Kelvin's Theorem and Alfvén's Theorem in Turbulent Flow

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In view of the infinite conductivity, every motion (perpendicular to the field) of the liquid in relation to the lines of force is forbidden because it would give infinite eddy currents. Thus the matter of the liquid is "fastened" to the lines of force... (H. Alfvén,1942)

Alfvén's Theorem

**Flux conservation:*

$$(d/dt)\Phi(S,t) \equiv (d/dt)\int_{S(t)} \mathbf{B}(t)\cdot d\mathbf{A} = 0$$

**"Frozen-in" magnetic field lines*

These properties are a consequence of the homogeneous Maxwell equations

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0, \;\; \partial\mathbf{B}/\partial t + \boldsymbol{\nabla}\times\mathbf{E} = \mathbf{0}$$

and the ideal Ohm's Law

$$\mathbf{E} + \mathbf{u}\times\mathbf{B} = \mathbf{0},$$

which imply the dynamical equation

$$\partial\mathbf{B}/\partial t = \boldsymbol{\nabla}\times(\mathbf{u}\times\mathbf{B}).$$

The condition for "frozen-in" lines of force follows:

$$[\partial\mathbf{B}/\partial t - \boldsymbol{\nabla}\times(\mathbf{u}\times\mathbf{B})]\times\mathbf{B} = \mathbf{0}$$

Helmholtz-Kelvin Theorem

**Conservation of circulation:*

$$(d/dt)\Gamma(S,t) \equiv (d/dt)\int_{S(t)} \boldsymbol{\omega}(t){\cdot}d\mathbf{A} = 0$$

**"Frozen-in" vortex-lines*

The incompressible Euler equations

$$\partial\boldsymbol{\omega}/\partial t = \boldsymbol{\nabla}\times(\mathbf{u}\times\boldsymbol{\omega}).$$

for vorticity $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\mathbf{u}$ implies both.

For a general collisionless plasma,

$$\partial\boldsymbol{\omega}_\pm/\partial t = \boldsymbol{\nabla}\times(\mathbf{u}_\pm\times\boldsymbol{\omega}_\pm),$$

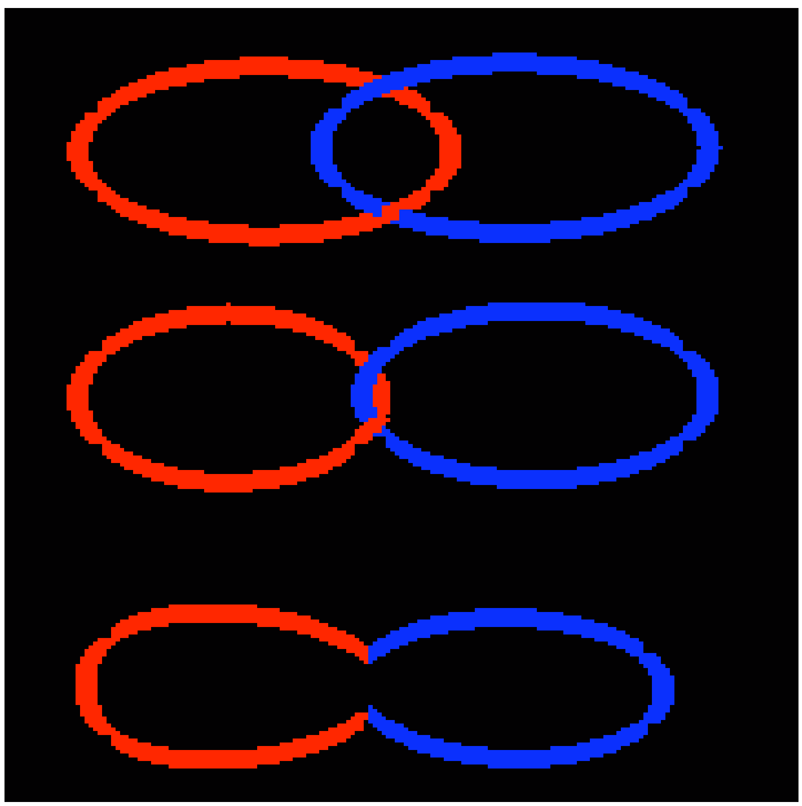
where $\mathbf{u}_\pm$ are the local mean velocities of ions (+) and electrons (−), and

$$\boldsymbol{\omega}_\pm = \boldsymbol{\nabla}\times\mathbf{u}_\pm + e_\pm\mathbf{B}/m_\pm$$

Conservation of Topology

If the plasma flow is *continuous*, then the "frozen-in" condition implies that magnetic line-topology is invariant in time.

Reconnection processes such as



are strictly **forbidden**.

Why Isn't Turbulent Plasma Rubberized?

