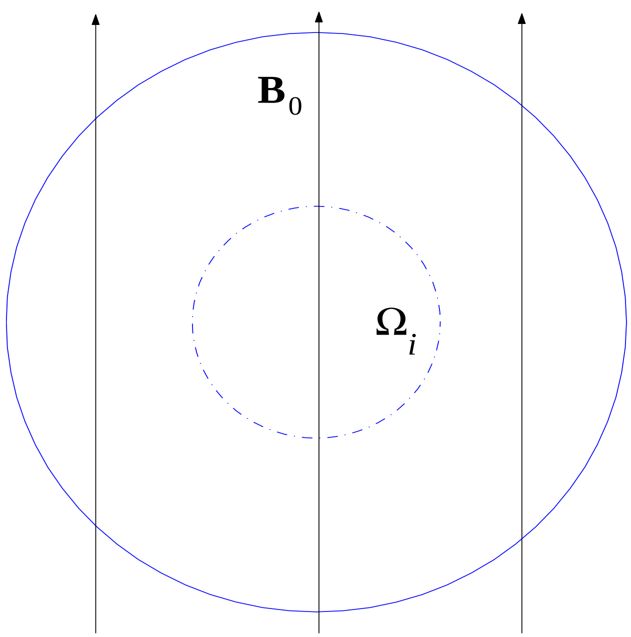
The MRI: A Physical Model

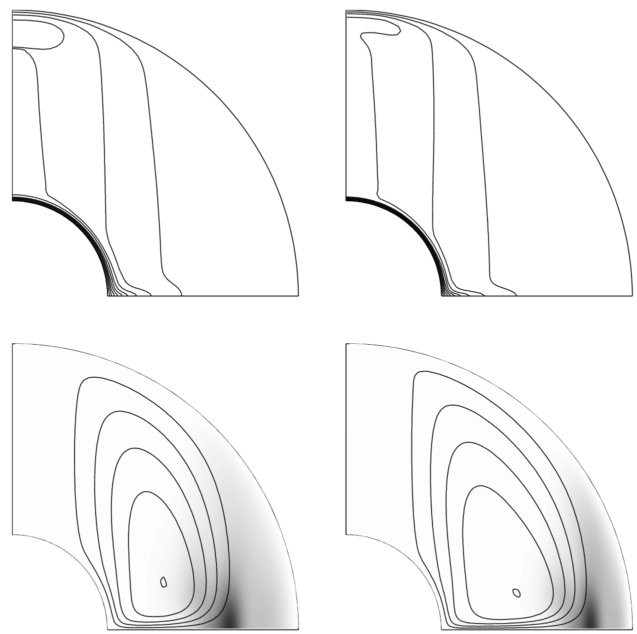
The magnetic tension force $\mathbf{B} \cdot \nabla \mathbf{B}/\mu\rho$ is not purely radial.
A.M. is therefore not conserved on individual fluid rings,
and the Rayleigh criterion simply does not apply.

- Angular momentum is indeed transferred *outward*.
- An azimuthal initial field $\mathbf{B}_0$ would not work.
- The azimuthal component of $\mathbf{b}$ is crucial though!

The MRI in spherical Couette flow?

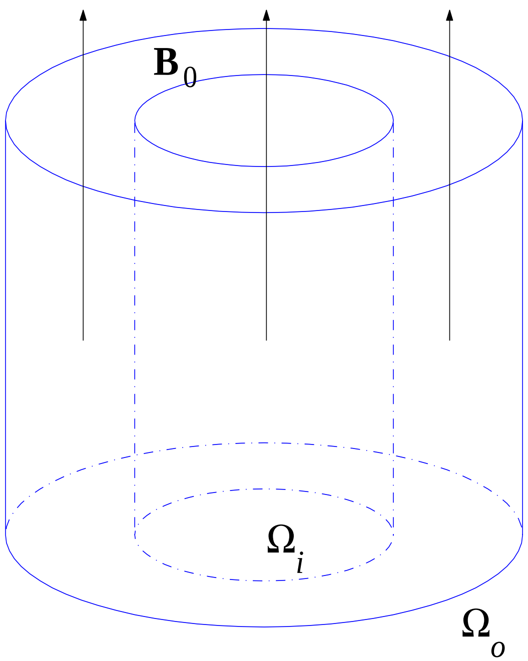
Sisan *et al.*, Phys. Rev. Lett. **93**, 114502 (2004)





