Kolmogorov Spectra and Multi-Scaling of Stochastic Aggregation

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Plan.

- The model of cluster-cluster aggregation with inputs ("Mass Model").
- Scaling analysis of mass model (MM).
- Stochastic Smoluchowski equation.
- Perturbative RG treatment of MM in $d \leq 2$. Numerical confirmation of theoretical results.
- Non-perturbative confirmation of multi-scaling.
- Multi-scaling and fluctuations of the mass flux.
- Conclusions.

References: PRL **94**, 194503 (2005); cond-mat/0510389 (to appear in the special issue of Physica D on coagulation, 2006)

Cluster-Cluster Aggregation with Deposition (MM).

Mass Model. Particles of mass m_0 are deposited into $\mathbf{R}^d$ at a rate of J kilograms per unit volume per second. Deposited particles diffuse at rate D and coagulate on contact at rate λ conserving mass.

Questions:

- What is the average mass distribution of particles in the steady state, $C_1(m) = \lim_{t\to\infty} E\left(P_t(\vec{x}, m)\right)$, where $P_t(\vec{x}, m)$ is local mass distribution?
- What is the p. d. f. of n-particle configurations in MM, $C_n(m_1, \ldots m_n) =$ $\lim_{t\to\infty} E\left(P_t(\vec{x}+\vec{l_1}, m_1)\cdots P_t(\vec{x}+\vec{l_n}, m_n)\right)$?

Scaling analysis of Mass Model.

- **Dimensions of relevant quantities:** $[C_n] = M^{-n} L^{-dn}$, $[J] = \frac{M}{L^d T}$, $[D] = \frac{L^2}{T}$, $[P] = \frac{1}{L^d M}$. Can set $D = 1$. Then $C_n(m) = F(m, m_0, J, \lambda, l)$.
- **Kolmogorov-Zakharov spectra in** $d > 2$: For typical masses » m_0, C_n is a function of mass, reaction rate λ and average flux J only. Also, $C_n \sim J^{n/2}$. Therefore,

$$C_n(m) \sim m^{-\frac{3}{2}n},$$

- **Smoluchowski theory for** $d < 2$: Reaction is diffusion-limited $\Rightarrow C_n = F(m, l, J)$ For $n = 1$, no l-dependence, hence

$$C_n(m) \sim m^{-\gamma(1)},$$

where $\gamma(1) = \frac{2d+2}{d+2}$. $n > 1$?

The Mass Model on the Lattice

Consider a lattice in d dimensions with particles of integer masses.
$N_t(\mathbf{x}, m)$=number of particles of mass m on site $\mathbf{x}$ at time t.

Diffusion

$$\begin{aligned} N_t(\mathbf{x}, m) &\to N_t(\mathbf{x}, m) - 1 \\ N_t(\mathbf{x}+\mathbf{n}, m) &\to N_t(\mathbf{x}+\mathbf{n}, m) + 1 \end{aligned}$$

Rate: $DN_t(\mathbf{x}, m)/2d$

Aggregation

$$\begin{aligned} N_t(\mathbf{x}, m_1) &\to N_t(\mathbf{x}, m_1) - 1 \\ N_t(\mathbf{x}, m_2) &\to N_t(\mathbf{x}, m_2) - 1 \\ N_t(\mathbf{x}, m_1 + m_2) &\to N_t(\mathbf{x}, m_1 + m_2) + 1 \end{aligned}$$

Rate : $\lambda K(m_1, m_2) N_t(\mathbf{x}, m_1) N_t(\mathbf{x}, m_2)$

Injection

$$N_t(\mathbf{x}, m) \to N_t(\mathbf{x}, m) + 1$$

Rate: J

Relation Between Interacting Particle Systems and Stochastic Evolution Equations.

- Evolution of a Markovian IPS is governed by Master Equation for the probability measure on the space of all configurations $\{N(\vec{x},m)\}_{\vec{x}\in\mathbf{Z}^d, m\in\mathbf{R}^+}$.
- Master equation is linear and is of the first order of time.
- Its solution admits path integral representation.
- If interactions are local and binary, corresponding interacting field theory can be mapped to a stochastic evolution equation for an effective field $\tilde{P}(\vec{x},m,t)$ using Hubbard-Stratonovich transformation.
- All correlation functions of the IPS can be expressed in terms of correlation functions of the field $\tilde{P}$.

