SIMPLE MODELS OF TURBULENT TRANSPORT

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Turbulent transport of pollutants, heat, chemical agents or magnetic fields is a non-equilibrium phenomenon playing an important role in environmental issues, meteorology, engineering or astrophysics. The simplest mathematical models of turbulent transport study advection of scalar or vector quantities by random velocities with prescribed statistics. Such reduced systems show many expected features of the full-fledged nonlinear models of developed hydrodynamic turbulence, including cascades of conserved quantities and intermittency. Mathematically, they provide examples of random non differentiable dynamical systems. Analysis of the simplified models of advection allows to relate the transport properties of flows to non standard behaviors of fluid particles that is made possible by the lack of regularity of turbulent velocities. The models exhibit a new robust mechanism of intermittency involving hidden statistically conserved quantities. Its understanding has led to a perturbative calculation of the intermittency exponents.

1. ADVECTION-DIFFUSION EQUATIONS

For small intensities, the transport of a scalar quantity $\theta(t,\boldsymbol{r})$ (e.g. temperature) by a d-dimensional velocity field $\boldsymbol{v}(t,\boldsymbol{r})$ may be described by the **advection-diffusion equation**

$$\partial_t\theta \,+\, (\boldsymbol{v}\cdot\boldsymbol{\nabla})\theta \,-\, \kappa\boldsymbol{\nabla}^2\theta \,=\, f \tag{1}$$

where κ is the (molecular) diffusivity constant and $f(t,\boldsymbol{r})$ is the scalar source. In absence of diffusion, i.e. for $\kappa = 0$, the equation may be solved by characteristics provided by trajectories of the **Lagrangian flow**. The latter are defined by the ODE

$$\frac{d\boldsymbol{R}}{ds} \,=\, \boldsymbol{v}(s,\boldsymbol{R}) \tag{2}$$

that describes the motion of virtual fluid particles, or of real ones, suspended in the fluid.

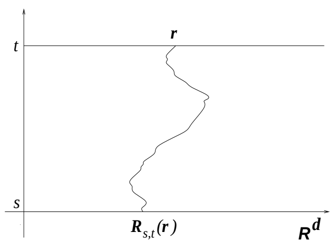

Figure 1. Lagrangian trajectory passing at time t through point $\boldsymbol{r}$

Explicitly, the time evolution of the scalar takes the form

$$\theta(t) \;=\; p_{t,s}\,\theta(s) \;+\; \int\limits_s^t p_{t,\sigma}\, f(\sigma)\, d\sigma \tag{3}$$

with the kernel of the evolution operator concentrated on the integral curve of Eq. (2) passing at time t through point $\boldsymbol{r}$, see Fig. 1:

$$p_{t,s}(\boldsymbol{r}, d\boldsymbol{\rho}) \;=\; \delta(\boldsymbol{\rho} - \boldsymbol{R}_{s,t}(\boldsymbol{r}))\, d\boldsymbol{\rho}\,. \tag{4}$$

In particular, in absence of the source, $\theta(t, \boldsymbol{r})$ is constant along the Lagrangian trajectories.

In the presence of diffusion, Eq. (1) may be solved similarly with the use of noisy Lagrangian trajectories obeying the stochastic ODE

$$d\boldsymbol{R} \;=\; \boldsymbol{v}(t, \boldsymbol{r})\, ds \;+\; \sqrt{2\kappa}\, d\boldsymbol{W} \tag{5}$$

where $\boldsymbol{W}(s)$ is a d-dimensional Brownian motion. The scalar evolution is still given by Eq. (3) provided that we replace operator $p_{t,s}$ by $p^{\kappa}_{t,s}$ with the kernel

$$p^{\kappa}_{t,s}(\boldsymbol{r}, d\boldsymbol{\rho}) \;=\; \Big\langle \delta(\boldsymbol{\rho} - \boldsymbol{R}_{s,t}(\boldsymbol{r}))\, d\boldsymbol{\rho} \Big\rangle \tag{6}$$

where the average is over the Brownian motion $\boldsymbol{W}$. Kernels $p^{\kappa}_{t,s}(\boldsymbol{r}, d\boldsymbol{\rho})$ form the transition probabilities of the velocity-dependent non-stationary Markov process defined by Eq. (5).

