Dynamo Theory

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Sir Joseph Larmor (1919)

"Such internal motion induces an electric field acting on the moving matter: and if any conducting path around the Solar axis happens to be open, an electric current will flow round it, which may in turn increase the inducing magnetic field. In this way it is possible for the internal cyclic motion to act after the manner of the cycle of a self-exciting dynamo, and maintain a permanent magnetic field from insignificant beginnings, at the expense of some of the energy of the internal circulation."

Electric conductor moving in a magnetic field
=> electromotive force => electric current =>
magnetic field

Possibility of self-excited field, as in a usual electrical dynamo.

Equations

$$\nabla \cdot \boldsymbol{E} = \rho/\varepsilon,$$
$$\nabla \cdot \boldsymbol{B} = 0,$$
$$\nabla \times \boldsymbol{B} = \mu \boldsymbol{J} + \mu\varepsilon\partial_t \boldsymbol{E},$$
$$\nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B}.$$

Maxwell's equations

$$\boldsymbol{J} = \sigma(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}).$$

Ohm's law

Neglecting the displacement current $\mu\varepsilon\partial_t \boldsymbol{E}$,
(speeds much less than c) gives the induction equation

$$\partial_t \boldsymbol{B} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) - \nabla \times (\eta \nabla \times \boldsymbol{B}).$$

with the magnetic diffusivity

$$\eta = (\sigma\mu)^{-1}$$

Simplifications

For constant magnetic diffusivity and incompressible fluid flow $\nabla \cdot \boldsymbol{u} = 0$, we are left with:

$$\partial_t \boldsymbol{B} + \boldsymbol{u} \cdot \nabla \boldsymbol{B} = \boldsymbol{B} \cdot \nabla \boldsymbol{u} + \eta \nabla^2 \boldsymbol{B}. \qquad \nabla \cdot \boldsymbol{B} = 0,$$

Note analogy with vorticity equation: magnetic field line stretching competes with diffusion. By analogy we can define a magnetic Reynolds number R=UL/η with

$$\partial_t \boldsymbol{B} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) + R^{-1} \nabla^2 \boldsymbol{B}, \quad \nabla \cdot \boldsymbol{B} = 0,$$

Can measure growth in magnetic energy in volume V:

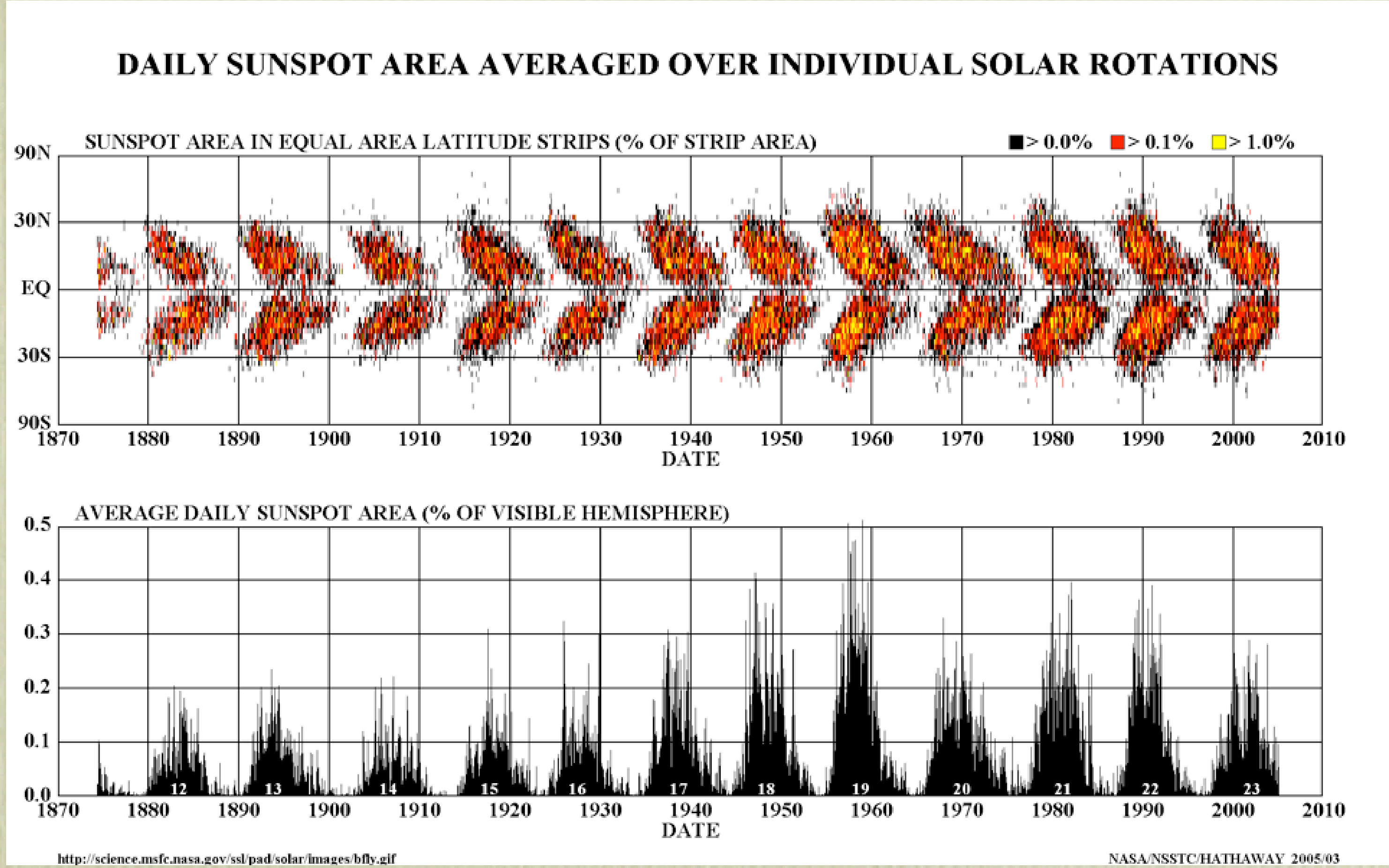
$$\mathcal{E}_V(t) = \frac{1}{2\mu} \int_V |\boldsymbol{B}|^2 \, dV,$$

Dynamos

- We have a `dynamo' if the magnetic field is sustained: magnetic energy does not tend to zero as t tends to infinity for some value of R.

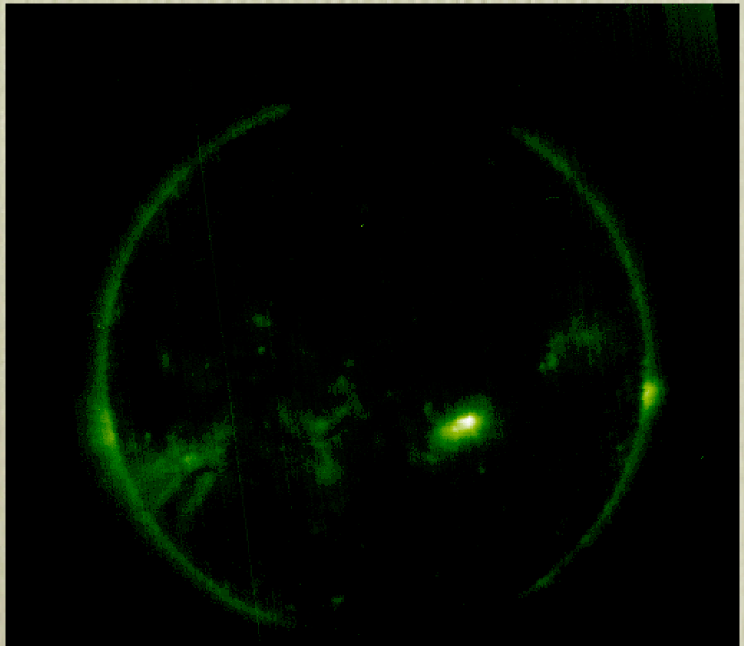
- Kinematic problem: given a flow u(x,y,z,t), how fast does the magnetic energy grow? Linear, eigenvalue problem - lots of theory, clean issues.

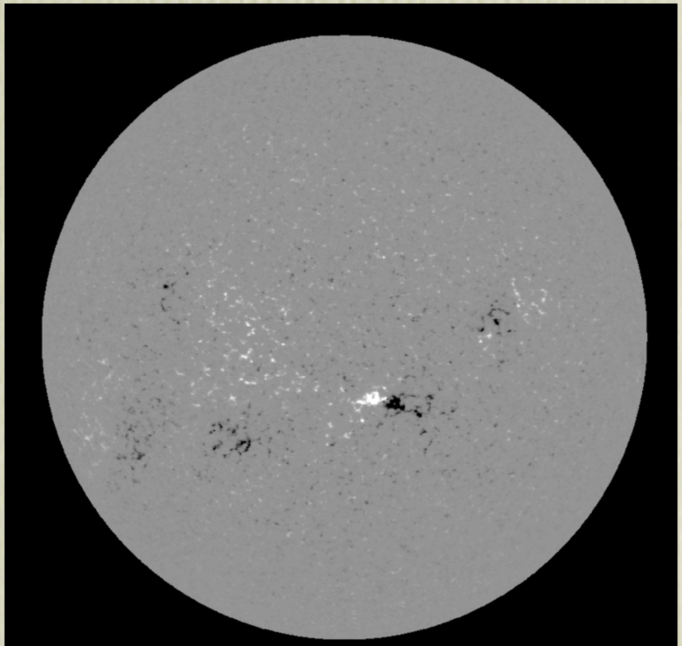
- Dynamical problem: given a mechanism for driving a flow (convection, shear, paddles) how does the field grow and saturate? Nonlinear, chaotic, issues of (MHD) turbulence. Usually requires numerical treatment - little theory.