Stochastic Smoluchowski Equation.

$$\begin{aligned}(\partial_t - D\Delta)\,\tilde{P}_m \;=\; & \frac{\lambda}{2}\int_0^\infty dm_1 dm_2\; \tilde{P}_{m_1}\tilde{P}_{m_2}\,\delta(m-m_1-m_2) \\ & - \frac{\lambda}{2}\int_0^\infty dm_1 dm_2\; \tilde{P}_{m}\tilde{P}_{m_1}\,\delta(m_2-m-m_1) \\ & - \frac{\lambda}{2}\int_0^\infty dm_1 dm_2\; \tilde{P}_{m}\tilde{P}_{m_2}\,\delta(m_1-m_2-m) \\ & + \frac{J}{m}\delta(m-m_0)\tilde{P}_m - 2i\sqrt{\lambda}\xi(\mathbf{x},t)\tilde{P}_m.\end{aligned}$$

- Noise, $\xi(\mathbf{x},t)$, is standard Gaussian and white.
- $\left\langle \tilde{P}_m \right\rangle_\xi = \langle P_m \rangle_{\rm RD}$,
  $\left\langle (\tilde{P}_m)^n \right\rangle_\xi = \langle P_m (P_m - 1)\ldots(P_m - n + 1)\rangle_{\rm RD}$.
- In the low density limit, $C_n(m) = \frac{1}{n!}\left\langle (\tilde{P}_m)^n \right\rangle_\xi$

SSE and $A + A \to A$ system: an exact mapping.

- Integro-differential SSE can be converted into an SPDE by Laplace Transform:

$$R_\mu(\mathbf{x},t) = \int_0^\infty dm\, P_m(\mathbf{x},t) - \int_0^\infty dm\, P_m(\mathbf{x},t)\, e^{-\mu m}$$

- SSE becomes

$$(\partial_t - D\Delta)R_\mu = -\lambda R_\mu^2 + J\frac{(1-e^{-m_0\mu})}{m_0} + 2i\sqrt{\lambda}R_\mu \xi(\mathbf{x},t)$$

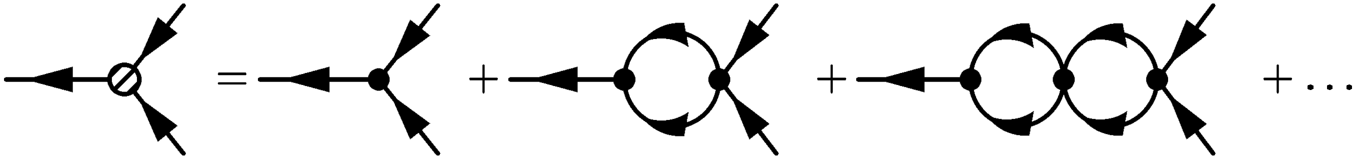
- To compute n-point correlation functions of $\tilde{P}$, one has to compute correlation functions of fields $R_{\mu_1}, \ldots R_{\mu_n}$, which are pairwise correlated via the common noise term.

Loop Expansion.

- Perturbative solution of SSE w. r. t. noise is equivalent to loop expansion around mean field (Smoluchowski) solution.
- In $d \leq 2$, expansion parameter is $g(m)^{\frac{2}{d+2}}$, where $g(m) = \lambda \left(\frac{Dm}{J}\right)^{\frac{\epsilon}{d+2}}$ is a dimensionless aggregation rate. Here $\epsilon = 2 - d$ and m is a typical mass.
- If $d < d_c = 2$, loop expansion becomes useless as $m \to \infty$.
- A re-summation of loop expansion has to be performed. Unlike Navier-Stokes turbulence, this can be done for MM using the formalism of perturbative RG.

Renormalisation of the Aggregation Rate

- There are no diagrams correcting the propagator. Hence there is no field renormalization.
- Only vertex functions with two incoming lines and one or two outgoing lines are relevant.