In random velocities, the transition probabilities $p^{\kappa}_{t,s}$ become themselves random objects so that the noisy Lagrangian flow described by Eq. (5) involves two levels of randomness: that of the noise and that of the velocities. It resembles in this respect a random walk in random environment. We shall be interested in the **passive advection** when the back-reaction of the scalar on the velocities may be neglected so that the velocity statistics is fixed and independent of the (initial) scalar configuration. The basic questions studied concern the induced statistics of the scalar at later times. Here is a sample of relevant questions:

- How do the initial scalar configurations (random or deterministic) evolve in absence or in presence of sources?
- Do the dissipative effects due to the presence of the molecular diffusion persist when $\kappa \to 0$?
- Do there exist stationary states of the scalar evolution?
- Do they exhibit cascade-like transfer of quantities formally conserved at $\kappa = 0$?
- How far from Gaussian is the statistics of scalar differences. Does it show intermittency?
- What is the effect of compressibility of velocities on the scalar statistics?

These are all questions that may be also asked about the velocity statistics itself in flows governed by the Navier-Stokes equations. They are notoriously hard to answer there. The point is that in the context of the scalar advection they are much easier to analyze if we assume that the velocities are distributed according to a simple synthetic statistics that possesses only the most crude features of the turbulent state. The surprise is that even such an oversimplified approach to the scalar advection leads to a number of far from equilibrium behaviors that are believed to be present also in real turbulent flows. Their understanding in the simpler context may then shed light on the origin of such phenomena in more realistic situations involving turbulence.

The feature of the turbulent velocity statistics that is not too difficult to reproduce in synthetic velocity ensembles is the scaling of the velocity structure functions

$$\langle\, |\boldsymbol{v}(t,\boldsymbol{r}) - \boldsymbol{v}(t,\boldsymbol{r}+\boldsymbol{\rho})|^N \,\rangle \;\propto\; \rho^{\zeta_N} \tag{7}$$

over the inertial range of scales $\eta \ll \rho \ll \ell$ between the viscous scale η and the integral scale ℓ. The celebrated Kolmogorov's mean field theory [23] predicts $\zeta_N = N/3$ and $\ell/\eta \propto (Re)^{3/4}$. It is a reasonable first approximation to the experimental data for $n \ll 10$ and the Reynolds numbers $Re \gg 10^3$. Flows with high Reynolds numbers are abundant in nature.

2. KRAICHNAN MODEL of ADVECTION

The model of passive advection that has been studied most extensively over the last ten years involves the Gaussian, stationary, homogeneous and isotropic ensemble of velocities with vanishing mean and covariance

$$\langle\, v^i(t,\boldsymbol{r})\, v^j(t',\boldsymbol{r}')\,\rangle \;=\; \delta(t-t')\, D^{ij}(\boldsymbol{r}-\boldsymbol{r}')\,. \tag{8}$$

Such an ensemble was first considered in 1968 by Kraichnan [25] in the context of advection of scalar and by Kazantsev [22] in that of magnetic field. The spatial part of the covariance $D^{ij}(\boldsymbol{\rho})$ is taken as a rotationally invariant tensor that exhibits for $\eta \ll \rho \ll \ell$ the scaling behavior

$$D^{ij}(\boldsymbol{0}) - D^{ij}(\boldsymbol{\rho}) \;\propto\; \rho^{\xi} \tag{9}$$

with $0 < \xi \leq 2$. Clearly, the even structure functions of velocities (smeared over time) scale then according to (7) with $\zeta_N = N\xi/2$. If D^{ij} is smooth then the behavior (9) changes for $\rho \ll \eta$ to

$$D^{ij}(\boldsymbol{0}) - D^{ij}(\boldsymbol{\rho}) \;\propto\; \rho^{2}\,. \tag{10}$$

If additionally one fixes the expression

$$\wp \;=\; \frac{\partial_i\partial_j D^{ij}(\boldsymbol{\rho})}{\partial_i\partial_i D^{jj}(\boldsymbol{\rho})} \tag{11}$$

to be a constant between 0 and 1, the form of the spatial covariance D^{ij} is determined in the scaling region. Since $\wp$ is equal to $\langle(\boldsymbol{\nabla}\cdot\boldsymbol{v})^2\rangle/\langle(\boldsymbol{\nabla}\boldsymbol{v})^2\rangle$, it measures the degree of compressibility. In particular, for $\wp = 0$ the velocities are incompressible and for $\wp = 1$ they are gradient fields.