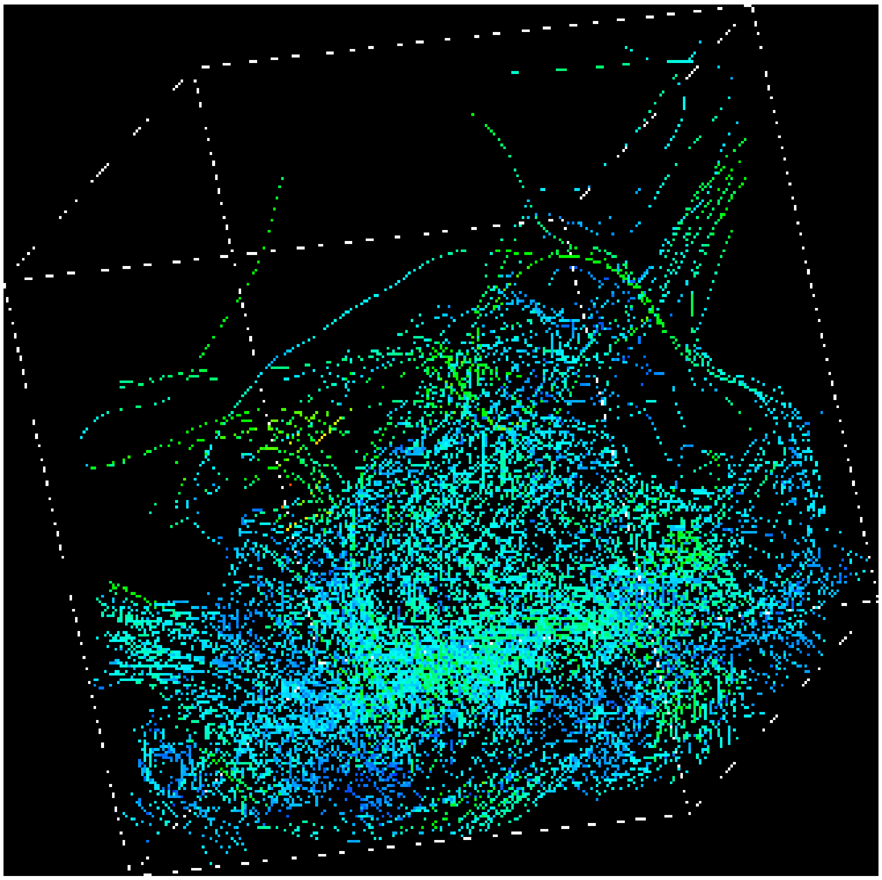


FIGURE: A tangle of magnetic field lines in incompressible MHD turbulence

If Alfvén's Theorem were applicable to turbulent plasmas, there would be an infinite set of topological constraints, like the entanglements of polymer chains in rubber.

Resistive (Non-Ideal) Reconnection

At small scales, an Ohmic resistance (or other non-ideality) will act, and

$$\partial \mathbf{B}/\partial t = \boldsymbol{\nabla}\times(\mathbf{u}\times\mathbf{B}) + \eta \triangle \mathbf{B}.$$

Thus,

$$(d/dt)\Phi(S,t) = -\eta \oint_{C(t)} \mathbf{J}(t)\cdot d\mathbf{x}$$

where $\mathbf{J} = \boldsymbol{\nabla}\times\mathbf{B}$ is the electric current and $C(t)$ is the boundary curve of $S(t)$.

However, effects of plasma resistivity are too small to explain observed rates of reconnection of large-scale magnetic fields.

Example: Transequatorial Coronal Loop. A loop connecting two active regions across the solar equator has length $L \approx 3 \times 10^5$ km and forms in < 5 days. The time for this reconnection to occur by plasma resistivity is about 6×10^9 yr!! (Dere, 1996)

Large-Scale "Coarse-Grained" Fields

Define the local-averaging operation

$$\overline{\mathbf{b}}_\ell(\mathbf{x}) = \int d\mathbf{r}\; G_\ell(\mathbf{r})\mathbf{b}(\mathbf{x}+\mathbf{r})$$

$$\overline{\mathbf{u}}_\ell(\mathbf{x}) = \int d\mathbf{r}\; G_\ell(\mathbf{r})\mathbf{u}(\mathbf{x}+\mathbf{r})$$

retaining the length-scales $> \ell$. Here G is smooth, compactly supported, $G \geq 0$ and

$$\int d\mathbf{r}\, G(\mathbf{r}) = 1,$$

with $G_\ell(\mathbf{r}) = \ell^{-3} G(\mathbf{r}/\ell)$.

The fields $\overline{\mathbf{b}}_\ell, \overline{\mathbf{u}}_\ell$ are collective variables, averaged over fluid parcels of radius $\approx \ell$.

These fields are the only variables that are directly observable by an experimentalist, corresponding to "coarse-grained" measurements at a space resolution $\approx \ell$.

Large-Scale Ohm's Law

The "bare" Ohm's Law

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \eta \mathbf{J}$$

is "renormalized" to

$$\overline{\mathbf{E}}_\ell + \overline{\mathbf{u}}_\ell \times \overline{\mathbf{B}}_\ell = -\boldsymbol{\varepsilon}_\ell + \eta \overline{\mathbf{J}}_\ell$$

where

$$\boldsymbol{\varepsilon}_\ell = \overline{(\mathbf{u} \times \mathbf{b})}_\ell - \overline{\mathbf{u}}_\ell \times \overline{\mathbf{b}}_\ell$$

is the *subscale (or turbulent) EMF.*

If the magnetic energy

$$E_B = \frac{1}{2}\|\mathbf{B}\|_2^2 = \frac{1}{2}\int d\mathbf{x}\, |\mathbf{B}(\mathbf{x})|^2$$

is finite, then

$$\|\eta \overline{\mathbf{J}}_\ell\|_2 \leq (const.)(\eta/\ell)\|\mathbf{B}\|_2$$

and $\eta \overline{\mathbf{J}}_\ell$ is negligible for small η or large ℓ.

Large-Scale Magnetic-Flux Balance

Define the *large-scale magnetic flux* as

$$\overline{\Phi}_\ell(S,t) \equiv \int_{\overline{S}_\ell(t)} \overline{\mathbf{B}}_\ell(t)\cdot d\mathbf{A},$$

where $\overline{S}_\ell(t)$ is the surface at time t advected by $\overline{\mathbf{u}}_\ell$ and started as S at time t_0.