The MRI in cylindrical Taylor-Couette flow

Rüdiger & Zhang 2001, Ji *et al.* 2001



$$\Omega(r) = c_1 + c_2/r^2$$

$$\hat{\mu} = \frac{\Omega_o}{\Omega_i}, \quad Re = \frac{\Omega_i r_i^2}{\nu}$$

$$Ha = \frac{B_0 r_i}{\sqrt{\rho\mu\eta\nu}}, \quad Pm = \frac{\nu}{\eta}$$

$$\mathbf{u} = \nabla \times (\psi\, \hat{\mathbf{e}}_\phi) + v\, \hat{\mathbf{e}}_\phi \qquad \mathbf{b} = \nabla \times (a\, \hat{\mathbf{e}}_\phi) + b\, \hat{\mathbf{e}}_\phi$$

Look for modes $\sim \exp(ikz + \gamma t)$

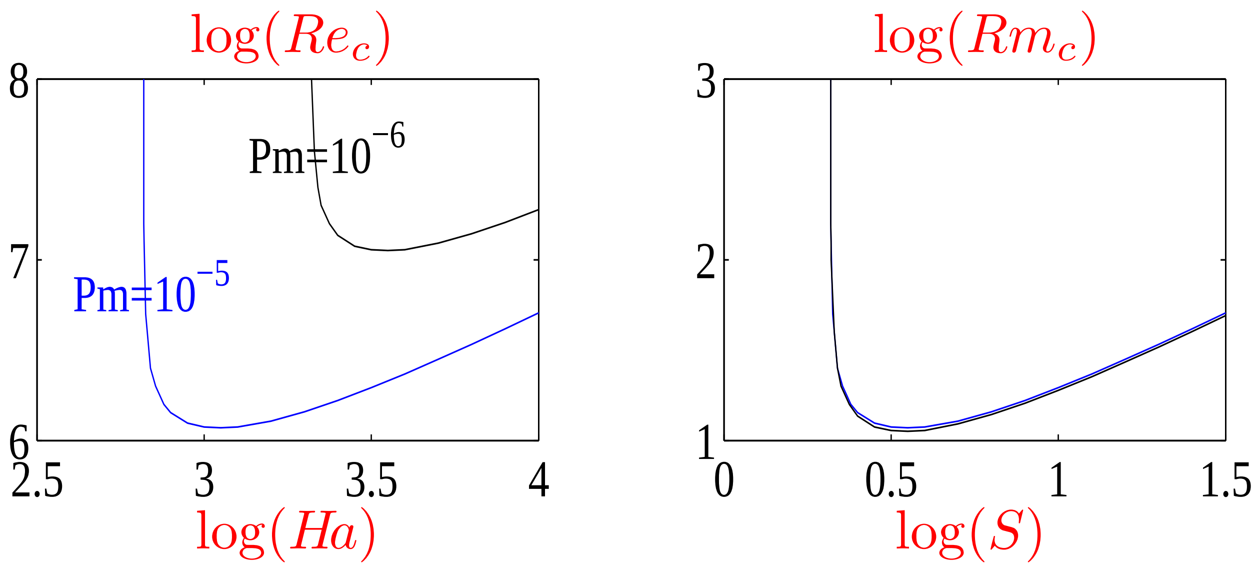
$$Re\, \gamma\, v = D^2 v + Re\, ik\, r^{-1} (\Omega r^2)'\, \psi + Ha^2\, ik\, b$$

$$Re\, \gamma\, D^2 \psi = D^4 \psi - Re\, 2ik\, \Omega\, v + Ha^2\, ik\, D^2 a$$

$$Pm\, Re\, \gamma\, b = D^2 b - Pm\, Re\, ik\, \Omega' r\, a + ik\, v$$

$$Pm\, Re\, \gamma\, a = D^2 a + ik\, \psi$$

$$\text{Fix} \quad r_i/r_o = 1/2, \;\; \hat{\mu} = \Omega_o/\Omega_i = 0.3$$



$$Re_c \sim Pm^{-1} \quad \Longrightarrow \quad Pm\, Re = \Omega_i r_i^2/\eta \equiv Rm \approx 10$$

$$Ha_c \sim Pm^{-1/2} \quad \Longrightarrow \quad Pm^{1/2}\, Ha = v_A r_i/\eta \equiv S \approx 3$$