From the mathematical point of view, studying the Lagrangian flow in the Kraichnan ensemble of velocities defined above is a problem in random (differentiable) dynamical systems, the velocity fields providing random, time-dependent vector fields. This is a well developed domain with a rich panoply of available methods mostly borrowed from the theory of deterministic hyperbolic dynamical systems, see e.g. [2]. (One may also consider finite volume versions of the Kraichnan model to deal with flows on compact spaces, as often presumed in the dynamical system theory). This approach permits to capture well the behavior of the

Lagrangian flow on distances of order of the viscous scale and smaller where the fluid particle trajectories separate exponentially with rates described by the **Lyapunov exponents**. The information about the Lagrangian flow in this regime may, in turn, be translated into that about passive advection in the so called Batchelor regime of scales where the velocity fields are smoothed by viscous effects. On distances much larger than the viscous scale η, however, different mechanisms dominate the dynamics of fluid particles. They are the main topic of this contribution.

To see clearly the difference between the two regimes, it is useful to consider the idealized version of the Kraichnan model where the scaling (9) extends to arbitrarily small ρ. In this situation, the Batchelor regime would correspond to the case when $\xi = 2$ with typical velocity fields that are smooth in space. This model is indeed a show-case for the theory of random differentiable dynamical systems, in particular for studying chaos in random flows, see again the book [2]. The model with $\xi < 2$, on the other hand, has typical velocities which are only Hölder continuous with exponent $\xi/2$ (more exactly, with any smaller exponent). It provides an example of a **non differentiable dynamical system**. Study of such systems is a nascent field that requires different tools than those developed for the hyperbolic dynamical systems. It reveals behaviors even more dramatic than in chaotic dynamics, as we shall see on the example of the Kraichnan model. Developing a theory of such systems seems crucial for understanding passive advection in fully developed turbulence where it is reasonable to attempt a description directly at infinite Reynolds number or vanishing viscous scale η (in turbulent atmosphere, for example, η is a fraction of millimeter).

3. PHASES of the KRAICHNAN MODEL

The Lagrangian flow in the Kraichnan model with scaling (9) extending to arbitrarily small ρ exhibits a rich phase structure [12, 20, 29, 13]. One may distinguish five different phases of the flow, depending on the values of the parameters ξ and $\wp$, see Fig. 2.

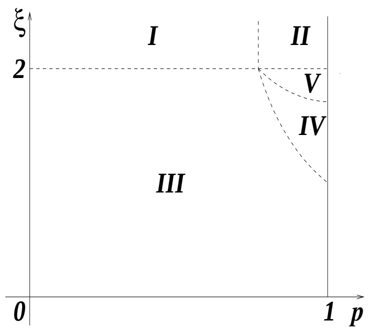

Figure 2. Phases of the Lagrangian flow in the Kraichnan model

Their precise location is as follows:

- $I.$ for $\xi = 2$ and weak compressibility degree $\wp < d/4$;
- $II.$ for $\xi = 2$ and strong compressibility degree $\wp > d/4$;

- *III.* for $\xi < 2$ and weak compressibility degree $\wp < (d-2+\xi)/(2\xi)$,
- *IV.* for $\xi < 2$ and moderate compressibility degree $(d-2+\xi)/(2\xi) < \wp < d/\xi^2$,
- *V.* for $\xi < 2$ and strong compressibility degree $d/\xi^2 < \wp$.

In the model with $\xi = 2$, i.e. with velocities smooth in space, Eq. (2) is interpreted as a stochastic ODE because velocities are white in time. With, invariably, Itô or Stratonovich convention, it gives rise to a random flow of diffeomorphisms $\boldsymbol{R}_{s,t}$ [26]. The Lyaponov exponents of such flow were first computed in [27] and [5]. They depend on the compressibility degree. The top exponent is positive in phase I and negative in phase II. Consequently, the Lagrangian flow is chaotic in phase I and is attracted to random points in phase II. Different properties of scalar transport in the two phases were first analyzed in [11, 12].