Ignoring small non-ideal terms,

$$(d/dt)\overline{\Phi}_\ell(S,t) = \oint_{\overline{C}_\ell(t)} \boldsymbol{\varepsilon}_\ell(t)\cdot d\mathbf{x},$$

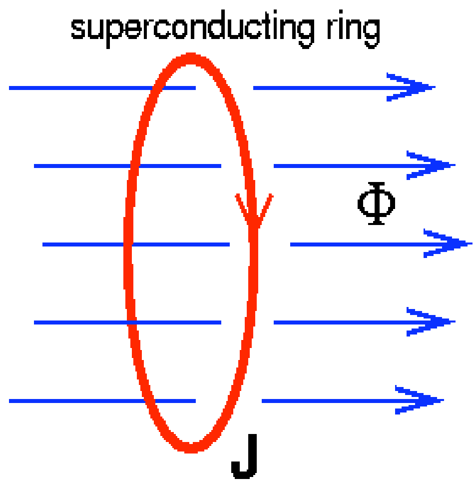
and Alfvén's Theorem is violated by the non-linear effects due to the subscale EMF.

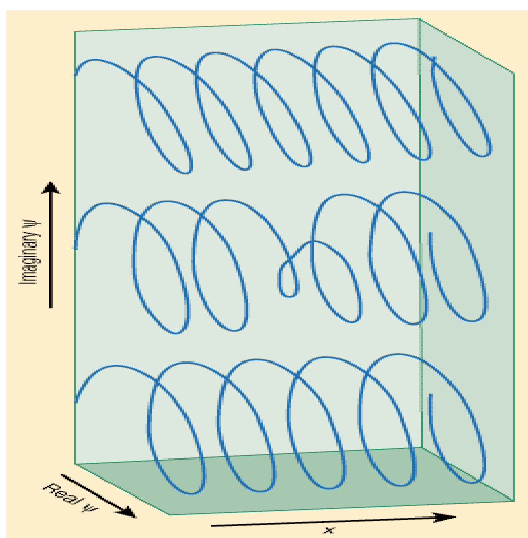
Physically, the lines of the large-scale magnetic field gain a transverse "slip velocity"

$$\Delta\overline{\mathbf{u}}_\ell^\perp = \boldsymbol{\varepsilon}_\ell \times \overline{\mathbf{B}}_\ell / |\overline{\mathbf{B}}_\ell|^2$$

relative to the large-scale plasma velocity $\overline{\mathbf{u}}_\ell$, due to the EMF of the small-scale modes.

Analogy: Phase-Slip in Superconductors





The migration of a quantized flux line out of the ring induces by quantum phase-slip a voltage pulse around the ring of superconducting material.

Persistent Nonlinear MHD Effect?

Is it possible that

$$\lim_{\ell\to 0} \oint_{\overline{C}_\ell(t)} \boldsymbol{\varepsilon}_\ell(t)\cdot d\mathbf{x} \neq 0?$$

If so, then one may take $\eta \to 0$ first, and then $\ell \to 0$, and Alfvén's Theorem will be violated for *ideal* MHD!

This is analogous to Onsager's 1949 result on inviscid energy dissipation by incompressible Euler equations in hydrodynamic turbulence, due to nonlinear energy cascade.

At a small "dissipation" length-scale ℓ_d,

$$\oint_{\overline{C}_\ell(t)} \boldsymbol{\varepsilon}_\ell(t)\cdot d\mathbf{x} \approx \eta \oint_{\overline{C}_\ell(t)} \overline{\mathbf{J}}_\ell(t)\cdot d\mathbf{x}.$$

For $\ell \lesssim \ell_d$, Alfvén's Theorem is violated by non-ideal plasma effects, but for $\ell \gtrsim \ell_d$ it is violated by nonlinear MHD effects.

Theorem 1 (Eyink-Aluie, 2006) *If the total energy is finite,*

$$\frac{1}{2}\int d\mathbf{x}\,[|\mathbf{u}(\mathbf{x})|^2 + |\mathbf{b}(\mathbf{x})|^2] < \infty,$$

then

$$\lim_{\ell\to 0}\int d\mathbf{x}\,|\boldsymbol{\varepsilon}_\ell(\mathbf{x})| = 0.$$

This result implies that

$$\boldsymbol{\varepsilon}_\ell(\mathbf{x}) \to 0 \ \ a.e.$$

with respect to 3D Lebesgue measure, for a subsequence of $\ell \to 0$.

However, a loop C has zero 3D Lebesgue measure, so that it is possible that

$$\lim_{\ell\to 0}\oint_C \boldsymbol{\varepsilon}_\ell{\boldsymbol\cdot} d\mathbf{x} \neq 0$$

for certain loops C.

Theorem 2 (Eyink-Aluie, 2006) *In order that*

$$\lim_{\ell\to 0}\oint_C \boldsymbol{\varepsilon}_\ell{\boldsymbol\cdot}d\mathbf{x}\neq 0,$$

it is necessary that at least one of the following three conditions hold:

(i) the curve C is non-rectifiable, or

(ii) either $\mathbf{u}$ or $\mathbf{b}$ is unbounded on C, or

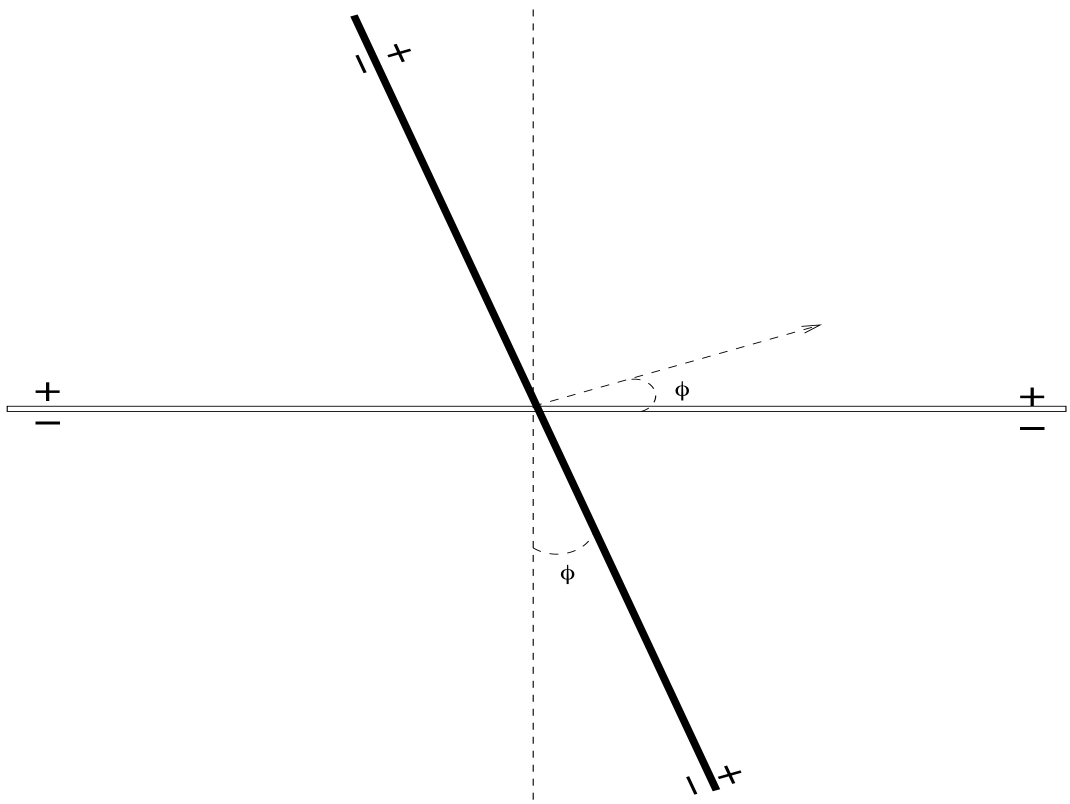
(iii) the set $\mathcal{D}=\mathcal{D}_{\mathbf{u}}\cap\mathcal{D}_{\mathbf{b}}$ of discontinuities of both $\mathbf{u}$ and $\mathbf{b}$ must satisfy

$$H^1(C\cap\mathcal{D})>0,$$

where H^1 is 1-dimensional Hausdorff measure.

For a rectifiable (finite-length) loop C, *both* $\mathbf{u}$ *and* $\mathbf{b}$ must be irregular—discontinuous on a positive length or even unbounded —to get a non-vanishing result.

Example: A vortex sheet (white strip) of strength Δu_0 in the xz-plane and a current sheet (black strip) of strength Δb_0 in the plane obtained by rotating the yz-plane by angle ϕ around the z-axis.



On the line of intersection (z-axis)

$$\boldsymbol{\varepsilon}_\ell(0,0,z) = \frac{\Delta u_0 \Delta b_0}{2\pi} \sigma(\varphi) \cos(\varphi)\, \hat{\mathbf{z}},$$

independent of ℓ, where σ is the 2π-periodic function defined by

$$\sigma(\varphi) = \begin{cases} \varphi & -\pi/2 < \varphi < \pi/2 \\ \pi - \varphi & \pi/2 < \varphi < 3\pi/2 \end{cases}$$

Comparison: Kelvin Theorem

For hydrodynamics, the large-scale circulation obeys

$$(d/dt) \oint_{\overline{C}_\ell(t)} \overline{\mathbf{u}}_\ell \cdot d\mathbf{x} = \oint_{\overline{C}_\ell(t)} \mathbf{f}_\ell(t) \cdot d\mathbf{x},$$

where

$$\mathbf{f}_\ell = \overline{(\mathbf{u}\times\boldsymbol{\omega})}_\ell - \overline{\mathbf{u}}_\ell \times \overline{\boldsymbol{\omega}}_\ell$$

is the *subscale (or turbulent) vortex-force*.

Eyink (CRAS, 2006) proves that

$$\mathbf{f}_\ell(\mathbf{x}) = O(\ell^{2h-1})$$

if $\mathbf{u}$ has Hölder exponent h at point $\mathbf{x}$:

$$|\mathbf{u}(\mathbf{x}+\boldsymbol{\ell}) - \mathbf{u}(\mathbf{x})| = O(\ell^h).$$

$\lim_{\ell\to 0} \mathbf{f}_\ell(\mathbf{x}) \neq 0$ requires only that $h \leq 1/2$.

Since 3D hydrodynamic turbulence has Hölder exponent $h = 1/3$ in the mean-field K41 sense, it is very easy to violate Kelvin's Theorem!

Cascade of Circulations

PDF & RMS of subscale torque are nearly independent of $k_c = 2\pi/\ell$ in the turbulent inertial-range: the cascade of circulations is persistent in scale.