Solar dynamo: 11 year solar cycle

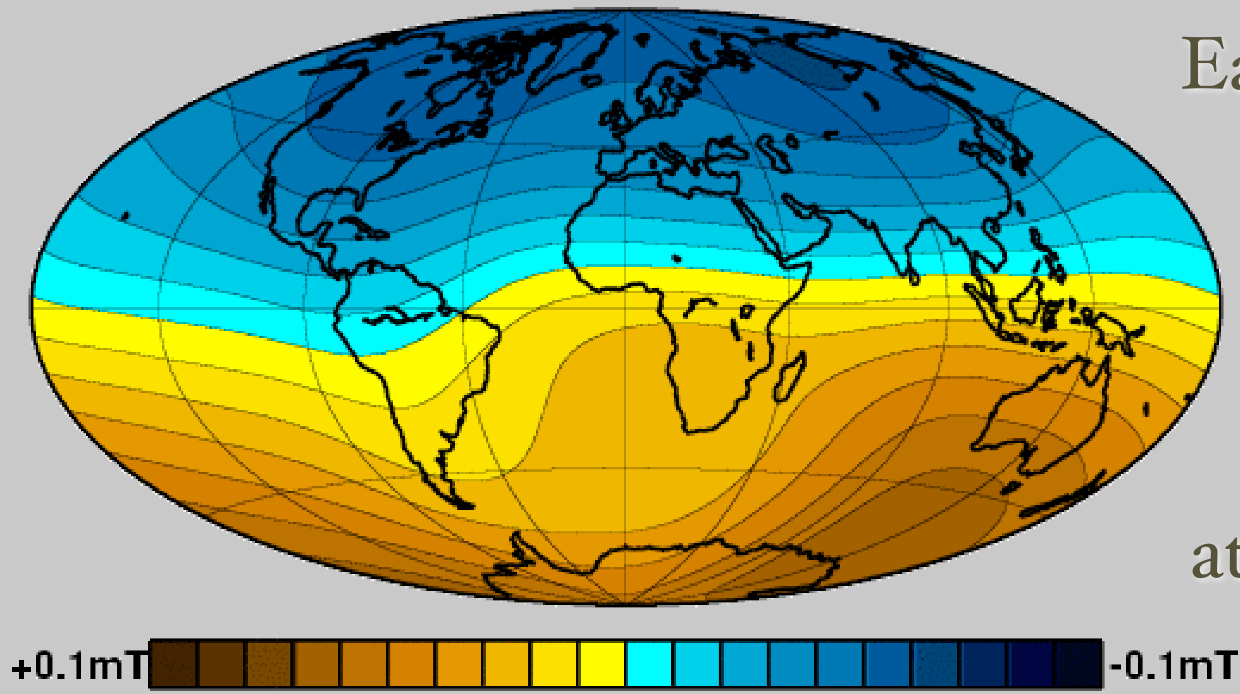
NASA/NSSTC/Hathaway 2005



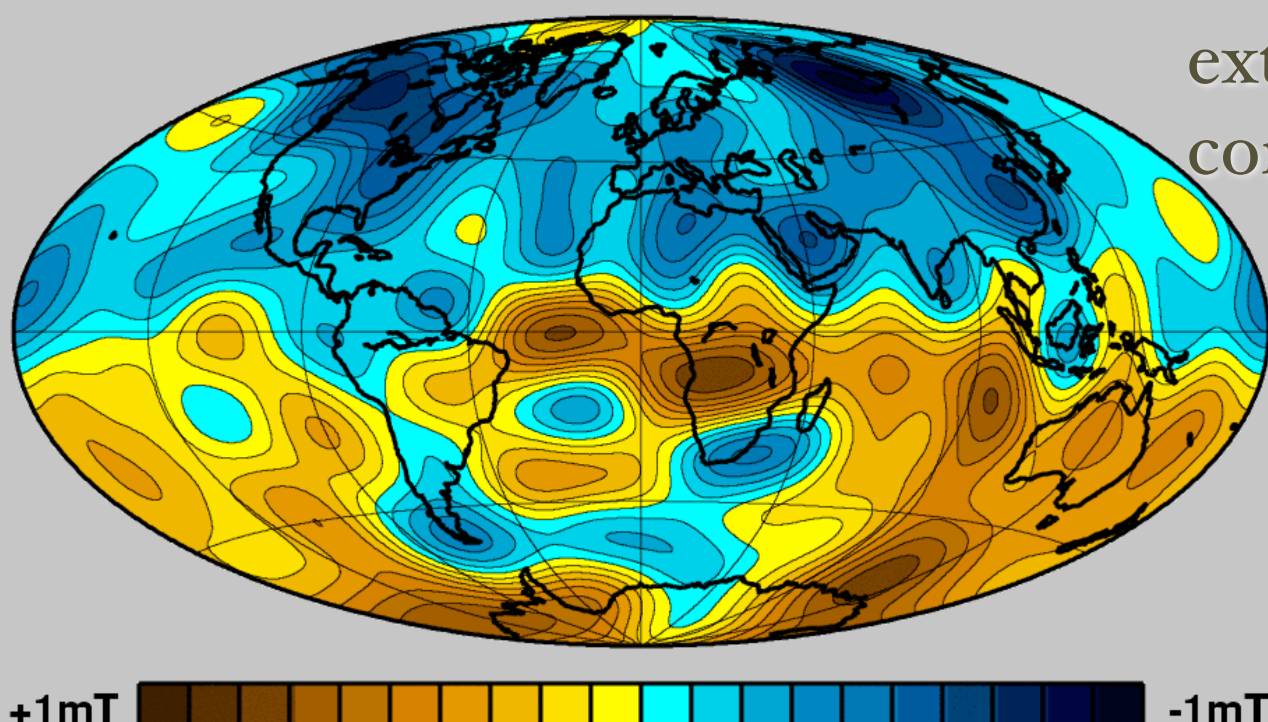


NASA

Earth's magnetic field:



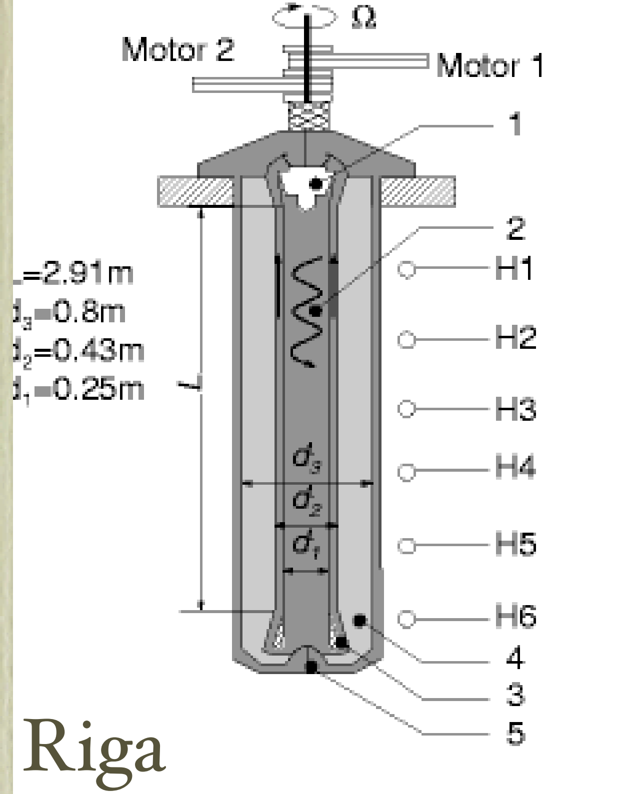
at the surface,



extrapolated down to the core-mantle boundary

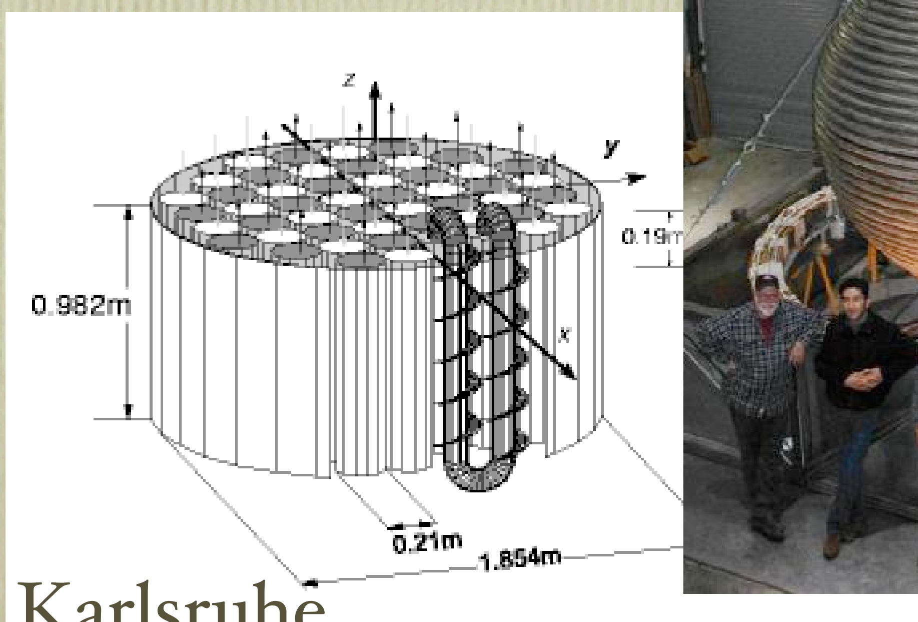
Holne, Potsdam

Madison



Riga

Perm



Karlsruhe

Maryland

Cadaraches

Dynamo experiments

Why are dynamos interesting?

- Many astrophysical applications: galaxies, stars, planets, earth, accretion discs.
- Require complex flows (anti-dynamo theorems).
- Interesting linear problems (kinematic regime).
- Immensely challenging 3-d nonlinear problems.
- Wide scope: theory, numerics, observation and experiment.
- Classical problem: William Gilbert, De Magnete 1600.

Anti-dynamo theorems

- A magnetic field of the form B(x,y,t) cannot be maintained by dynamo action.
- A \`planar' flow, of the form (u(x,y,z,t),v(x,y,z,t),0) cannot maintain a magnetic field (Zeldovich's theorem).
- An axisymmetric magnetic field, of the form B(r,ϑ,t) in (r, ϑ,ϕ) coordinates, cannot be maintained by dynamo action (Cowling's theorem).
- A \`toroidal' flow, of the form (0,v(r, ϑ,ϕ,t),w(r, ϑ,ϕ,t)) in (r, ϑ,ϕ) coordinates, cannot maintain a magnetic field.

In short: we can't achieve a dynamo with a \`simple' flow or \`simple' field.

Upper bounds

Under the induction equation

$$\partial_t \boldsymbol{B} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) - \nabla \times (\eta \nabla \times \boldsymbol{B}).$$

magnetic energy

$$\mathcal{E}_V(t) = \frac{1}{2\mu} \int_V |\boldsymbol{B}|^2 \, dV,$$

obeys

$$\mu \partial_t \mathcal{E}_V = -\eta \int_{V_\mathrm{i}} |\nabla \times \boldsymbol{B}|^2 \, dV - \int_{V_\mathrm{i}} \boldsymbol{u} \cdot (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} \, dV$$

In a sphere of radius a (insulating boundary conditions), a dynamo requires $R = a u_{\mathrm{max}}/\eta \geq \pi$. (Childress).

In a sphere of radius a, a dynamo requires $R = a^2 e_{\mathrm{max}}/\eta \geq \pi^2$, (Backus), where e_{max} is the largest eigenvalue of the rate-of-strain matrix $e_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$

Key points: need stretching, R large enough.

Smooth Ponomarenko dynamo

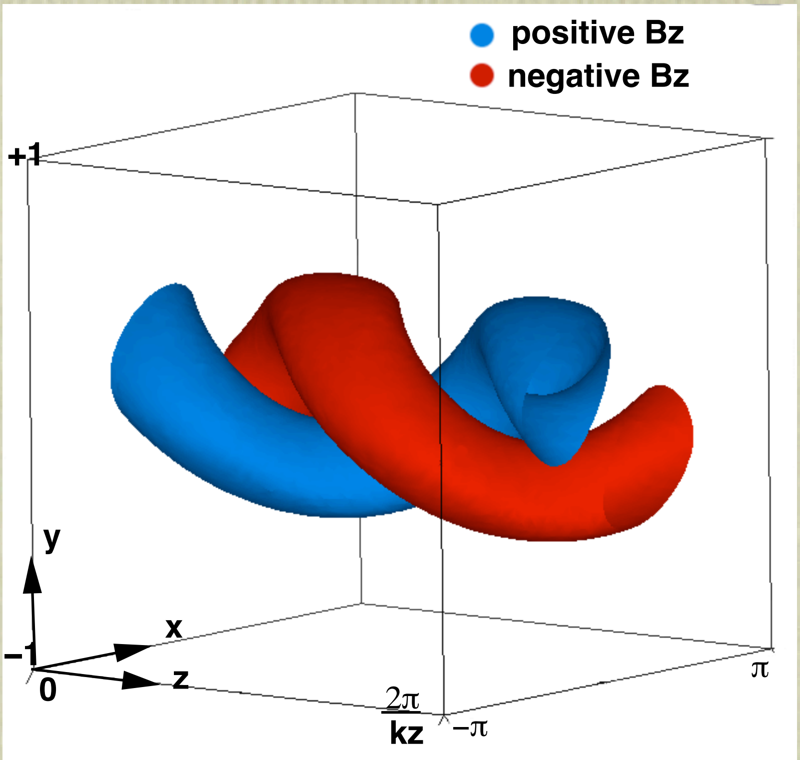
Simplest dynamo to study, $\boldsymbol{u} = \hat{\boldsymbol{\theta}}\, r\Omega(r) + \hat{\boldsymbol{z}}\, U(r),$ in cylindrical polars (r,ϑ,z).