In the model with $\xi < 2$, i.e. with velocities only Hölder continuous in space, thus non-Lipschitz, the definition of the Lagrangian flow is more tricky. It has been at the center of the mathematical work by Le Jan-Raimond [28, 29, 30]. A possible way to proceed here is to use the regularizing effect of the noise by replacing the trajectory equation (2) by its noisy version (5) that defines a Markov process with transition probabilities $p^\kappa_{t,s}(\boldsymbol{r}, d\boldsymbol{\rho})$ even for rough velocities, in particular in almost all realizations of the Kraichnan model with arbitrary parameters. Formally,

$$p^\kappa_{t,s} \;=\; T\,\exp\Big\{-\int\limits_s^t [\boldsymbol{v}(\sigma)\cdot\boldsymbol{\nabla} \;+\; \tilde\kappa\boldsymbol{\nabla}^2]\,d\sigma\Big\} \tag{12}$$

where T denotes the time ordering and the integrals are taken with the Itô convention. $\tilde\kappa = \kappa + \frac{1}{2}D_0$ where $D_0\delta^{ij} = D^{ij}(\boldsymbol{0})$ (this contribution is the Itô term). Following [28] one may show that the "Wiener chaos" expansion of the right hand side into the power series in $\boldsymbol{v}$, when integrated against a bounded continuous function on space, converges in the L^2 norm of the velocity ensemble. The argument works for all $\kappa \geq 0$. The $\kappa = 0$ amplitudes are the limits of the ones with $\kappa > 0$. The limiting family of the Markov processes with transition probabilities $p^0_{t,s}(\boldsymbol{r}, d\boldsymbol{\rho})$ indexed by the velocities in the L^2 space of the Kraichnan ensemble has been constructed in [29]. The construction works for any value of $\xi \in (0, 2]$. For $\xi = 2$, the limiting process is deterministic with $p^0(\boldsymbol{r}, d\boldsymbol{\rho})$ related as in Eq. (4) to the flow of diffeomorphisms $\boldsymbol{R}_{s,t}$. What is the situation for $\xi < 2$? Is the limiting process deterministic or does it remain genuinely stochastic with $p^0(\boldsymbol{r}, d\boldsymbol{\rho})$ not concentrated on a single $(t, s, \boldsymbol{r})$-dependent point?

One needs proper tools to study such questions. They are provided by the joint N-particle transition probabilities that take into account both levels of randomness of the noisy Lagrangian flow:

$$P^{N,\kappa}_{t,s}(\boldsymbol{r}_1,\dots,\boldsymbol{r}_N;\, d\boldsymbol{\rho}_1,\dots,d\boldsymbol{\rho}_N) \;=\; \Big\langle \prod_{n=1}^N p^\kappa_{t,s}(\boldsymbol{r}_n, d\boldsymbol{\rho}_n)\Big\rangle \tag{13}$$

where the average is over the velocity ensemble. In the Kraichnan model the transition probabilities $P^{N,\kappa}_{t,s}(\underline{\boldsymbol{r}}, d\underline{\boldsymbol{\rho}}) \equiv P^{N,\kappa}_{|t-s|}(\underline{\boldsymbol{r}}, d\underline{\boldsymbol{\rho}})$ are still Markovian and provide a family of N-particle stationary diffusion processes invariant under particle permutations [30]. The family is consistent in the sense that one may obtain the $(N-1)$-particle process from the N-particle one by forgetting what the last particle does. The generators of the N-particle processes are explicitly known: $P_t^{N,\kappa} = \mathrm{e}^{\,t\,M_{N,\kappa}}$, where

$$M_{N,\kappa} \;=\; \tfrac{1}{2}\sum_{n,m} D^{ij}(\boldsymbol{r}_n - \boldsymbol{r}_m)\,\nabla_{r_n^i}\nabla_{r_m^j} \;+\; \kappa\sum_{n}\boldsymbol{\nabla}^2\,. \tag{14}$$

are for $\kappa > 0$ elliptic operators. For $\kappa = 0$, however, their symbols loose ellipticity on the diagonals $\boldsymbol{r}_n = \boldsymbol{r}_m$. The heat kernels of $M_{N,0}$ were analyzed in the incompressible case $\wp = 0$ in [21]. For $\wp > 0$, operators $M_{N,0}$ are not symmetric and a direct analysis of their heat kernels is an open problem.

A lot of information about the Lagrangian flow may be already extracted from the 2-particle process reduced to the homogeneous and isotropic sector, i.e. describing the distance ρ between two (noisy) Lagrangian trajectories. Whether the flow is deterministic or genuinely stochastic depends on the limiting behavior of the transition probability $P_t^\kappa(r, d\rho)$ of the distance process:

$$\lim_{r\to 0}\,\lim_{\kappa\to 0}\;P_t^\kappa(r,d\rho) \;=\; \begin{cases} \delta(\rho)\,d\rho\,, \\ \\ \text{absolutely continuous} \\ \quad \text{w.r.t. } d\rho\,, \end{cases} \tag{15}$$

respectively. The process $\rho(s)$ is a diffusion on the half-line whose generator is obtained by restricting operator $M_{2,\kappa}$ to translation- and rotation-invariant functions. Extracting information about such diffusions is easy. In the weakly and moderately compressible phase, it is the second possibility that is realized [6, 20] and, as the result, the velocity-indexed Markov process with the transition probabilities $p^0_{t,s}(\boldsymbol{r}, d\boldsymbol{\rho})$ is genuinely stochastic, see Fig. 3.