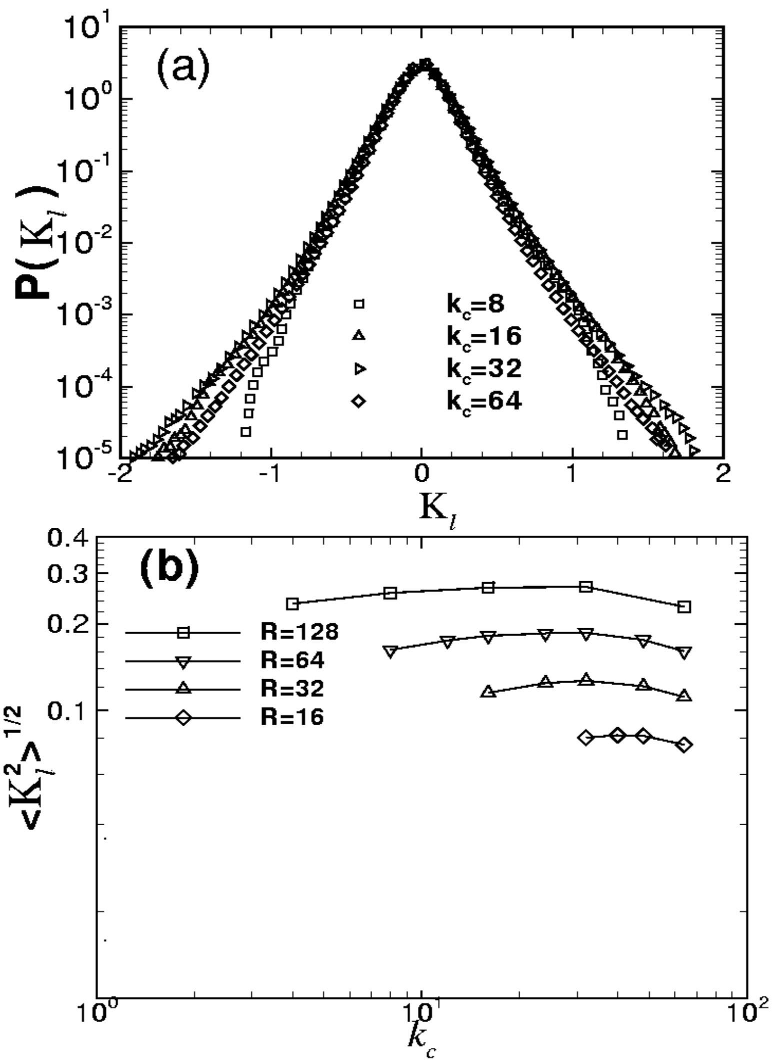
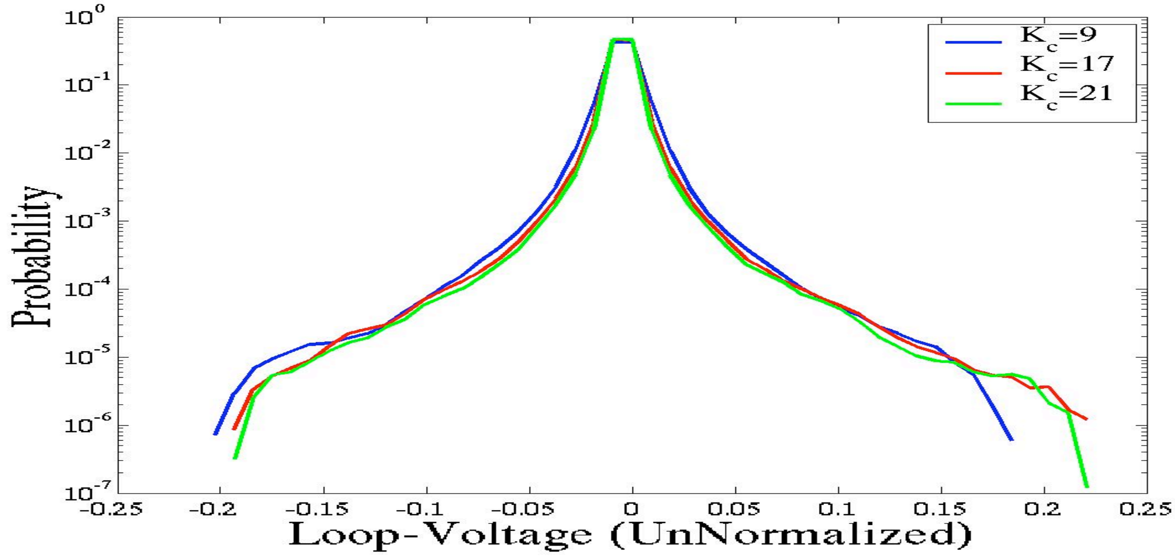
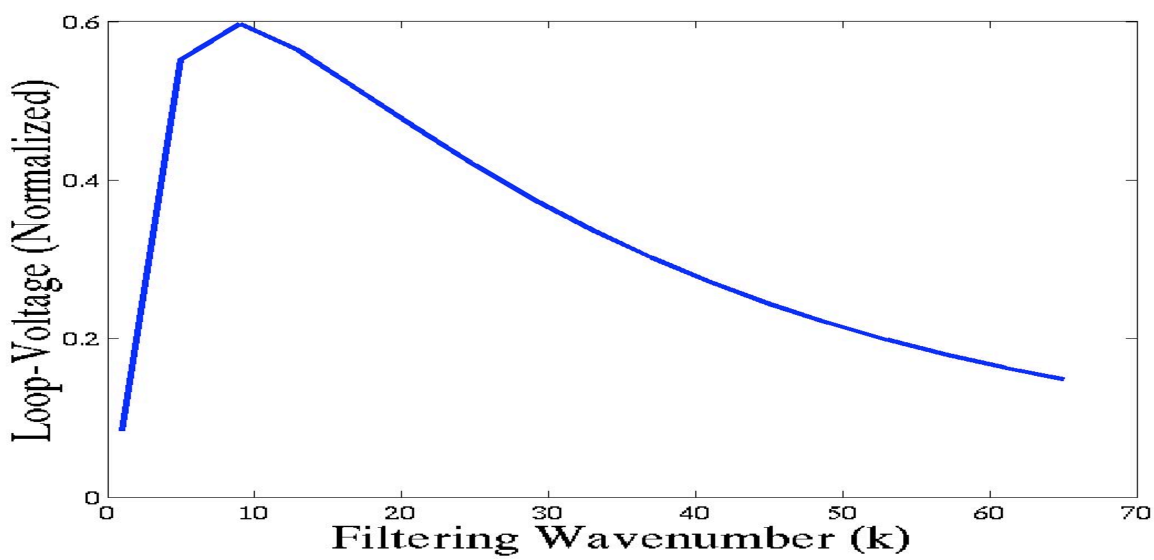