- Diagrams correcting λ form a geometric series and can be summed exactly:

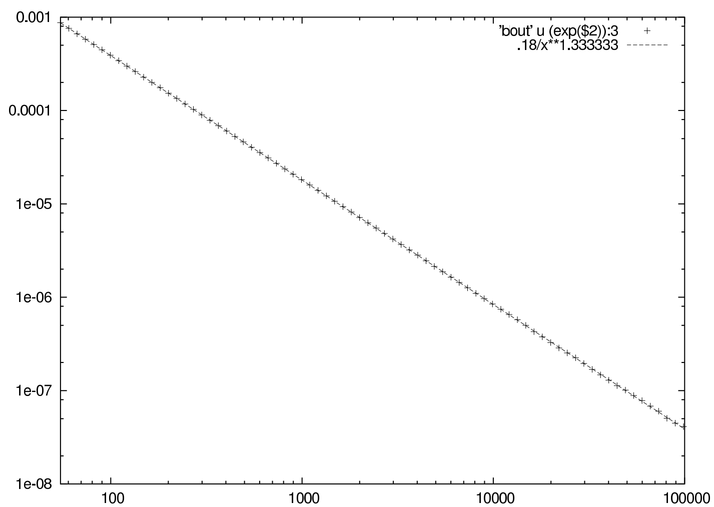
$$\lambda \to \lambda_R(m) = \frac{\lambda}{1 + \lambda C(\epsilon)(DJ^{-1}m)^{\frac{\epsilon}{d+2}}}$$

where $\epsilon = 2 - d$.

- The β-function is $\beta(g) = g^2 - gg^*$, where $g^* \sim \epsilon$.

Average mass distribution.

$m \to \infty$ asymptotics of C_1 can be determined exactly: Firstly, $Z_{\tilde{P}} = 1$. Secondly, as $m \to \infty$, $g(m, M) \to g^*$, where $g(m, M)$ is the running coupling.



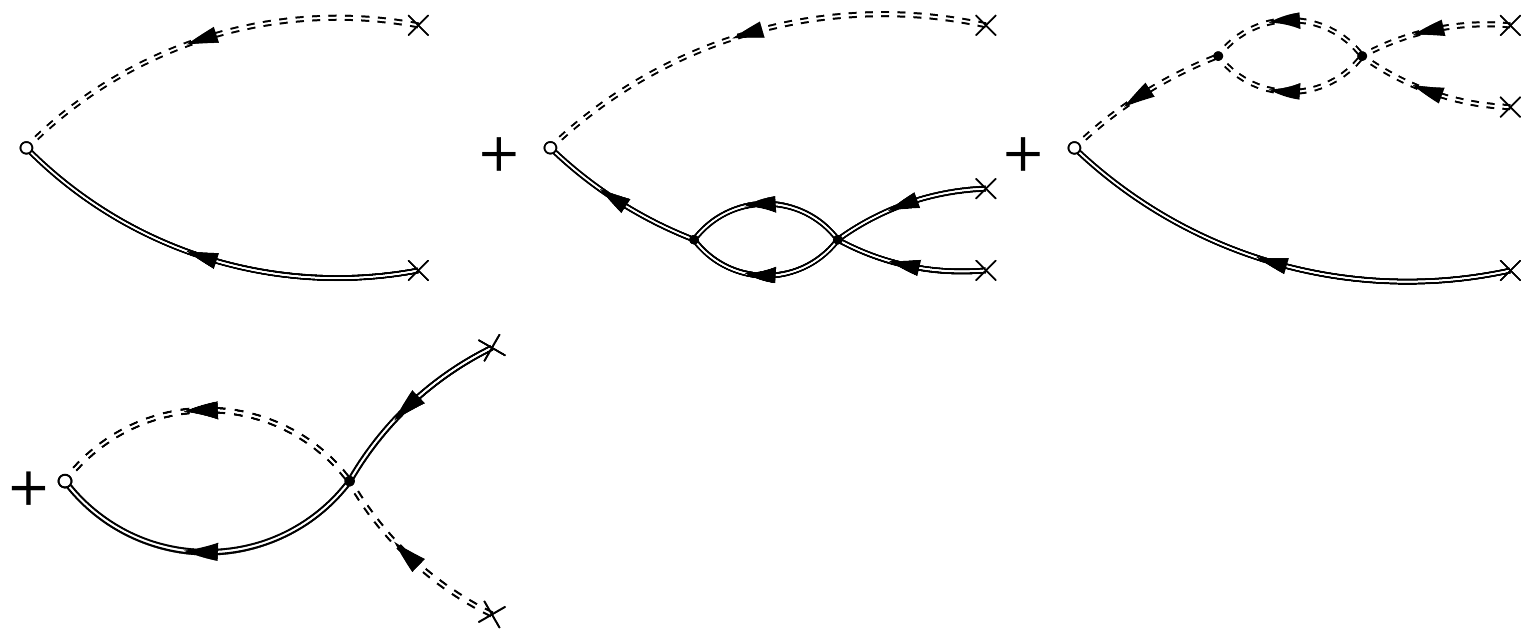
Results :