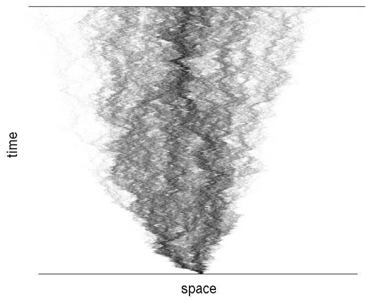

Figure 3. Lagrangian trajectories in a fixed velocity realization starting from the same point in the presence of spontaneous stochasticity

In other words, the stochasticity introduced by the addition of noise to the Lagrangian flow does not disappear when $\kappa \searrow 0$ but persists in the limit in (almost) all velocity realizations! This **spontaneous stochasticity** of the Lagrangian flow is of course possible only in spatially rough velocities where the theorem about the uniqueness of the solutions of the (stochastic) ODE does not hold. The difference between phases III and IV is that, although in both the trajectories starting from the same point separate to finite distance in a finite time due to spontaneous stochasticity, in phase IV they hit each other again in later times, as signaled by the recurrence of the distance process [29].

In the strongly compressible phase V, this is the first term of alternative (15) that is realized [20]. As the result, the deterministic flow is restored in the limit $\kappa \to 0$ with the transition probabilities $p^0_{t,s}(\boldsymbol{r}, d\boldsymbol{\rho})$ concentrated at a single point $\boldsymbol{R}_{s,t}(\boldsymbol{r})$. However, for positive distances r,

$$P^0_t(r, d\rho) \;=\; P^{reg}_t(r, d\rho) \,+\, \pi_t(r)\,\delta(\rho)\,d\rho \tag{16}$$

with $P^{reg}_t(r, d\rho)$ absolutely continuous w.r.t. $d\rho$ and $\pi_t(r) > 0$, unlike in phases III and IV where $P^0_t(r, d\rho)$ is absolutely continuous w.r.t. $d\rho$ for $r > 0$. The behavior (16) signals that, with finite probability, trajectories starting at different points coalesce in finite time, see Fig. 4.

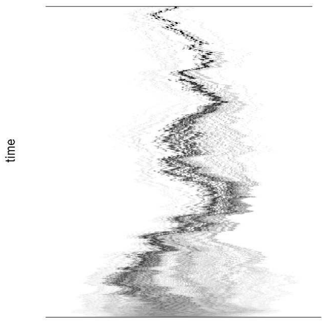

Figure 4. Coalescent Lagrangian trajectories in a fixed velocity realization

This means that the random maps $\boldsymbol{R}_{s,t}$ are not diffeomorphisms but rather many-to-one maps. This is again an unconventional behavior of trajectories possible only in spatially non-Lipschitz velocities.

As stressed by E-Vanden-Eijnden in [13], there are other physically relevant ways to define Lagrangian flow in non-Lipschitz velocities that may lead to different results. For example, mimicking viscous effects, one may smear velocity realizations on a small scale η, define the deterministic flow for smeared velocities, and consider its limit in an appropriately weak sense when $\eta \to 0$. If this procedure is applied to the transition probabilities for the inter-trajectory distance then, indeed, one obtains a different solution in the moderately compressible phase IV (but not in the other phases). It has similar characteristics as the one of phase V [13], see (16). In physical flows, the ratio of viscosity and molecular diffusivity defines the Prandtl number Pr. The limit when $\kappa \to 0$ in non-Lipschitz velocities corresponds then to $Pr = 0$ whereas the one when $\eta \to 0$ at zero diffusivity relates to $Pr = \infty$. One may consider also the case when one starts with both κ and η positive and sends them to zero fine-tuning the Prandtl number in the process, as first suggested in [14].

As shown in [18], an appropriate limiting procedure leads in phase IV to a sticky distance process which spends at zero a finite amount of time, although never a whole time-interval, and which shows a hybrid behavior between the to extremes of (15).