Figure. (a) PDF and (b) RMS of the subscale *loop-torque* $K_\ell(C) = -\oint_C \mathbf{f}_\ell \cdot d\mathbf{x}$, for square loops C of edge-length 64 in 1024^3 DNS of forced 3D hydrodynamic turbulence. (Chen et al., 2006)

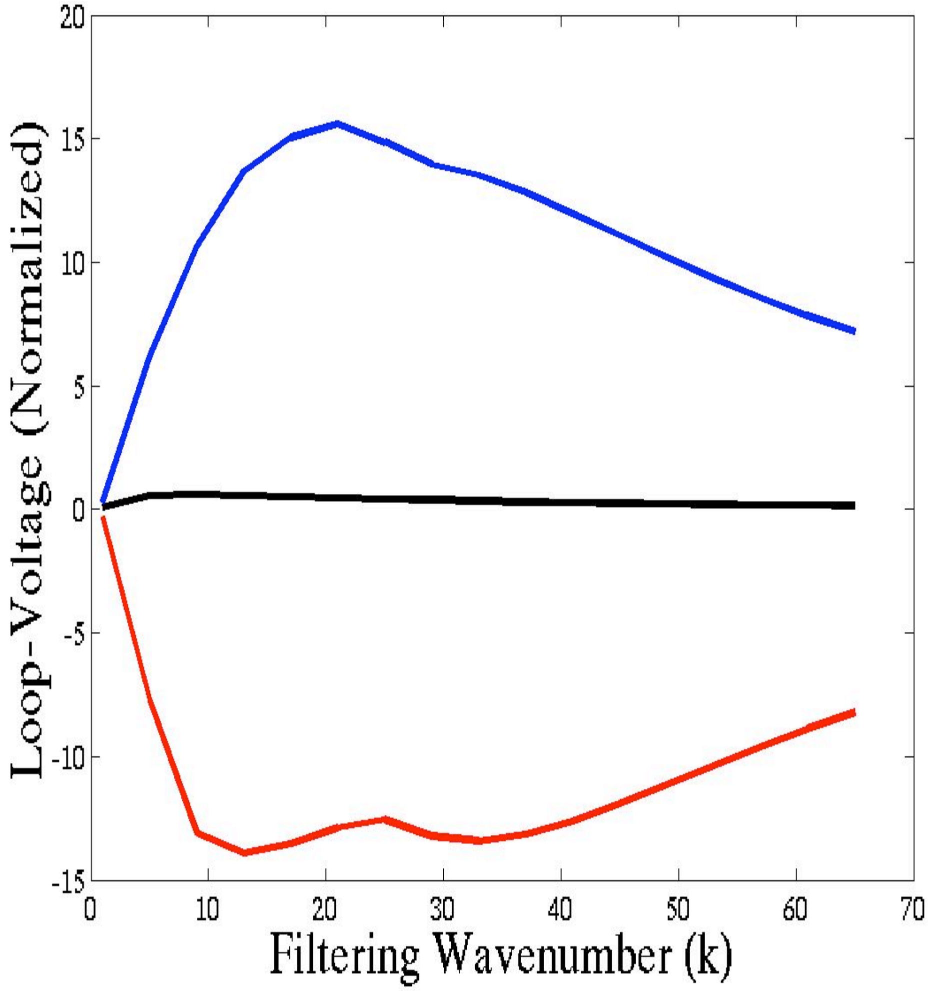
PDF's and RMS of Loop-Voltage





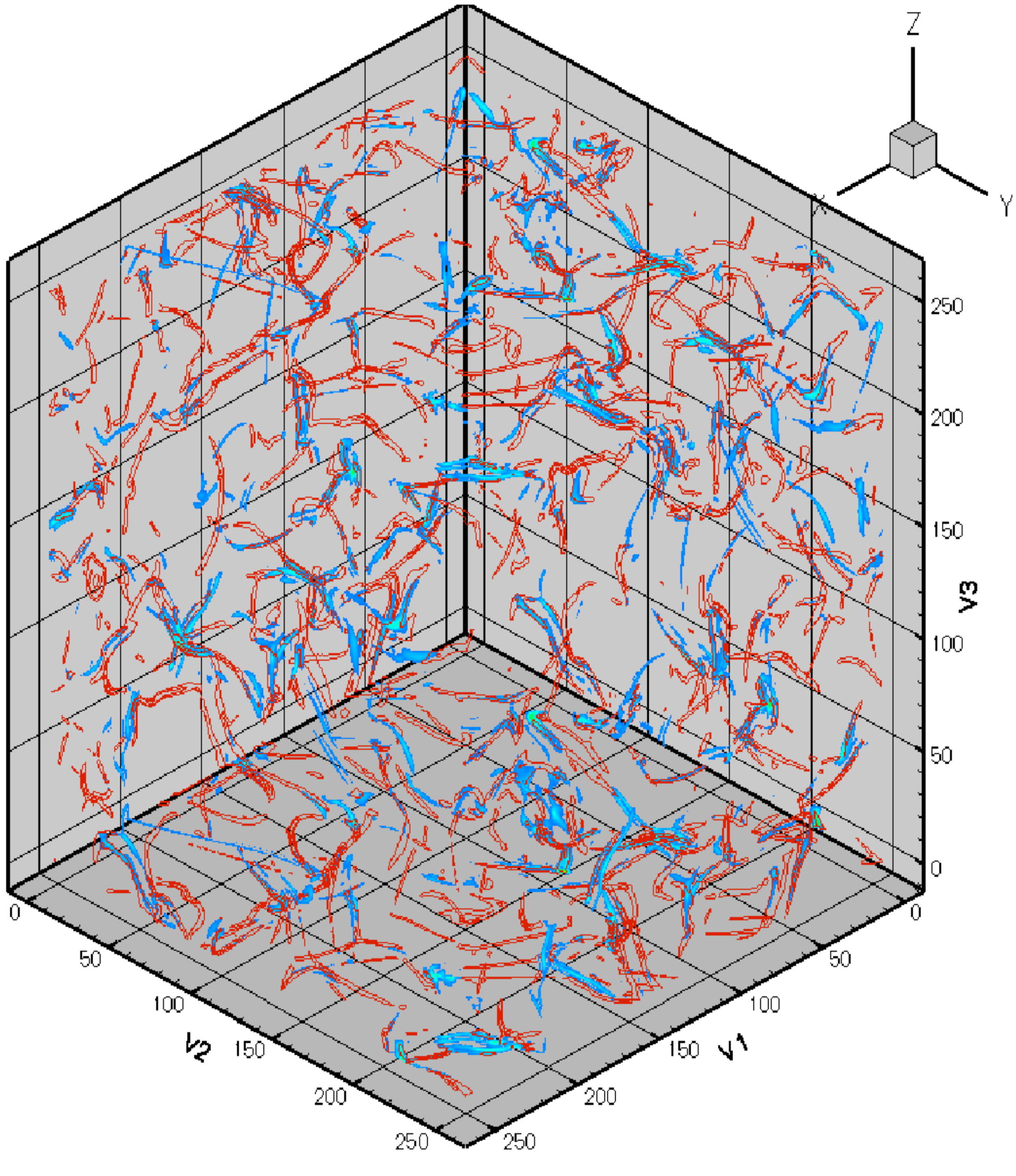