- $N_m = C_1 m^{-\frac{3}{2}}, \quad d > 2$
- $N_m = C_2 m^{-\frac{2d+2}{d+2}}, \quad d < 2$
- $N_m = C_3 \sqrt{\log m}\, m^{-\frac{3}{2}},\ d = 2$

- Kolmogorov-Zakharov theory is incorrect for $d < 2$.
- Smoluchowski (renormalized MF) theory is correct for $d \leq 2$.

The Origin of Multi-Fractal Scaling in $d \leq 2$.

- Loop expansion of $C_2(m)$ looks as follows:



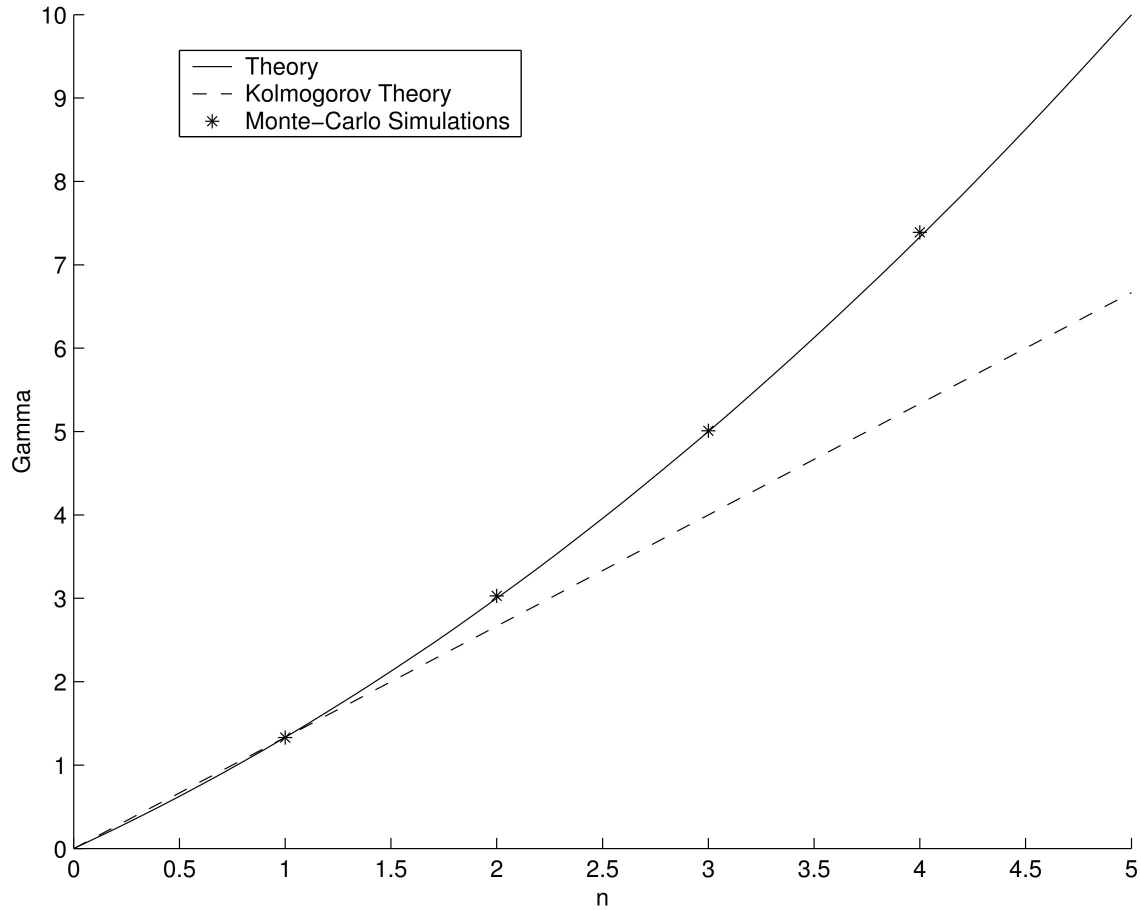
- Connected one-loop diagram is primitively divergent, but this divergence cannot be eliminated using coupling constant renormalization.
- As a result, $Z_{\tilde{P}^2} = 1 + \frac{C(\epsilon)}{\epsilon} g + O(g^2) \neq 1$. Analogously, $Z_{\tilde{P}^n} = 1 + \frac{n(n-1)}{2} \frac{C(\epsilon)}{\epsilon} g + O(g^2)$.