The consistent N-particle processes in phase IV with coalescence of trajectories were constructed directly by Le Jan-Raimond in [30]. Such processes should arise via the $Pr = \infty$ limiting procedure. The article [30] also sets stage for a general classification of the possible Lagrangian flows in the Kraichnan model. Such flows are assumed to be given by consistent families of N-particle diffusion processes with generators that coincide with operators $M_{N,0}$ away from the diagonals. Under mild regularity assumptions, such a family of N-particle processes allows to reconstruct via relation (13) the transition probabilities $p_{t,s}(\boldsymbol{r}, d\boldsymbol{\rho})$ indexed by elements of an abstract probability space possibly containing other random elements besides the velocity fields. The extra noise is expected to be "black" [35], i.e. of non-standard type (not white, not Poisson), as has been shown in a 1-dimensional limiting case with $\xi = 0$ [30]. Construction of the sticky N-particle flow as well as the full classification of possibilities and their physical interpretation is still an open problem.

4. CONSEQUENCES for SCALAR TRANSPORT

The transition probabilities $p^\kappa_{t,s}(\boldsymbol{r}, d\boldsymbol{\rho})$ of the noisy Lagrangian trajectories (indexed by the velocity realizations) provide via Eq. (3) the solution of the advection-diffusion equation (1) interpreted as the Stratonovich stochastic equation in the Kraichnan velocities. Different behaviors of the Lagrangian flow in the Kraichnan model with Hölder continuous velocities in the limit $\kappa \to 0$ lead to different transport properties in the purely convective regime of scalar advection. It is interesting to study the scalar evolution both in the decaying regime without sources and in the presence of stationary source. Here we shall briefly discuss only the latter case with the source $f(t, \boldsymbol{r})$ modeled by a Gaussian random field, independent of velocities, with mean zero and with covariance

$$\langle\, f(t,\boldsymbol{r})\, f(t',\boldsymbol{r}')\,\rangle \;=\; \delta(t-t')\,\chi(\tfrac{\boldsymbol{r}-\boldsymbol{r}'}{L})\,. \tag{17}$$

L describes the correlation length of the source and is assumed large.

In the presence of such a source, in the weakly compressible phase III and dimension bigger than two[a], there exists a stationary state of the scalar evolution (1) for any $\kappa \geq 0$ [13]. It is obtained from the product of the Gaussian measures for velocities and source by the map $(\boldsymbol{v}, f) \mapsto \theta$. The map is given by the relation

$$\theta(t,\boldsymbol{r}) \;=\; \int\limits_{-\infty}^{t} d\sigma \int p^\kappa_{t,\sigma}(\boldsymbol{r}, d\boldsymbol{\rho})\; f(\sigma,\boldsymbol{\rho})\,, \tag{18}$$

that is produced by solving Eq. (1) on semi-infinite time intervals for vanishing initial data, see (3). The integral converges in the L^2 norm of the Gaussian ensembles. The stationary state is unique in the sense that any initial state of the scalar with decaying correlations approaches it at long times. One of its basic characteristics is the persistence of the dissipative effects in the limit when $\kappa \to 0$, as signaled by the so called **dissipative anomaly**.

[a]The latter restriction is present already in forced diffusion and may be dropped in finite volume.

The anomaly states that the dissipation rate $\kappa(\nabla\theta)^2$ of the "scalar energy" $\frac{1}{2}\int\theta^2$ does not vanish (in mean) when $\kappa\to 0$:

$$\lim_{\kappa\searrow 0}\ \kappa\,\langle(\nabla\theta)^2\rangle\ >\ 0\,. \tag{19}$$

In fact the energy injected at scale L is transfered through the scales in a **direct cascade** process that may be assessed quantitatively. The dissipation takes place only at the shortest scales. Its rate is equal to the energy injection rate $\frac{1}{2}\,\chi(0)$. The persistence of dissipation is a consequence of the spontaneous stochasticity of the Lagrangian flow (deterministic flows cannot dissipate scalar energy unless it has infinite density). The scalar energy cascade in phase III resembles the direct cascade of the fluid kinetic energy $\frac{1}{2}\int\boldsymbol{v}^2$ expected to occur in high Reynolds number $3d$ Navier-Stokes turbulence [23]. In the intermediate phase IV, the mean energy density $\frac{1}{2}\langle\theta^2\rangle$ grows to infinity with time and the (unique) stationary state exist only for the scalar differences $\theta(t,\boldsymbol{r})-\theta(t,\boldsymbol{r}+\boldsymbol{\rho})\equiv\Delta\theta(t,\boldsymbol{\rho})$. It still exhibits dissipative anomaly and direct energy cascade.