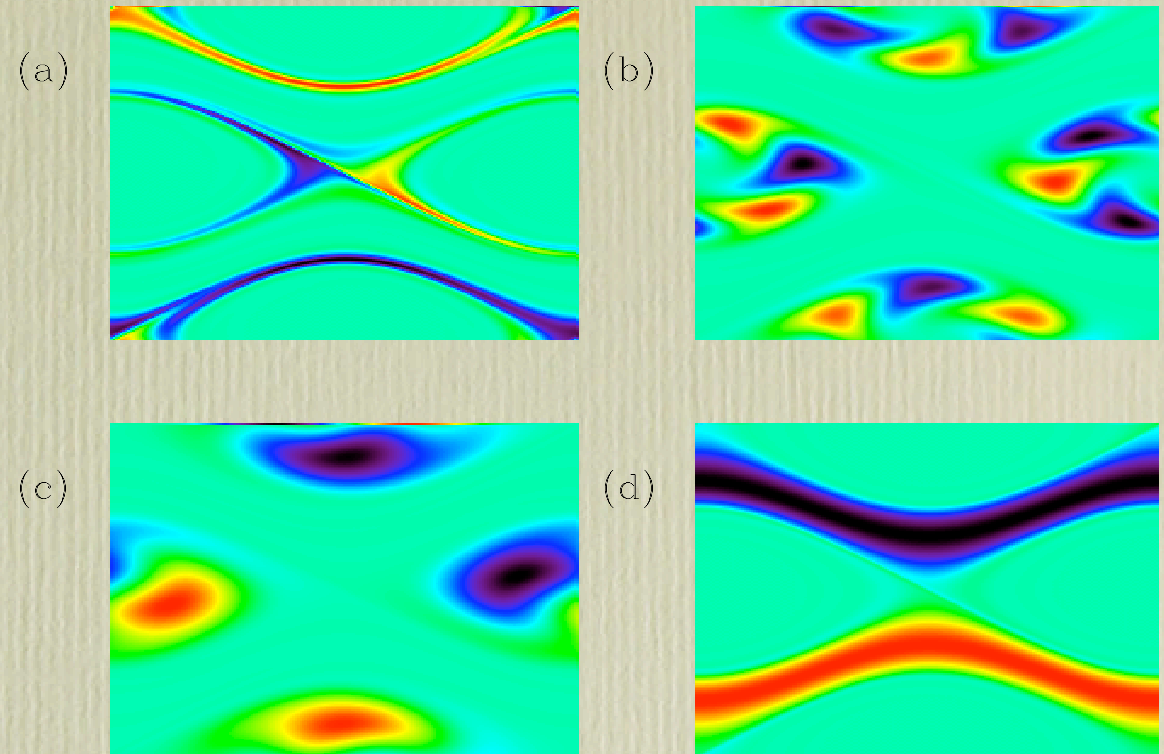
Solve $\partial_t \boldsymbol{B} + \boldsymbol{u}\cdot\nabla\boldsymbol{B} = \boldsymbol{B}\cdot\nabla\boldsymbol{u} + \varepsilon\nabla^2\boldsymbol{B},$ numerically, or approximately for small ϵ=1/R. Put $\boldsymbol{B} = \boldsymbol{b}(r)e^{\lambda t + im\theta + ikz}$

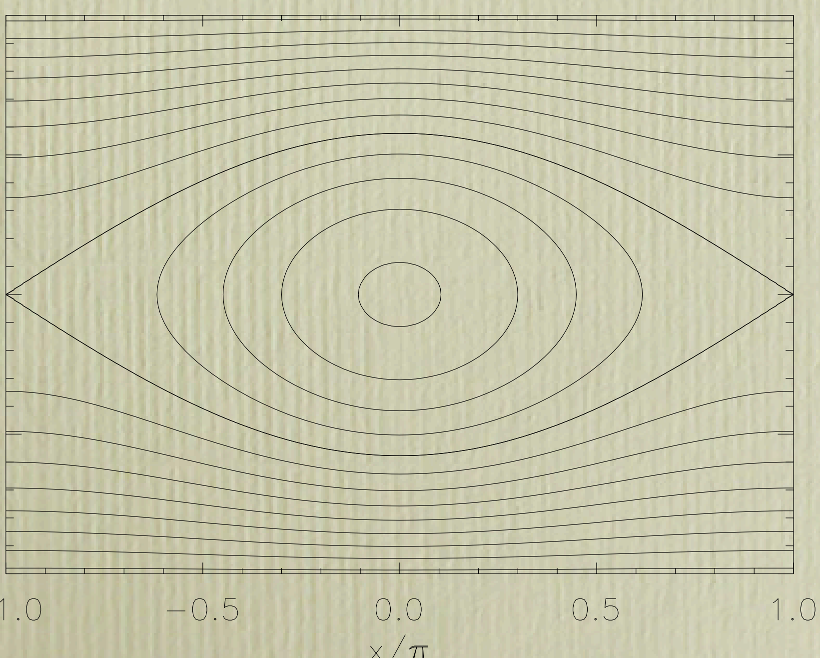
$$
\begin{aligned}
(\lambda + im\Omega(r) + ikU(r))b_r &= \varepsilon((\Delta_m - r^{-2})b_r - 2imr^{-2}b_\theta),\\
(\lambda + im\Omega(r) + ikU(r))b_\theta &= r\Omega'(r)b_r + \varepsilon((\Delta_m - r^{-2})b_\theta + 2imr^{-2}b_r),\\
(\lambda + im\Omega(r) + ikU(r))b_z &= U'(r)b_r + \varepsilon\Delta_m b_z.
\end{aligned}
$$

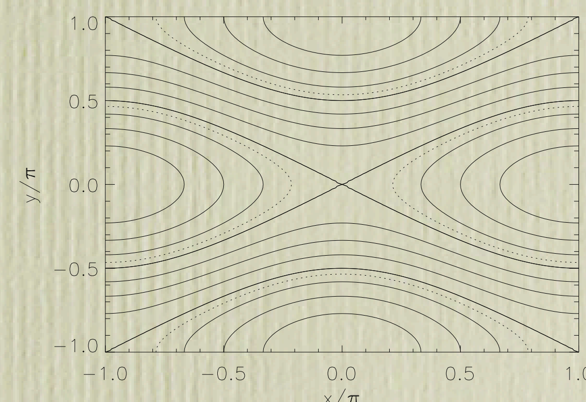
Mechanism: stretching of radial field, diffusion of azimuthal field. For small ϵ, a mode (m,k) localises at a resonant radius where mΩ'(a)+kU'(a)=0 and the growth rate is:

$$
\gamma = \mathrm{Re}\,\lambda \simeq \mp\sqrt{\frac{\varepsilon|m\Omega'(a)|}{a}} - (j+\tfrac{1}{2})\sqrt{\varepsilon|m\Omega''(a) + kU''(a)} - \varepsilon(m^2/a^2 + k^2)
$$


positive Bz
negative Bz
+1
y
x
z
−1
0
2π
kz
−π
π


(a)
(b)
(c)
(d)


1.0
−0.5
0.0
0.5
1.0
x/π


1.0
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y/π
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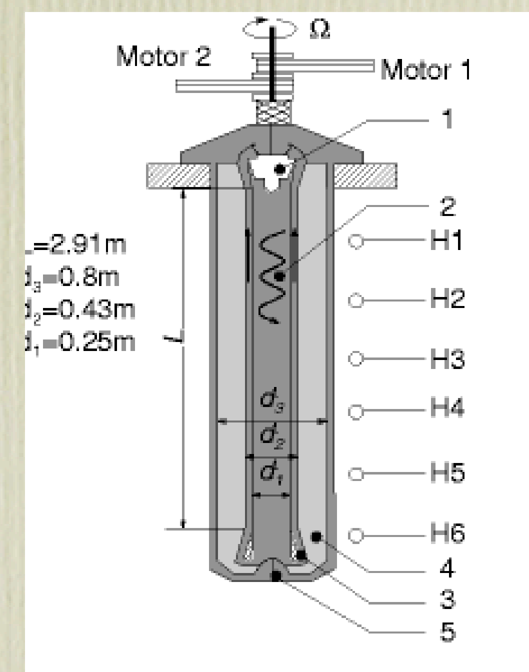
1.0
x/π


Ω
Motor 2
Motor 1
1
2
H1
H2
H3
H4
H5
H6
4
3
5

A geometrical condition

$$\gamma = \mathrm{Re}\,\lambda \simeq \mp\sqrt{\frac{\varepsilon|m\Omega'(a)|}{a}} - (j+\tfrac{1}{2})\sqrt{\varepsilon|m\Omega''(a)+kU''(a)} - \varepsilon(m^2/a^2+k^2)$$

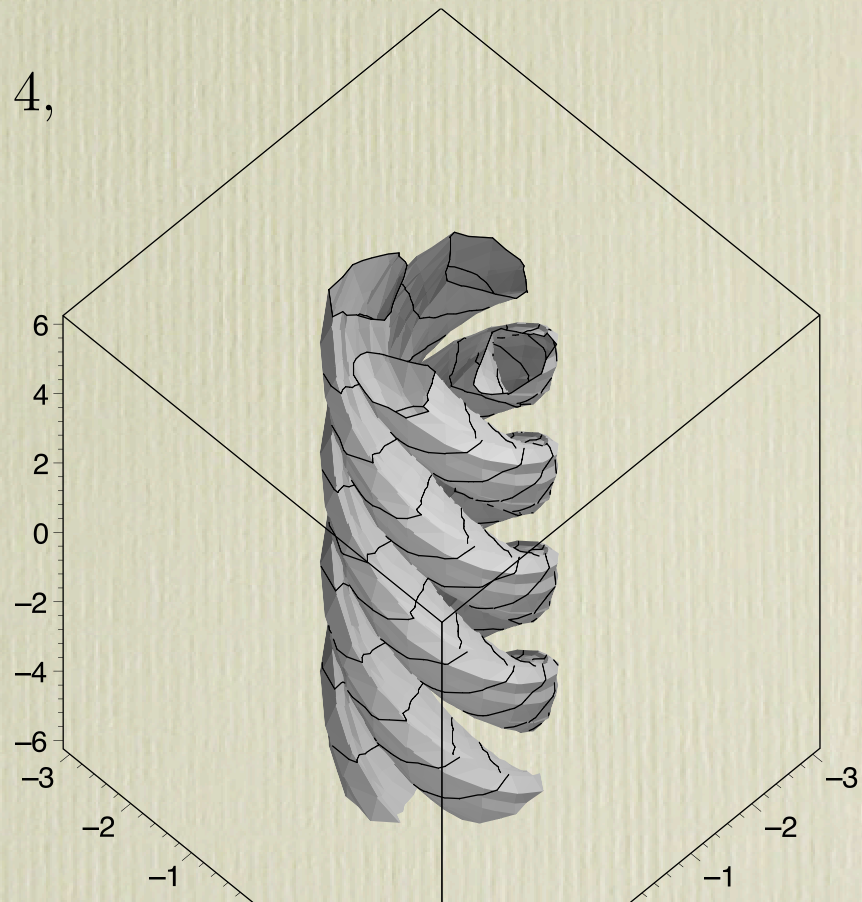
For growth, first term must dominate, which requires the purely geometrical condition to hold:

$$r\left|\frac{\Omega''(r)}{\Omega'(r)} - \frac{U''(r)}{U'(r)}\right| \equiv r\left|\frac{d}{dr}\log\left|\frac{\Omega'(r)}{U'(r)}\right|\right| < 4,$$