Multi-scaling MM.

- Solving Callan-Symancyk Equation at the fixed point of the RG flow we find: $C_n(m) \sim m^{-\gamma_n}$, where $\gamma_n = \frac{2d+2}{d+2}n + \epsilon\frac{n(n-1)}{2(d+2)} + O(\epsilon^2)$.
- In two dimensions, $C_n \sim ln(m)^{n(2-n)/2} m^{-\frac{3}{2}n}$.
- Solving for moments of P_m in terms of moments of $\tilde{P}_m$, one finds that $E(P_m^n) = E(P_m)$ (extreme anomalous scaling.)
- Self-similarity is violated due to anti-correlation between particles: $C_n(m)/C_1(m)^n \to 0$ as $m \to \infty$.

Numerical verification of multi-scaling of MM.

Comparison with numerics: $d = 1$.



Non-perturbative confirmation of multi-scaling of MM in $d \leq 2$.

- The answer for γ_1 is exact to all orders in ϵ as $Z_{\tilde{P}} = 1$ and β-function is known exactly.
- γ_2 is also exact to all orders in ϵ: averaging SSE,

$$(\partial_t - D\Delta)R_\mu = -\lambda R_\mu^2 + J\frac{(1-e^{-m_0\mu})}{m_0} + 2i\sqrt{\lambda}R_\mu \xi(\mathbf{x}, t),$$

  in Fourier space, one gets $C_2(\mu) \sim \mu$ for $\mu m_0 << 1$. Hence, $\gamma_2 = -3$ exactly. This is a counterpart of the 4/5 law (or CFR) in cluster-cluster aggregation, as explained by Rajesh.
- $\gamma_0 = 0$, $\gamma_1 = -\frac{2d+2}{d+2}$ and $\gamma_2 = -3$ do not lie on a straight line for $d < 2$.

Multi-scaling and fluctuations of the mass flux.

- $J(m) = \lambda \int_0^m dm_1 m_1 \int_0^{m_1} dm_2 \tilde{P}(\vec{x}, m_2)\tilde{P}(\vec{x}, m - m_2) + \ldots.$
- Assume that $E(J(m)^n) \sim m^{\mu_n}$.
- Then $-\frac{3}{2}n + \mu_{n/2} = \gamma_n$ (!).
- $\gamma_n = -\frac{3}{2}n + \delta\gamma(n)$, where $\delta\gamma_n = -\frac{\epsilon}{8}n(n-2) + O(\epsilon)$.
- RSS $\Rightarrow \mu_n = -\frac{\epsilon}{2}n(n-1) + O(\epsilon)$
- Intermittency of flux $\rightarrow$ multi-scaling.
- **Random cascade model of mass flux**: $J_m = J_M W_1 W_2 \ldots W_n$, $n = ln(M/m)$. W_k's are i. i. d.'s lognormal: $E(W_k) = 1$ $W_k = e_k^x$, where $E(x_k) = -\frac{\epsilon}{2}$, $var(x_k) = \epsilon$.

Conclusions.

- Stochastic aggregation models provide an excellent testing ground for theories of non-equilibrium statistical mechanics.
- There are analogies with strong turbulence including constant flux relation, bi-fractality, multi-scaling, refined self-similarity.
- Can we do anything with the non-constant kernel cases?
- Application of lessons learned here to other systems, in particular to wave turbulence (see CC's talk tomorrow).
- Large deviations analysis?
- Can we solve the model exactly in $d = 1$? Done for decaying $A + A \to A$ reaction (Ranjiva Munasinghe, RR, Roger Tribe, OZ, math.PR/0512179, to appear in CMP, 2006) using probabilistic methods. Hopefully, one can construct a "physicist's solution" using Bethe Anzats for hard core $1D$ bosons (John Cardy, OZ, in progress).