At strong compressibility, the transport properties become very different, as first realized for phase II of the smooth model in [11, 12]. In phase V of the $\xi<2$ model, the restoration of deterministic Lagrangian flow when $\kappa\to 0$ does not permit persistent dissipation. Indeed, here

$$\lim_{\kappa\searrow 0}\ \kappa\,\langle(\nabla\theta)^2\rangle\ =\ 0\,. \tag{20}$$

Instead, the scalar energy injected by source is transfered to long scales in an **inverse cascade** process and condenses in the constant mode growing linearly with time, although the stationary state for the scalar differences still exists [20]. Similar inverse cascade of fluid kinetic energy is expected in the $2d$ Navier-Stokes turbulence in infinite volume [24, 4].

5. INTERMITTENCY and ZERO MODES

There is another feature in which the scalar states resemble what is observed in hydrodynamic turbulence: the direct cascade shows strong short-scale **intermittency** of scalar differences whereas the inverse cascade is scale invariant. Intermittency of scalar is signaled by frequent occurrence at short distances of large deviations of $\Delta\theta(t,\boldsymbol{\rho})$, more frequent than in the normal distribution. It is reflected in the decay slower than Gaussian of the corresponding probability distribution function. It manifests itself in anomalous scaling of the scalar structure functions at $\kappa=0$ for small ρ:

$$S_N(\rho)\,\equiv\,\langle\,(\Delta\theta(t,\boldsymbol{\rho}))^N\,\rangle\ \propto\ \rho^{\zeta_N}\quad\text{ with }\quad\zeta_N<N\zeta_2/2\,. \tag{21}$$

Such a scaling implies that the undimensionalized moments $S_N(\rho)/[N!!\,S_2(\rho)^{N/2}]$, that are equal to 1 in the normal distribution, diverge when $\rho\to 0$ exhibiting long tails in the distribution of $\Delta\theta(t,\boldsymbol{\rho})$ that become more and more pronounced when ρ decreases. The Kraichnan model has been the first example where such an anomalous scaling, observed experimentally for velocity and temperature structure functions [34, 32], has been analytically demonstrated in a hydrodynamic context [10, 19, 33]. Here is the main idea.

The stationary equal time correlation functions $F_N(\underline{\boldsymbol{r}}) = \langle\,\theta(t,\boldsymbol{r}_1)\ldots\theta(t,\boldsymbol{r}_N)\rangle$ in the Kraichnan model satisfy (for even N) a chain of closed equations

$$M_{N,\kappa}\,F_N \;=\; F_{N-2}\otimes\chi_L \tag{22}$$

with $M_{N,\kappa}$ given by (14) and χ_L denoting the spatial covariance of source, see (17). Eqs. (22) determine F_N up to zero modes of the operators $M_{N,\kappa}$. It appears that in the limit $\kappa\to 0$ the contribution of the scaling zero modes $\phi_{N,i}$,

$$M_{N,0}\,\phi_{N,i} \;=\; 0\,, \qquad \phi_{N,i}(\lambda\underline{\boldsymbol{r}}) = \lambda^{\zeta_{N,i}}\,\phi_{N,i}(\underline{\boldsymbol{r}})\,, \tag{23}$$

to the scalar N-point function gives rise to terms in the N-point structure function that are dominant at small r. As the result, $\zeta_N = \min\limits_i \zeta_{N,i}$. (More exactly, one should consider only the non-degenerate zero modes depending non-trivially on all N positions.) The scaling zero modes of operators $M_{N,0}$ were computed by a singular perturbation expansion which gave[b]

$$\zeta_N \;=\; \begin{cases} \frac{N}{2}(2-\xi) - \frac{N(N-2)(1+2\wp)}{2(d+2)}\xi + O(\xi^2)\,, \\ \frac{N}{2}(2-\xi) - \frac{N(N-2)(1+2\wp)}{2d}\xi + O(d^{-2})\,. \end{cases} \tag{24}$$

Note the presence of terms violating the linear behavior in N responsible for the anomalous scaling of the structure functions. Numerical simulations [17] have confirmed the perturbative predictions and, in the incompressible model, the anomalous scaling for all values of $\xi<2$, see Fig. 5.