Holds for spiral Couette flow:

$$\Omega(r) = a + br^{-2}, \quad U(r) = c + d\log r,$$

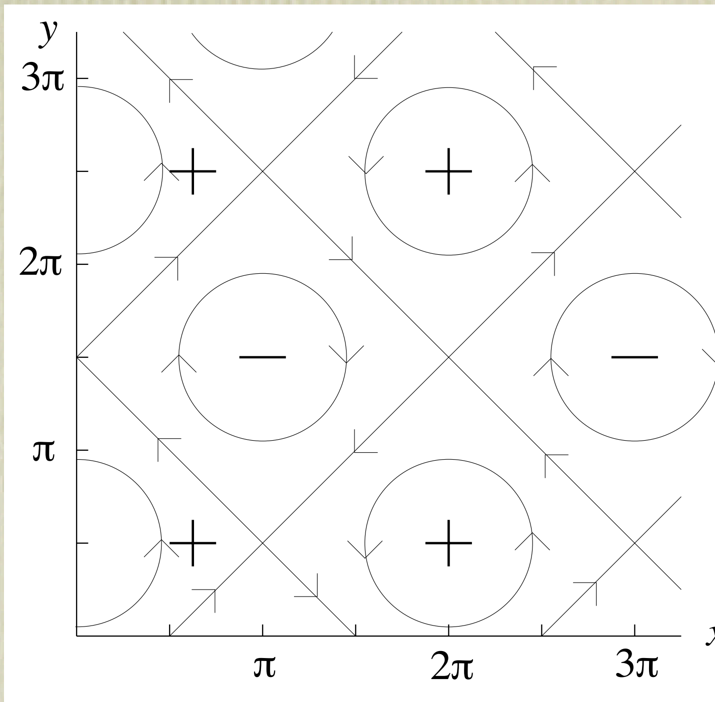
Generalisations for more general flows with stream surfaces.



G.O. Roberts flow

$$\boldsymbol{u} = (\cos y, \sin x, \sin y + \cos x) = (\partial_y \psi, -\partial_x \psi, \psi), \quad \psi = \sin y + \cos x,$$

$$\partial_t \boldsymbol{B} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) + R^{-1} \nabla^2 \boldsymbol{B}, \quad \nabla \cdot \boldsymbol{B} = 0,$$



Multiple scale analysis for small R,

$$\partial_t = R^{-1}\partial_\tau + R^3 \partial_T, \quad \nabla = \nabla_{\boldsymbol{x}} + R^2 \nabla_{\boldsymbol{X}}, \quad \boldsymbol{B} = \boldsymbol{B}_0 + R\boldsymbol{B}_1 + \cdots$$

Small scale flow generates B1 from Bo:

$$(\partial_\tau - \nabla_{\boldsymbol{x}}^2)\boldsymbol{B}_0 = 0,$$
$$(\partial_\tau - \nabla_{\boldsymbol{x}}^2)\boldsymbol{B}_1 = \nabla_{\boldsymbol{x}} \times (\boldsymbol{u} \times \boldsymbol{B}_0),$$

On large scales a `mean emf' sustains Bo

$$\partial_T \boldsymbol{B}_0 = \nabla_{\boldsymbol{X}} \times \langle \boldsymbol{u} \times \boldsymbol{B}_1 \rangle + \nabla_{\boldsymbol{X}}{}^2 \boldsymbol{B}_0,$$

$$\bar{\boldsymbol{\mathcal{E}}} \equiv \langle \boldsymbol{u} \times \boldsymbol{B}_1 \rangle, = \sum_{\boldsymbol{k},\omega} \frac{(i\boldsymbol{k} \cdot \boldsymbol{B}_0)(\hat{\boldsymbol{u}}^*_{\boldsymbol{k},\omega} \times \hat{\boldsymbol{u}}_{\boldsymbol{k},\omega})}{k^2 - i\omega} = \alpha_{mn} B_{0n}, \quad \alpha_{mn} = -\sum_{\boldsymbol{k},\omega} \frac{k_m k_n \hat{H}(\boldsymbol{k},\omega)}{k^4 + \omega^2}$$

$$H(\boldsymbol{k},\omega) = \hat{\boldsymbol{u}}^*_{\boldsymbol{k},\omega} \cdot i\boldsymbol{k} \times \hat{\boldsymbol{u}}_{\boldsymbol{k},\omega}$$

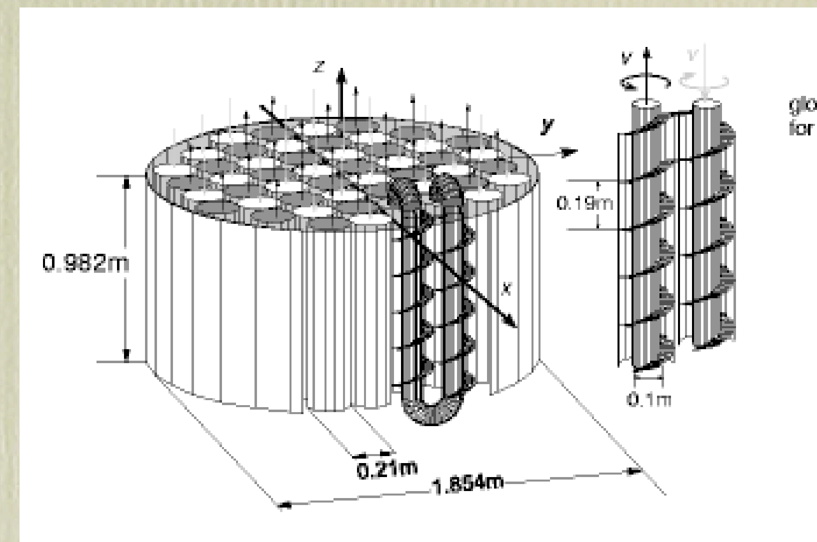
Alpha effect linked to helicity: $h = \boldsymbol{u} \cdot \boldsymbol{\omega}$

Large-scale modes obey

$$\partial_T \boldsymbol{B}_0 = \nabla_{\boldsymbol{X}} \times (\boldsymbol{\alpha} \cdot \boldsymbol{B}_0) + {\nabla_{\boldsymbol{X}}}^2 \boldsymbol{B}_0, \quad \nabla_{\boldsymbol{X}} \cdot \boldsymbol{B}_0 = 0.$$

with the alpha effect tensor

$$\boldsymbol{\alpha} \equiv (\alpha_{mn}) = \alpha \boldsymbol{I}_2, \quad \alpha = -1, \quad \boldsymbol{I}_2 \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$



amplifying magnetic fields of the form

$$\boldsymbol{B}_0 = (\pm \sin KZ, \cos KZ, 0), \;\; \boldsymbol{B}_0 = \boldsymbol{\beta} e^{iKZ + \Lambda T}$$

with growth rate $\Lambda = \mp K - K^2.$

Large scale modes are amplified even though the small-scale R<<1. In dimensional terms growth rates are

$$\gamma = \pm \alpha q - \eta q^2, \quad \alpha = -u_0^2/\eta k = -R u_0.$$

Alpha effect in a helical flow can destabilise large-scale magnetic fields.

Alpha effect

Key transport effect that can destabilise large-scale magnetic fields (Parker, Steenbeck, Krause, Radler, Moffatt).
For low small-scale magnetic Reynolds number:

$$\alpha_{mn} = -\eta \sum_{\boldsymbol{k},\omega} \frac{k_m k_n \hat{H}(\boldsymbol{k},\omega)}{\eta^2 k^4 + \omega^2},$$

Isotropic flow: $$\boldsymbol{\alpha} = \alpha \boldsymbol{I}, \quad \alpha = -\frac{\eta}{3}\int \frac{k^2 \hat{H}(k,\omega)}{\eta^2 k^4 + \omega^2}\, dk\, d\omega$$

Alpha effect can circumvent anti-dynamo theorems

$$\partial_t \boldsymbol{B} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B} + \alpha \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B}, \quad \nabla \cdot \boldsymbol{B} = 0.$$

Alpha effect and shear can give dynamo waves in an alpha-omega dynamo: modelling of solar, terrestrial, galactic fields. Idea immensely influential.

Problems with the alpha effect

- Only satisfactory formula is for low small-scale magnetic Reynolds number R (with link to helicity). For short correlation times though we have $\alpha = -\frac{1}{3}\tau\langle \boldsymbol{u} \cdot \nabla \times \boldsymbol{u} \rangle, \quad \beta = \frac{1}{3}\tau\langle |\boldsymbol{u}|^2 \rangle$
- At large R it is hard to calculate and may be irrelevant (small-scale fields grow faster than large-scale ones).
- Nonlinear feedback on alpha difficult to assess at large R. Also eddy diffusivity.
- Dynamo applications have large R (outside the laboratory)

Fast versus slow dynamos

Magnetic fields evolve on fast turn-over time-scales in many astrophysical objects at large R=1/ϵ; e.g. in the Sun R=10^8, 11-year cycle.

In the smooth Ponomarenko dynamo the fastest growing modes have growth rates $\gamma = O(\varepsilon^{1/3})$, for $m, k = O(\varepsilon^{-1/3})$

We call this a `slow' dynamo: growth rates go to zero as R tends to infinity.