Data from a 256^3 simulation of MHD turbulence. The PDF's of the loop-voltage narrow with decreasing ℓ and the RMS values decay rapidly, while the extreme values decrease more slowly.

Extreme Values of Loop-Voltage



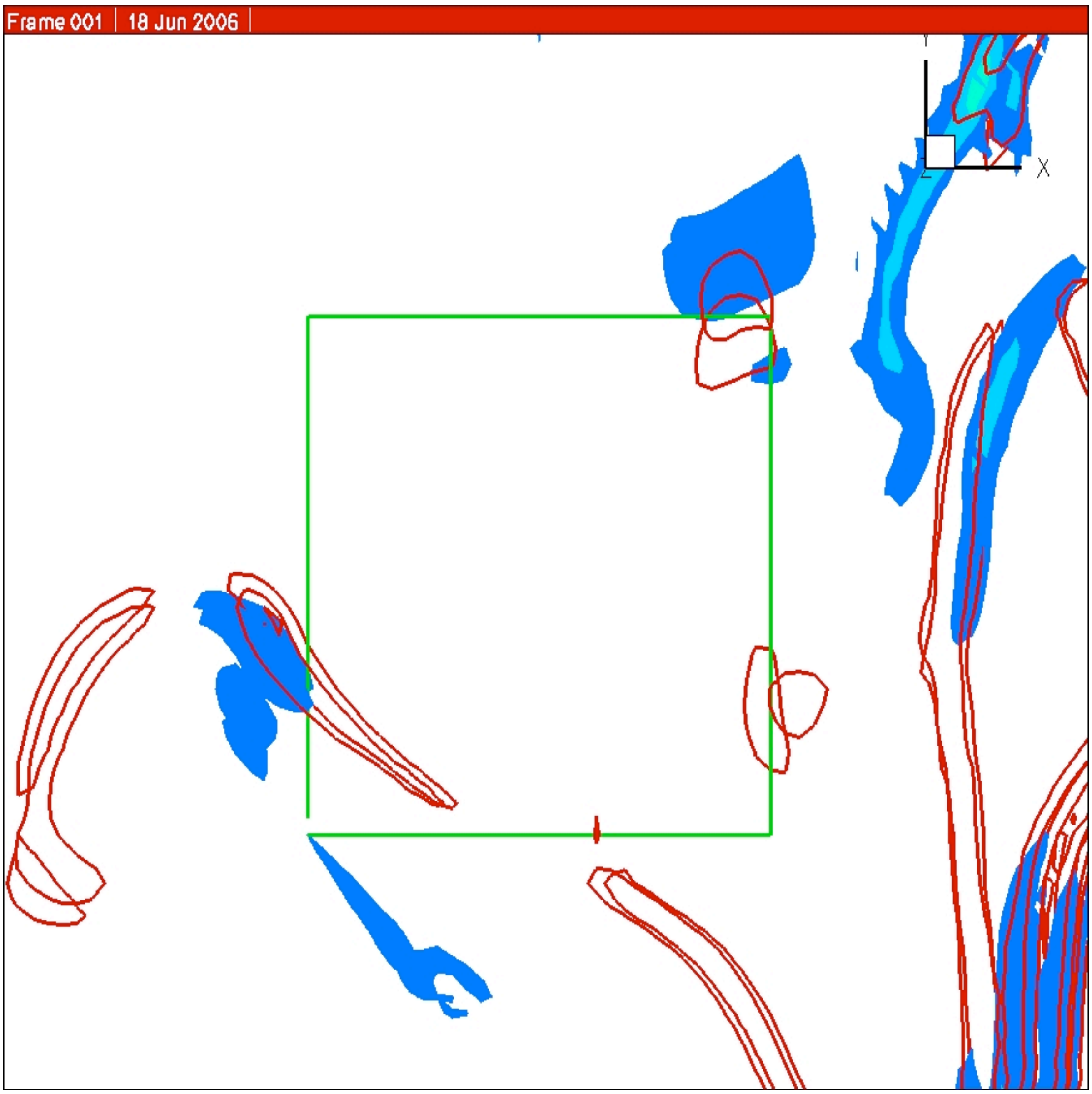
The maximum and minimum loop-voltages for all square loops of edgelength $R = \frac{\pi}{4}$ in the 256^3 simulation, plotted versus $k_c = 2\pi/\ell$.