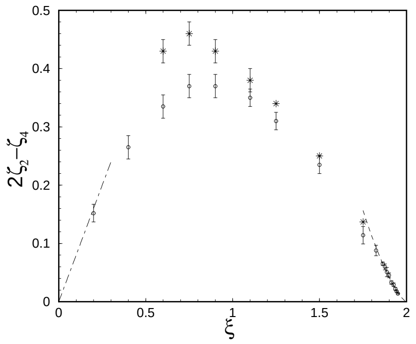

Figure 5. Anomalous part of the 4-point scalar structure function exponent in the $2d$ (higher points) and $3d$ (lower points) incompressible Kraichnan model. The line through zero represents the $3d$ leading perturbative prediction. From [17]

The zero modes of operators $M_{N,\kappa}$ have interesting interpretation [6]: when calculated on simultaneous positions of N Lagrangian particles, they provide statistically conserved quantities (i.e. they are martingales of the N-particle processes). The domination of the scalar correlations at short distances by quantities statistically conserved by the fluid particle dynamics has been also observed numerically in the velocities governed by the stochastically

[b]The first calculation of [10, 19] covered the case $N=4$ and $\wp=0$. The $O(\xi^2)$ and $O(\xi^3)$ contributions to ζ_N are also known [1]. See Sect. II.E.5 of [15] for more complete account.

forced $2d$ Navier-Stokes equation in the inverse cascade range [8]. A search for such hidden conservation laws of the Lagrangian dynamics in the $3d$ Navier-Stokes direct energy cascade is a challenge both for numerical simulations and for the new generation of particle tracking experiments.

It was also observed in the numerical simulations of scalar advection by the Kraichnan model velocities that the exponents ζ_N approach fast a finite value when N grows for fixed ξ [7]. There were attempts to understand analytically such behavior of scalar exponents by instanton type methods [3]. The saturation of ζ_N signals the appearance of values close to zero in the spectrum of Hölder exponents of the instantaneous scalar field realizations. Such values are far from the mean-field prediction $(1-\xi/2)$ for the Hölder exponent of the scalar field. The regions with small Hölder exponents form "cliffs" in the instantaneous scalar realizations. The saturation of scalar anomalous exponents and abundance of the cliffs in scalar configurations, see Fig. 6, was also seen numerically in the advection by the $2d$ inverse cascade Navier-Stokes turbulence [7] and, experimentally, for the temperature field in the $3d$ turbulent helium gas [32].

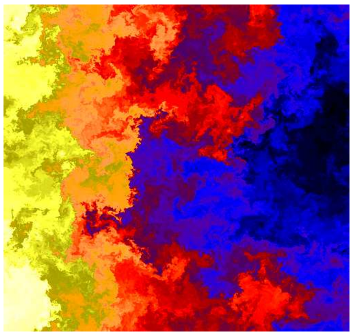

Figure 6. An instantaneous scalar configuration in the inverse $2d$ Navier-Stokes cascade. From [7]

6. CONCLUSIONS

The Kraichnan ensemble of Hölder continuous velocities provides an example of a **random non differentiable dynamical system**. It exhibits unconventional behaviors of the Lagrangian flows, impossible in smooth velocities. Such behaviors range from the **spontaneous stochasticity**, with possible **stickiness**, to **coalescence** of deterministic fluid particle trajectories. Different Lagrangian flow behaviors are reflected in different transport properties of scalar advection: **direct** versus **inverse cascade**, **intermittency** versus **scale invariance**. When present, intermittency is governed by the **zero modes**, the quantities statistically conserved by the fluid particle evolution. Although many of these aspects are under analytic control, the complete classification of possible Lagrangian flows in the Kraichnan model and a rigorous, non perturbative control of the zero modes are still open problems.

Similar and possibly other unconventional behaviors are expected in Gaussian ensembles of spatially Hölder continuous velocities time correlated in a scale-dependent way [9]. The rigorous control of such models is more difficult and is only at a preliminary stage [16].

The main aim of this exposition was to indicate how random non differentiable dynamical systems appear naturally as a mathematical setup for description of passive advection in higly turbulent flows and to list some of the possible intricate behaviors of such systems discovered in the example of the Kraichnan model. The theory of non-differentiable dynamical systems promises to be as interesting as that of the differentiable ones, although it has not been developed yet to a comparable degree. In particular, one lacks more precise information about both geometric and ergodic properties of the resulting flows. Beyond doubt, non differentiable dynamical systems deserve further study in the random and, possibly, in the non-random context.

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