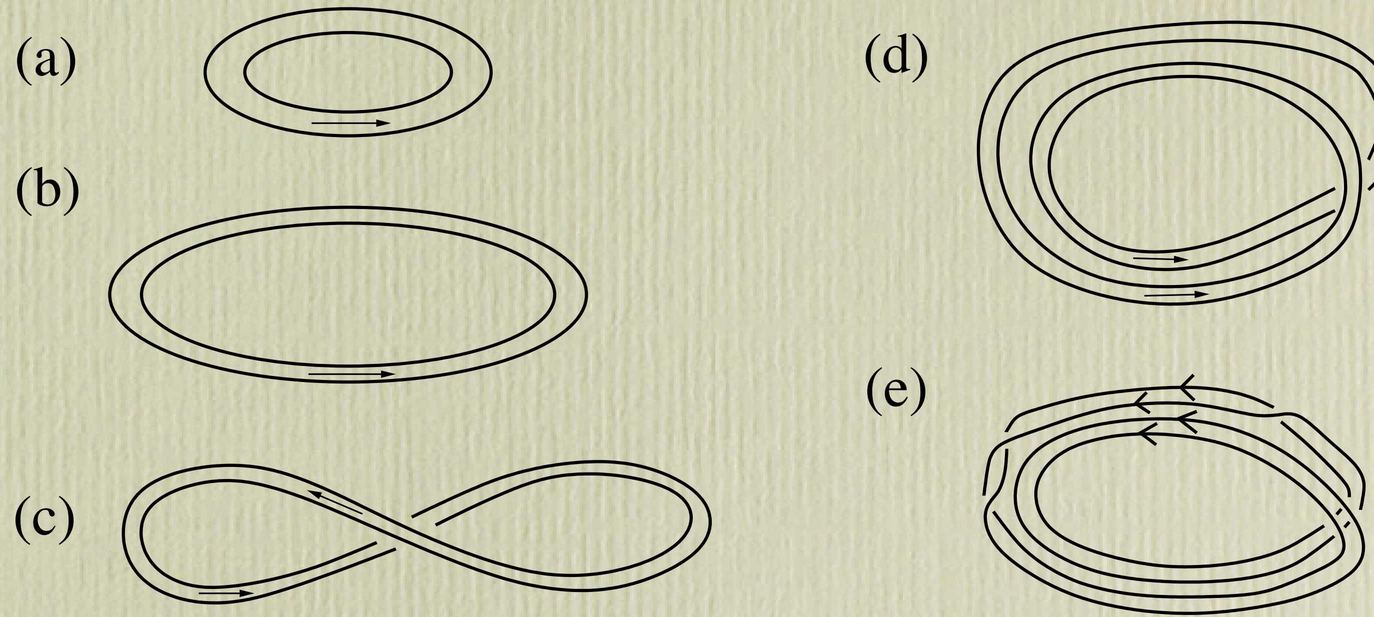
For the G.O. Roberts dynamo growth rates go to zero (Soward):

$$\gamma = O((\log\log\varepsilon^{-1})/\log\varepsilon^{-1}), \qquad k = O(\varepsilon^{-1/2}/\sqrt{\log\varepsilon^{-1}}).$$

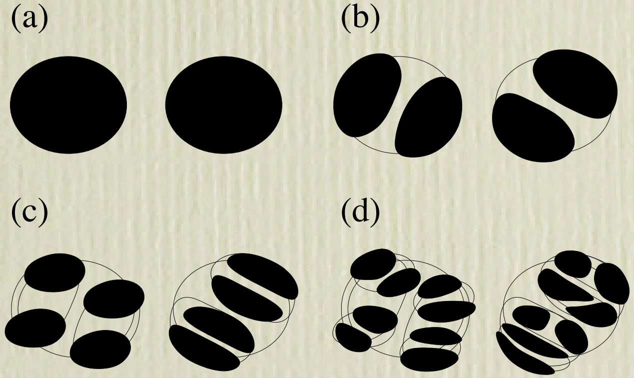
For a `fast' dynamo, growth rates remain bounded above zero no matter how large R is, that is how small the diffusivity ϵ is. The limiting maximum growth rate $\gamma_0 \equiv \lim_{\varepsilon\to 0}\gamma(\varepsilon)$ is positive.

Stretch, twist and fold

Fast dynamo paradigm (Vainshtein, Zeldovich). Take a flux tube:



Magnetic flux doubles each iteration.
If diffusion unimportant then field has limiting growth rate $\gamma_0 \simeq \log 2.$ and so would be a fast dynamo.

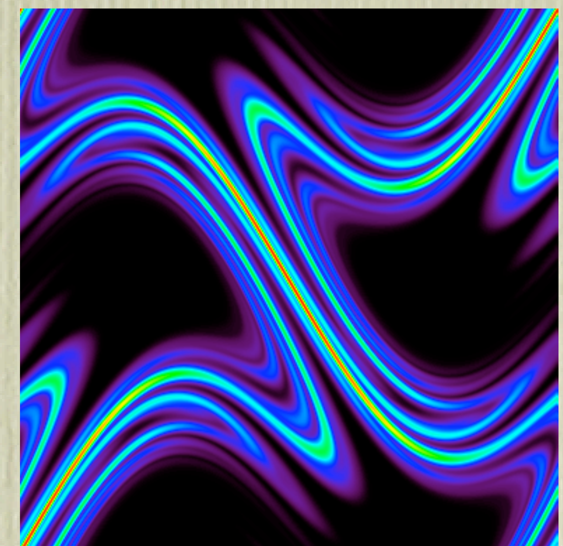


Otani, Galloway/Proctor flows

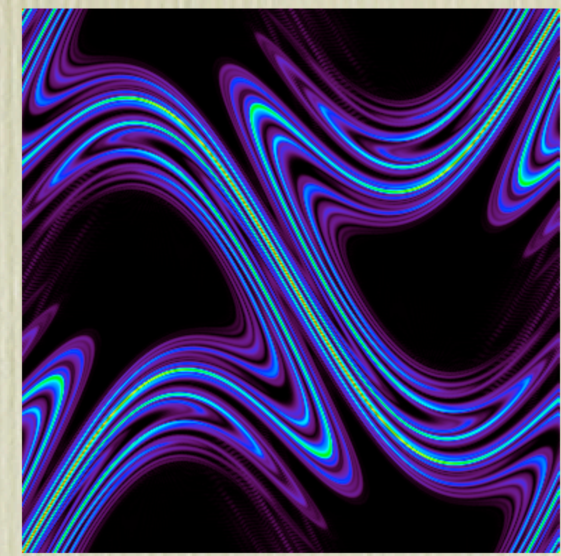
$$\mathbf{u}(x,y,t) = 2\cos^2 t\,(0, \sin x, \cos x) + 2\sin^2 t\,(\sin y, 0, -\cos y),$$
$$\mathbf{B}(x,y,z,t) = e^{ikz+\sigma t}\,\mathbf{b}(x,y,t),$$

Eigenfunctions for ϵ=5×10^(-4), ϵ=5×10^(-5), and k=0.8 (snap-shots).

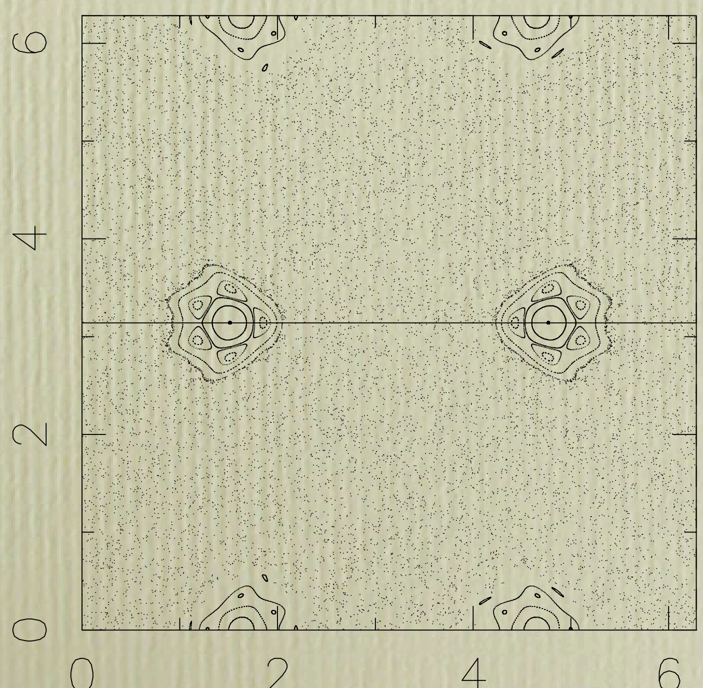
(a)



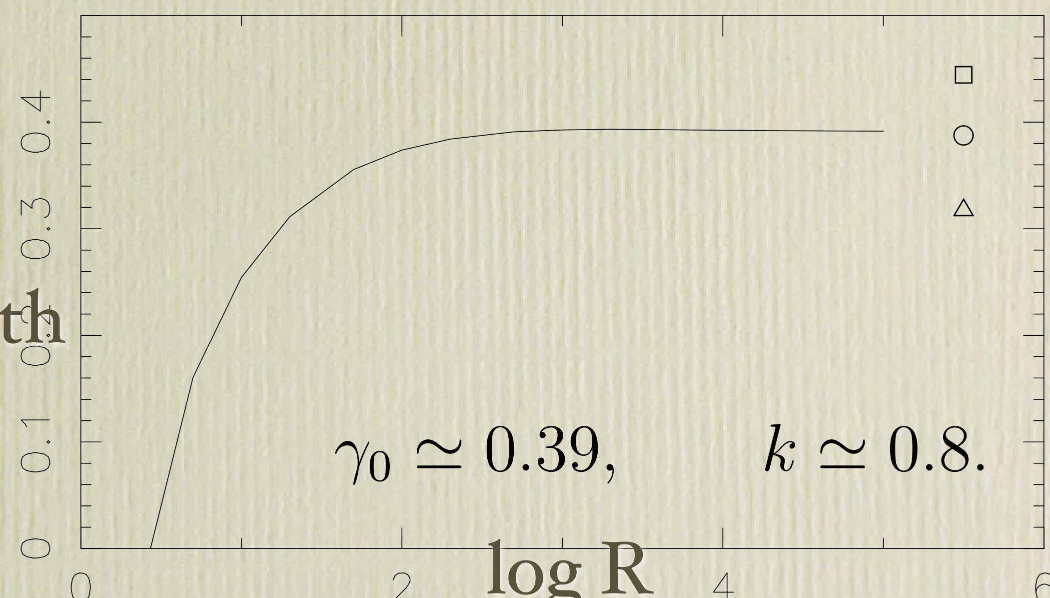
(b)



Poincare section



Growth rate



$\gamma_0 \simeq 0.39, \qquad k \simeq 0.8.$

log R

ABC flows

$$\mathbf{u} = (C \sin z + B \cos y, A \sin x + C \cos z, B \sin y + A \cos x),$$

Probably a fast dynamo (Arnold, Galloway, Frisch, Dorch, Archontis, Nordlund).

Fully three-dimensional flow and field complicates study; numerically very demanding.

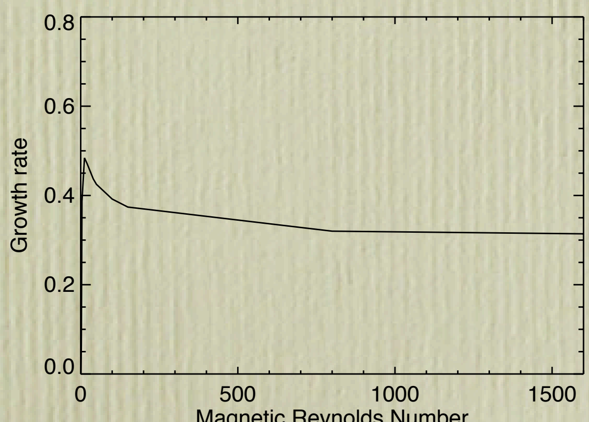


Fig. 2. Growth rate of the magnetic energy (normalized by the rate of strain) versus the magnetic Reynolds number for the kinematic *ABC* dynamo with $A = B = C = 1$ and $k = 2$.