Current Sheets and Vortex Sheets



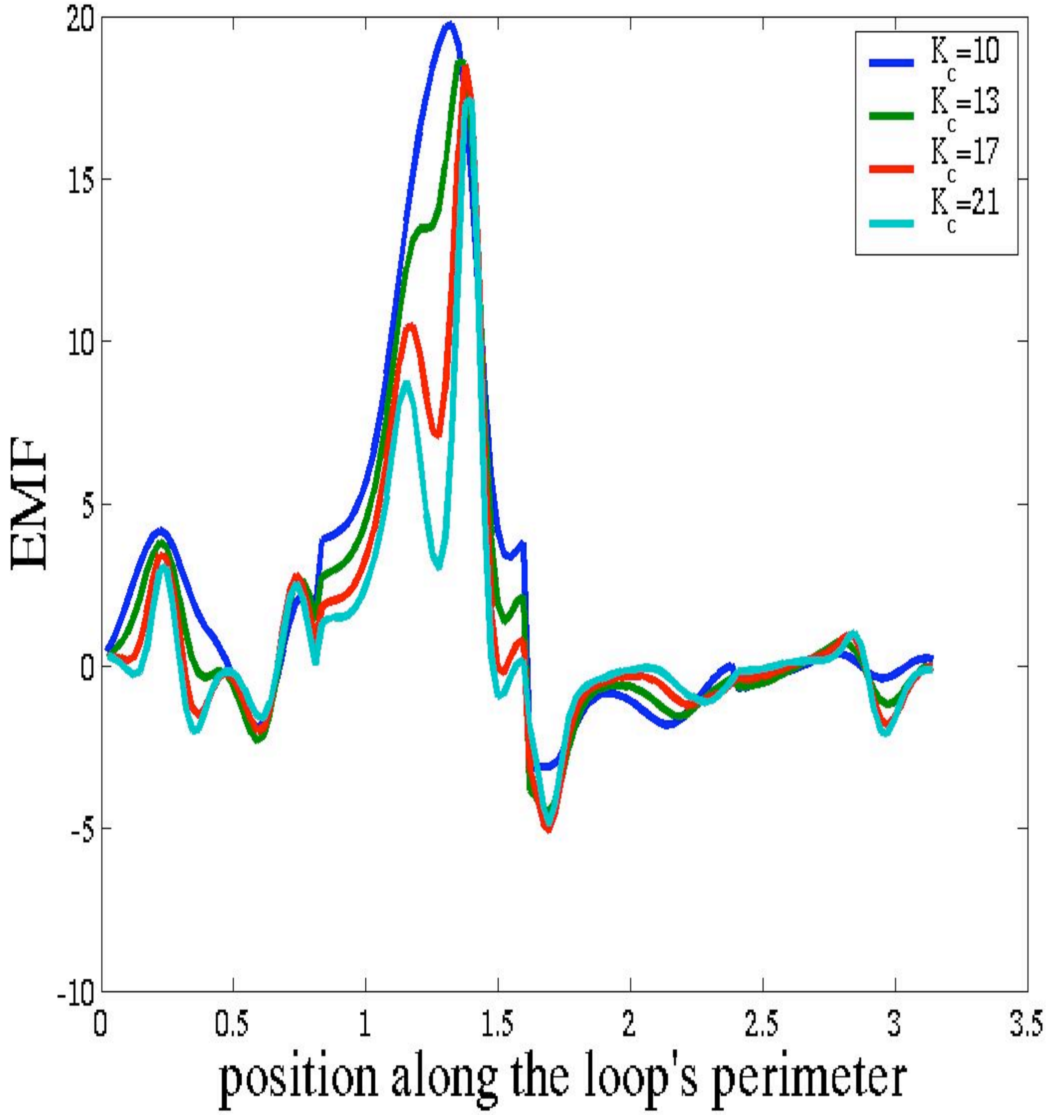
Visualization of intense ribbons of current (blue) and vorticity (red), above a threshold of 2 rms.

Loop with Maximum Voltage



The loop with maximum voltage (green), plotted together with current (blue) and vorticity (red) above 2 rms.

EMF Around the Loop



The parallel component of the turbulent EMF, plotted as a function of the length around the loop. The large peak near $s = 60$ corresponds to the intersection with the strong current and vortex sheets in the upper right corner of the previous plot.

Conclusions

1) Resistivity or other plasma non-ideality is not necessary to explain the breakdown of Alfvén's Theorem at high magnetic Reynolds numbers. Nonlinear MHD effects should be dominant at large length-scales $\ell \gg \ell_d$.

2) Numerical simulations at low resolution are consistent with this picture. The predicted physical effect—related to quantum phase-slip in superconductors—should be observable in high-resolution MHD simulations and laboratory plasma experiments at moderately high magnetic Reynolds numbers.

3) Theoretical and modelling efforts should focus on understanding the turbulent EMF generated by small-scale plasma motions.

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