Eq. for b : $Pm\, Re\, \gamma\, b = D^2 b - \underline{Pm\, Re\, ik\, \Omega' r\, a} + ik\, v$

Take the imposed field to be $\mathbf{B}_0 = \hat{\mathbf{e}}_z + \beta\,(r_i/r)\,\hat{\mathbf{e}}_\phi$

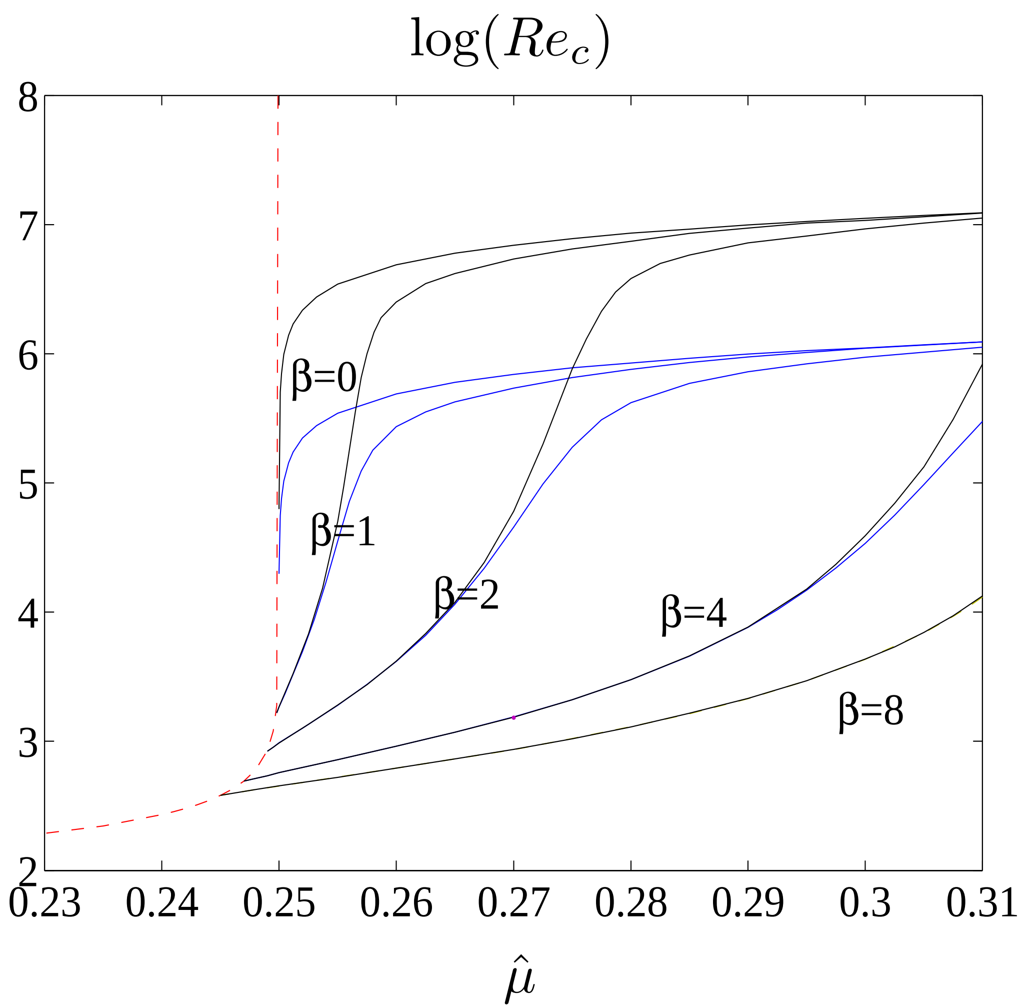
Hollerbach & Rüdiger 2005

$$Re\,\gamma\,v = D^2 v + Re\,ik\,r^{-1}(\Omega r^2)'\,\psi + Ha^2\,ik\,b$$

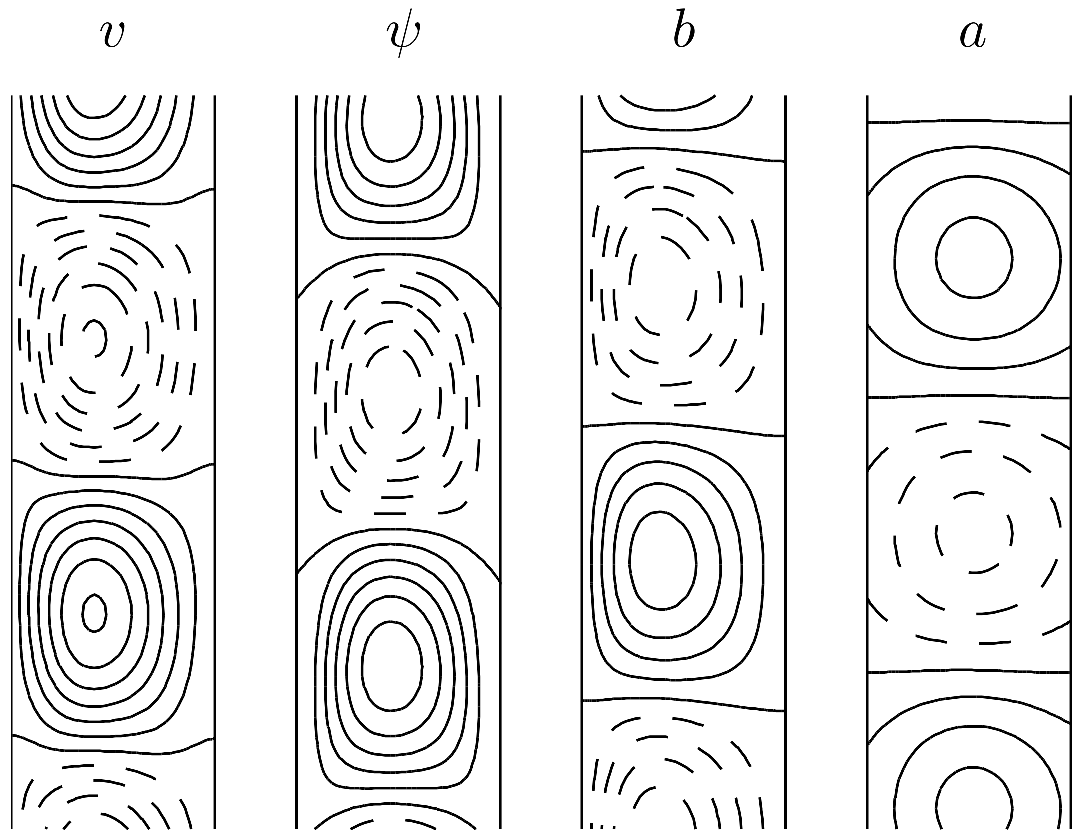
$$Re\,\gamma\,D^2\psi = D^4\psi - Re\,2ik\,\Omega\,v + Ha^2\,ik(D^2 a + 2\beta r^{-2} b)$$

$$Pm\,Re\,\gamma\,b = D^2 b - Pm\,Re\,ik\,\Omega' r\,a + ik\,v - 2ik\,\beta r^{-2}\psi$$

$$Pm\,Re\,\gamma\,a = D^2 a + ik\,\psi$$


log(Re_c)
8
7
6
5
4
3
2
0.23
0.24
0.25
0.26
0.27
0.28
0.29
0.3
0.31
μ̂
β=0
β=1
β=2
β=4
β=8

$Pm = 10^{-5}$ $Pm = 10^{-6}$



Drift speed $0.066\,\Omega_i r_i$

‘PROMISE’

Potsdam ROssendorf Magnetic InStability Experiment

P=Rüdiger; RO =Stefani, Gundrum, Gerbeth; ...