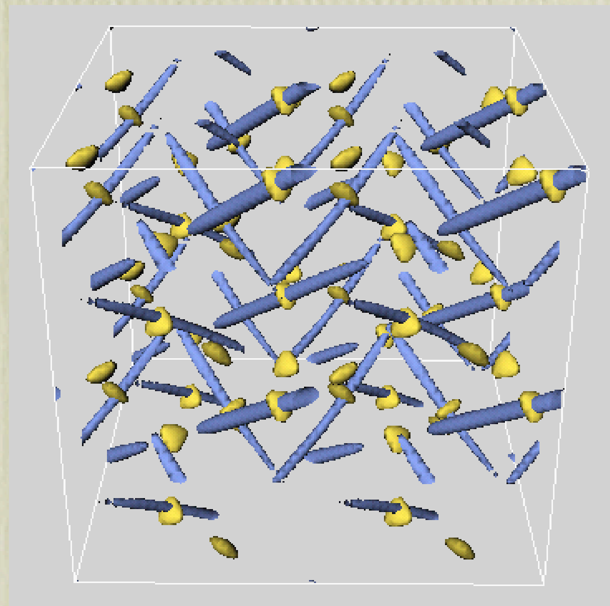


Fig. 5. Magnetic field strength isosurfaces (dark) and stagnation points of the flow (light). The magnetic field is concentrated along flux cigars centered at the α-type stagnation points of the flow. The stagnation points with no flux cigars centered at them are the β-type points. In

Figures from Archontis, Dorch, Nordlund 2003

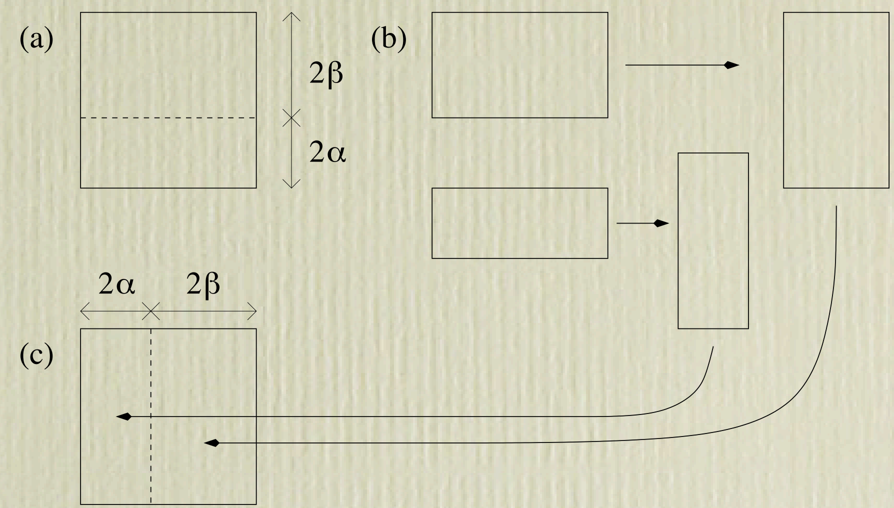
Klapper-Young upper bound

- Only rigorous general result on fast dynamos.
- For a smooth flow, the limiting fast dynamo growth rate γ_0 cannot exceed the topological entropy h.
- h is the maximum of the rates of growth of material lines or surfaces in the flow in 3-d.
- Makes precise the notion that `chaos', and chaotic stretching, is needed for fast dynamo amplification.
- Growth rate can (and does) exceed Liapunov exponent.

Fast dynamos based on maps

Apply a mapping with field frozen (no diffusion) and then allow diffusion to act (with no motion). Allows models based on maps instead of flows. For example a baker's map:

$$M(x,y) = \begin{cases} (\alpha(x+1)-1, \alpha^{-1}(y+1)-1) & (y < \Upsilon), \\ (\beta(x-1)+1, \beta^{-1}(y-1)+1) & (y \geq \Upsilon) \end{cases}$$



ι $\Upsilon = -1 + 2\alpha \equiv 1 - 2\beta.$ $\beta = 1 - \alpha.$

Magnetic field maps according to

$$Tb(x) = \begin{cases} \alpha^{-1} b(\alpha^{-1}(x+1)-1) & (x < \Upsilon), \\ \beta^{-1} b(\beta^{-1}(x-1)+1) & (x \geq \Upsilon). \end{cases}$$

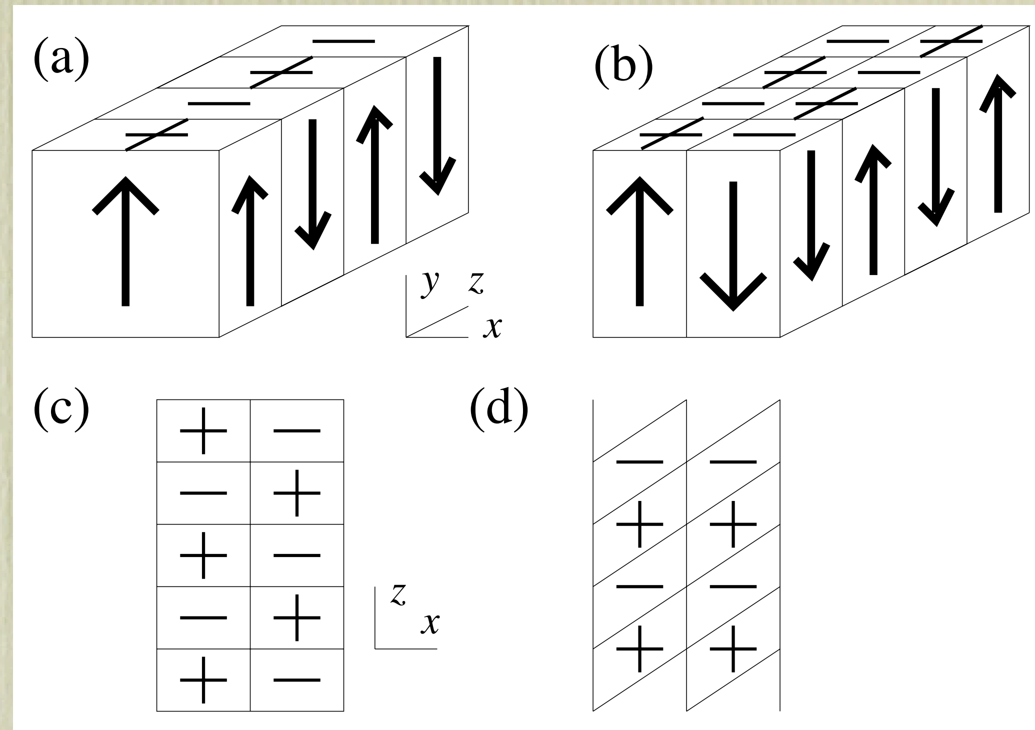
Ignoring diffusion, growth rate is $\gamma_0 = \log 2$ (fluxes double each iteration) and this exceeds the Liapunov exponent

$$\lambda_{\text{Liap}} = \alpha \log \alpha^{-1} + \beta \log \beta^{-1}.$$

(Finn, Ott)

Stretch, fold and shear

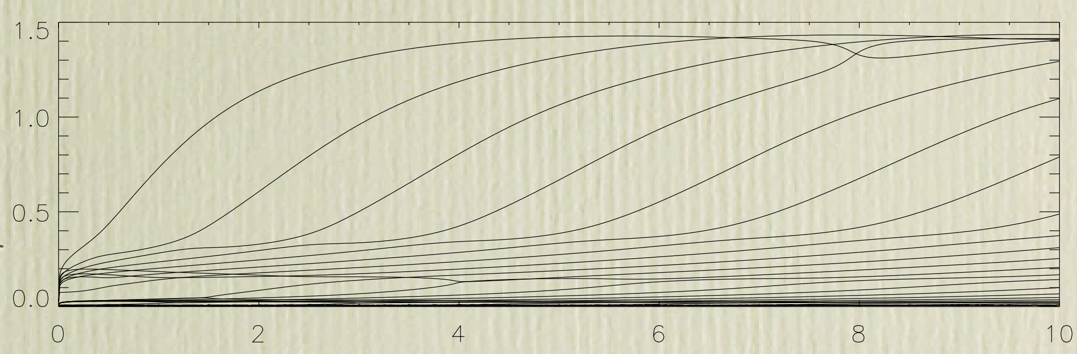
Map idealising process seen in Otani's example (Bayly, Childress).



$$\mathbf{B}(x,y,z) = e^{ikz} b(x)\hat{\boldsymbol{y}} + \text{complex conjugate}$$

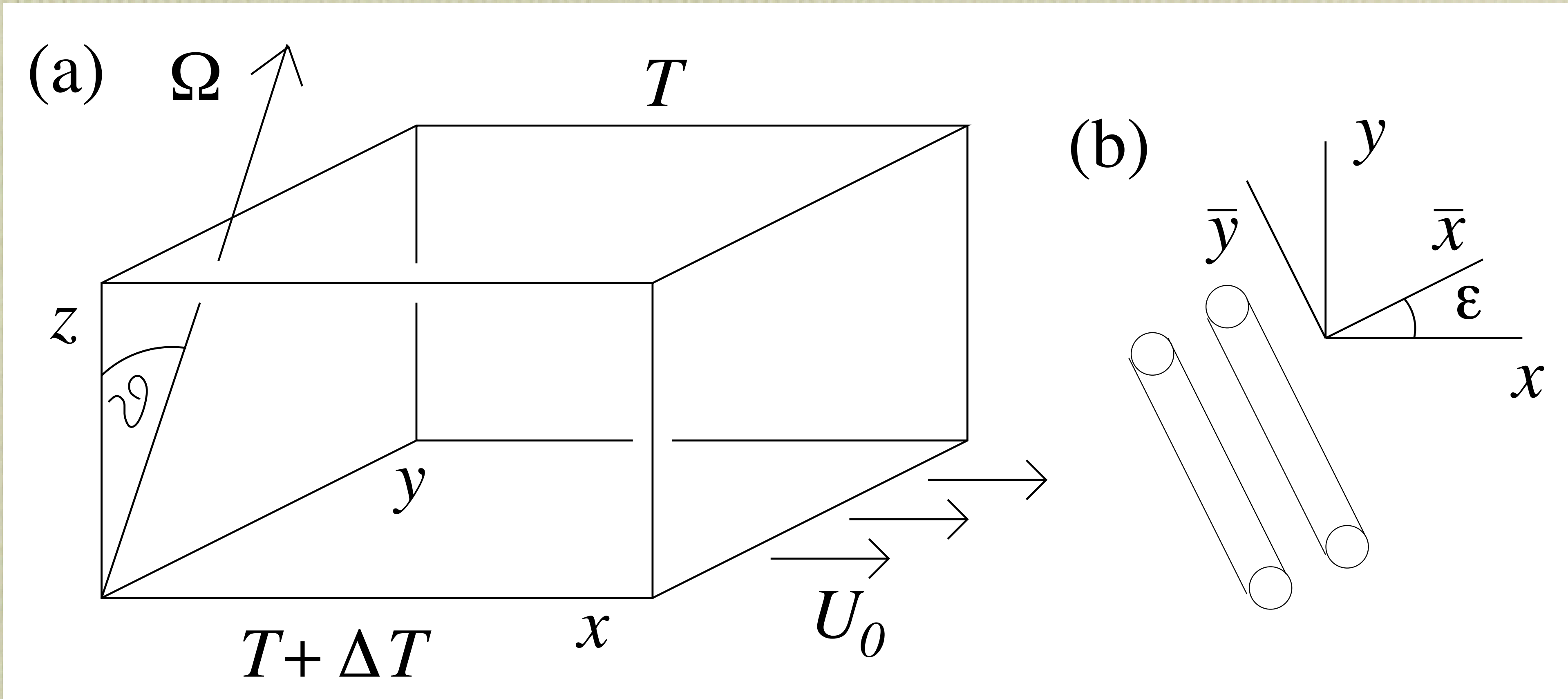
$$Tb(x) = \begin{cases} 2e^{-i\alpha kx} b(1+2x) & (x<0), \\ -2e^{-i\alpha kx} b(1-2x) & (x \geq 0). \end{cases}$$

log(γ) plotted against shear parameter α for insulating boundary conditions (and weak diffusion). Growth for α>π/2.

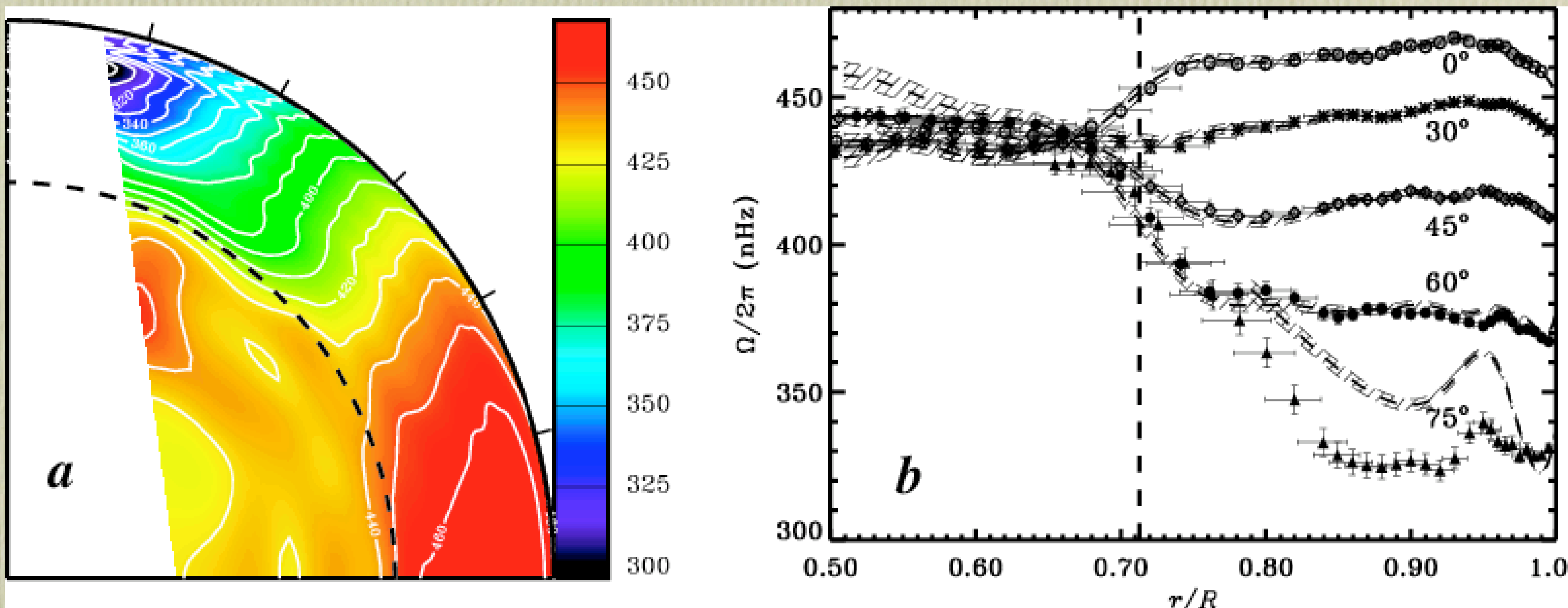


Eigenfunctions develop fine scales for small diffusion.

Dynamos driven by shear and convection

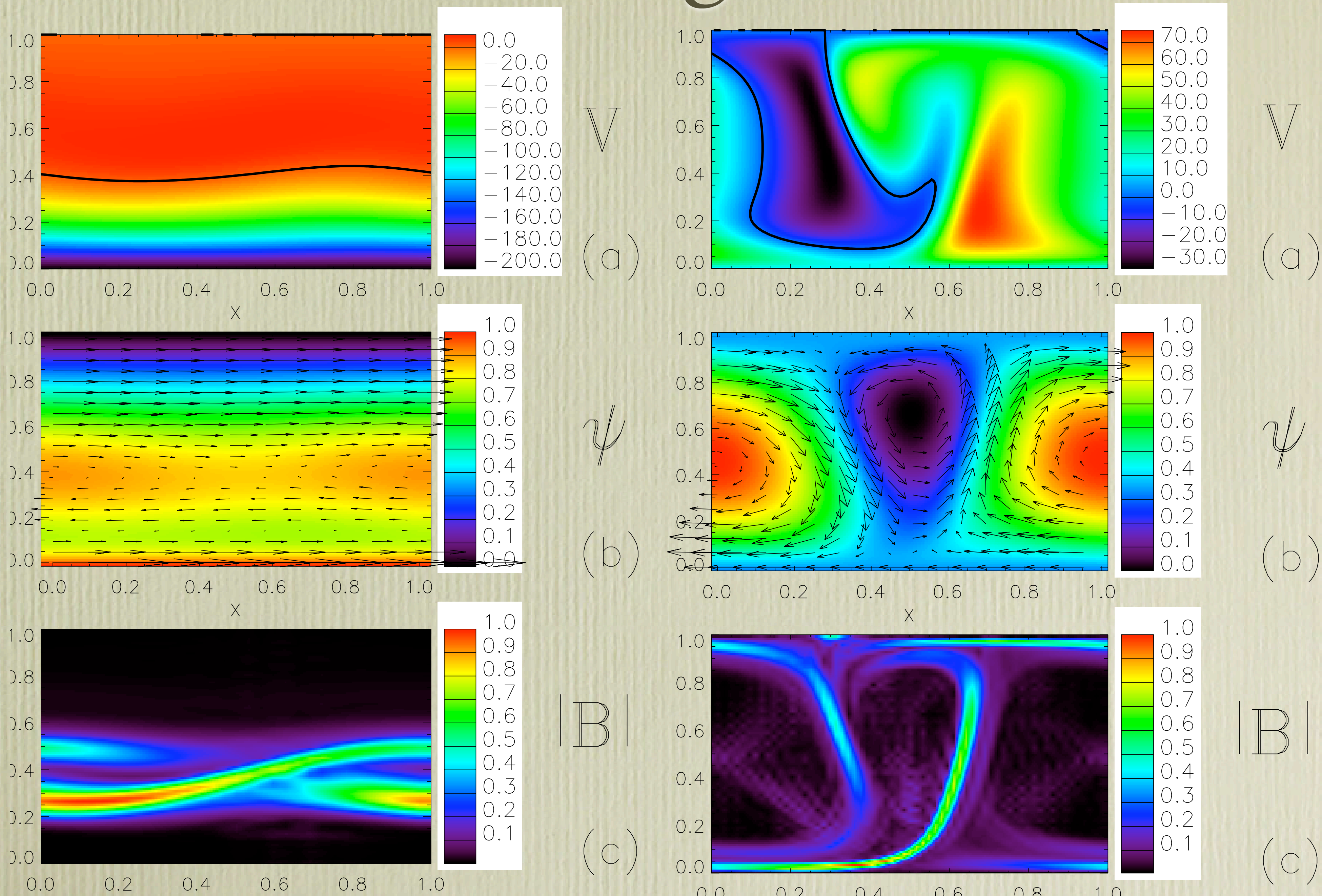


Flow driven by shear or convection in a rotating frame. Ekman layer can become unstable giving cat's eye rolls (Ponty, Gilbert, Soward).



Thompson, Christensen-Dalsgaard.

Linear regimes



Ekman instability

Convective instability

Equations and parameters: $\{\mathrm{Re}, \tau, \mathrm{Pm}, \vartheta, \varepsilon, k_{\bar{x}}, k_{\bar{y}}\}$..

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{U} + \tau \widehat{\boldsymbol{\Omega}} \times \mathbf{U} = -\nabla \Pi + (\nabla \times \mathbf{B}) \times \mathbf{B} + \nabla^2 \mathbf{U},$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{U} \times \mathbf{B}) + \mathrm{Pm}^{-1} \nabla^2 \mathbf{B},$$

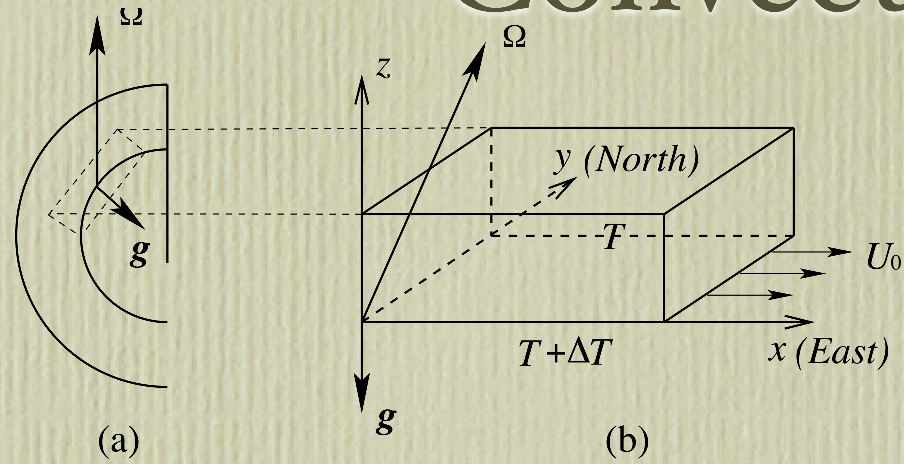
$$\nabla \cdot \mathbf{U} = 0, \quad \nabla \cdot \mathbf{B} = 0$$

$$\vartheta, \quad \mathrm{Re} = \frac{U_0 h}{\nu}, \quad \tau = \frac{2\Omega h^2}{\nu}, \quad \mathrm{Pm} = \frac{\nu}{\eta},$$

The Magnetic Reynolds number becomes a diagnostic defined by:

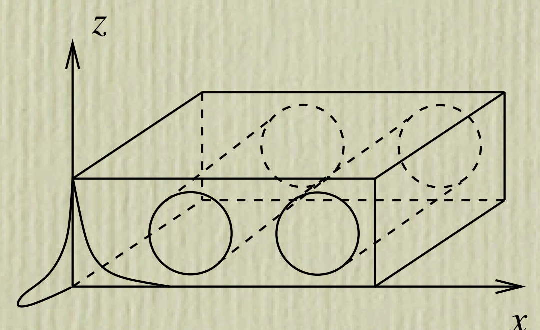
$$\mathrm{Rm} = \mathrm{Pm}\, U, \qquad U \equiv \sqrt{2E_K}, \quad E_K = \tfrac{1}{2} \langle \mathbf{U}^2 \rangle$$

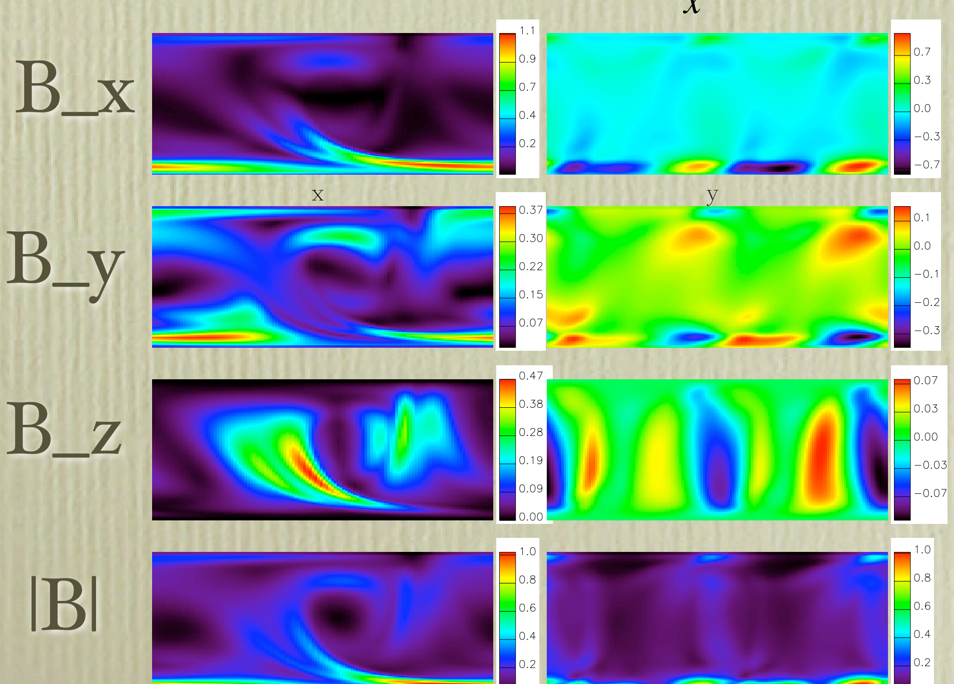
Convective example



Convective roll axes here aligned with horizontal component of rotation.

$$\mathrm{Ra} = 7500 \simeq 2\,\mathrm{Ra}_c, \quad \mathrm{Re} = 30, \quad \tau = 200,$$
$$\mathrm{P} = 1, \quad q = 50, \quad \vartheta = 67.5^\circ, \quad k_x = 4.30, \quad k_y = 1.0,$$





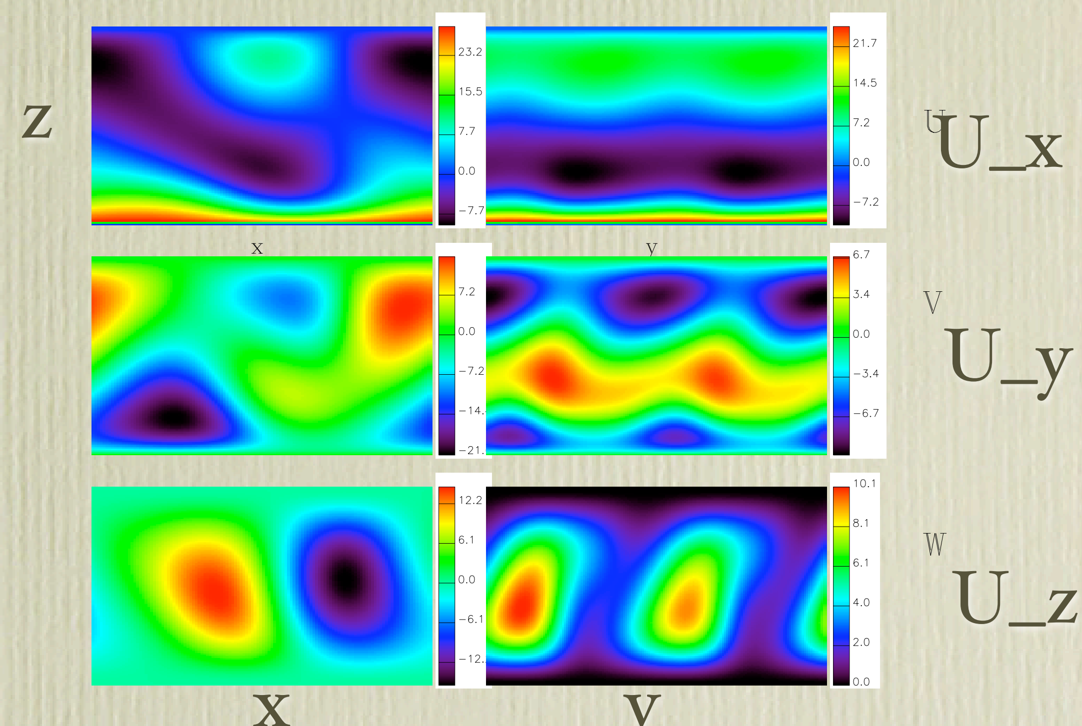
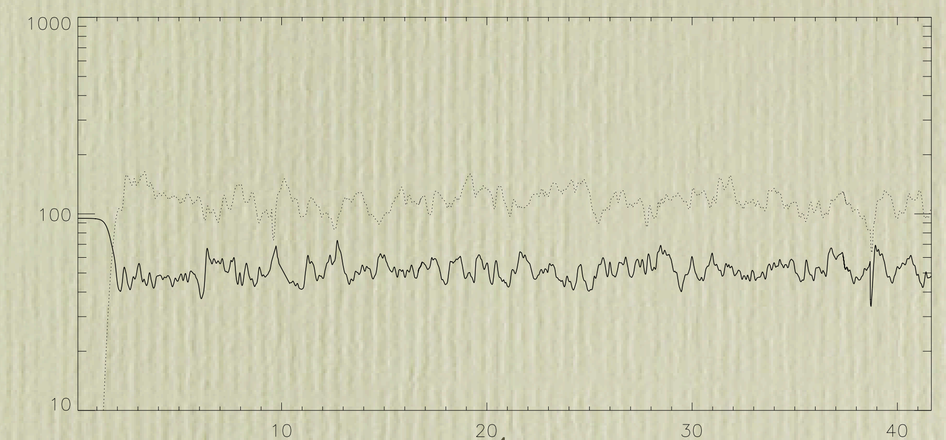


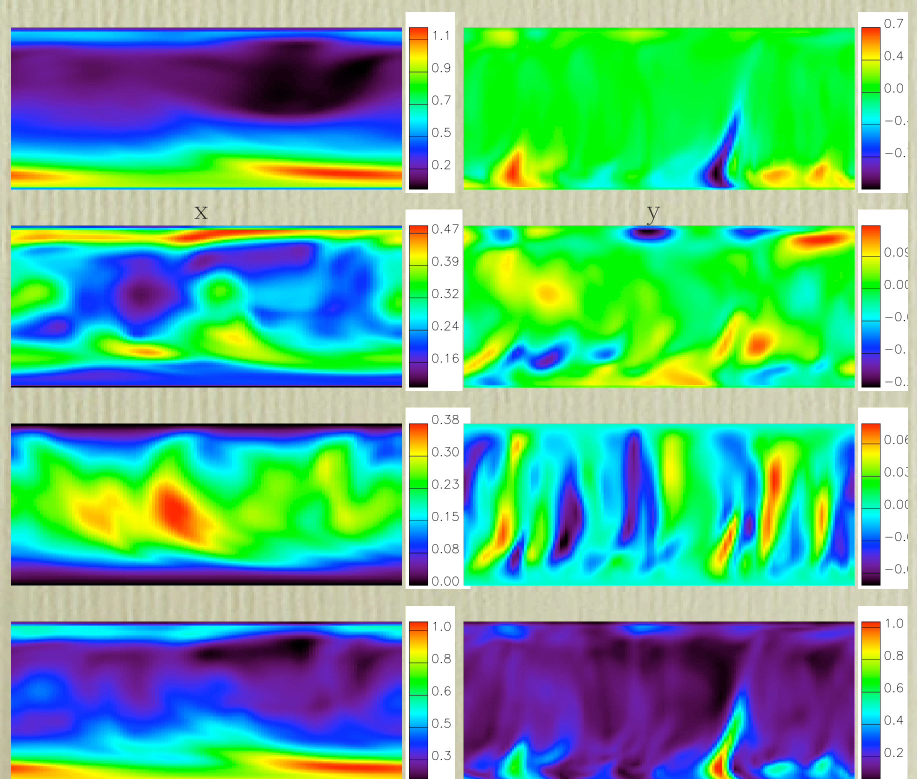
Fig. 2. Convective flow in the kinematic regime, with parameter values (3.1), (3.2). Plotted are the flow components (U, V, W) of $\boldsymbol{U}$ in the (x, z)-plane (left-hand panels) and (y, z)-plane (right-hand panels).

Dynamical regime

E_K, E_M

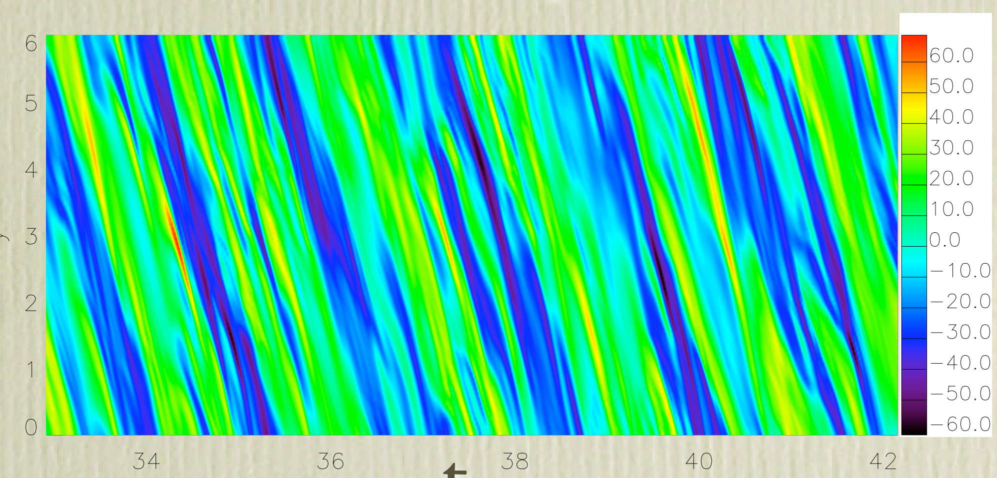


t



Alpha-omega dynamo: waves and phases

y



t

$$\hat{B}_x(y,t) = \langle B_x(x,y,z_{\max},t)\rangle_x$$

Open issues

- How does the solar dynamo function? Is the tachocline important or not? Do any/all of convection, shear instability, and waves play a role?
- How useful is the alpha effect in modelling Solar magnetic fields and in other dynamo applications?
- How does the Sun generate quite fine-scaled field at the surface, yet showing clear (Hale) polarity laws?
- What is the nature of nonlinear alpha-effect suppression at large R? Is it even a useful concept?
- Can any rigorous results be proven about the existence of fast dynamos in `realistic' flows (rather than maps)?