Oleg, Sergey thanks for the great conference!

Massimo congratulations !

Role of viscous processes in inertial range dynamics

Victor Yakhot

Department of Aerospace and Mechanical Engineering,

Boston University, Boston 02215

PACS numbers:

J. Schumacher, K.R. Sreenivasan, T Gotoh.

$$\frac{\partial E(k)}{\partial t} + \frac{\partial J(\mathbf{k})}{\partial \mathbf{k}} = \mathbf{f}(\mathbf{k}) - 2\nu k^2 E(\mathbf{k})$$

$$\overline{f(\mathbf{k})^2} \propto \mathcal{P}\delta(k - k_0)/k^{d-1}; \;\; 1/k_0 = 1/L$$

$$\nu \to 0; \;\; \frac{\partial J(k)}{\partial \mathbf{k}} = 0; \;\; J(k) = const = \mathcal{P} = \mathcal{E}$$

K41: The IR dynamics are independent upon viscosity and viscous processes.

"We therefore conclude that, for the large (IR) eddies which are the basis of any turbulent flow, the viscossity is unimportant and may be equated to zero, so that the motion of these eddies obeys Euler's equation. ...

The viscosity of the fluid becomes important only for the smallest eddies, whose Reynolds number is comparable with unity ".

"... we may say that none of the quantities pertaining to the eddies of sizes $r >> \eta_K$ can depend on ν (more exactly, these quantities cannot be changed if ν varies but other conditions of the motion are unchanged)."

Landau and Lifshitz, "Fluid Mechanics".

This seems to be correct. Does it means that the viscous processes do not play role in the IR dynamics"

K41

$$S_{n,m} = \overline{(\delta_r u)^n (\delta_r v)^m} = A_{n,m} (\mathcal{E} L)^{\frac{n}{3}} (\frac{r}{L})^{\xi_{n,m}}$$

The dissipation rate is the only dynamically relevant parameter in the inertial range:

$$-\mathcal{E} = -\nu \overline{\mathbf{u} \cdot \frac{\partial^2 \mathbf{u}}{\partial x_i^2}} = -\nu \ lim_{r \to \eta} \frac{\partial^2}{\partial x_j^2} \overline{u_i(x) u_i(x+r)} =$$

$$\nu \ lim_{r \to \eta} \frac{1}{2} \frac{\partial^2}{\partial r^2} S_{2,0}(r) \propto \nu \mathcal{E}^{\frac{2}{3}} \eta^{\xi_2 - 2}$$

$$\xi_2 = 2/3; \quad , \eta_K \approx (\frac{\nu^3}{\mathcal{E}})^{\frac{1}{4}}$$

.

$$u_{rms} = L = 1; \ \mathcal{E} = 1; \ \nu = 1/Re$$

$$\eta_K \approx L R^{-\frac{3}{4}}$$

.

Since $\mathcal{E} = \mathcal{P}$, we can say that viscosity disappeared S_ns. In Kolmogorov's phenomenology the dissipation scale is a constant number.

Some exact relations for isotropic, homogeneous and incompressible velocity field.

$$\frac{1}{r^{d-1}}\frac{\partial}{\partial r}r^{d-1}S_{2,0} = \frac{d-1}{r}S_{0,2}(r)$$

$$\overline{\phi(x)u_i(x')} = 0$$

$$\frac{1}{r^{d+1}}\frac{\partial}{\partial r}r^{d+1}S_{3,0} = (-1)^d\frac{12}{d}\mathcal{E}$$

The NS equations give:

$$S_{3,0}(r) = -\frac{12}{d(d+2)}\mathcal{E}r$$

$$S_{3,0}/S_{1,2} = 3$$

From this Kolmogorov concluded that

$$\xi_n = n/3$$

.

This seems to be incorrect.

VY, 2001

$$\frac{\partial S_{n,0}}{\partial r}+\frac{d-1}{r}S_{n,0}=\frac{(n-1)(d-1)}{r}S_{n-2,2}-(n-1)\overline{\delta_r a(\delta_r u)^{n-2}}$$

$$\mathbf{a}=-\nabla p+\nu\nabla^2\mathbf{u}$$

We will assume that

$$S_{n,0}\propto S_{n-2,2}\propto r^{\xi_n}$$

At the large scales, the PDF is a gaussian, so

$$\frac{2+\xi_2}{2}S_{2n,0}=(2n-1)S_{2n-2,2}=(2n-1)S_{2n-2,0}S_{0,2}$$

$$S_{2n,0}=(2n-1)S_{2n-2,0}S_{2,0};\qquad S_{2n-2,2}=S_{2n-2,0}S_{0,2}$$

$$1.35S_{2n,0}\approx(2n-1)S_{2n-2,2}$$

Relations between the moments. (VY 2001; Hill (2002); Kurien-Srenivasan). In the isotropic and homogeneous turbulence the Navier-Stokes equations lead to the following exact relations for the structure functions:

$$\frac{\partial S_{2n,0}}{\partial r}+\frac{d-1}{r}S_{2n,0}=\frac{(2n-1)(d-1)}{r}S_{2n-2,2}+(2n-1)\overline{\delta_r a_x(x)(\delta_r u)^{2n-2}} \tag{1}$$

The closure problem. This equation includes both velocity and Lagrangian acceleration increments and is not closed. 1. Even orders. In the IR, the dissipation term disappears by the symmetry and a $\approx -\nabla p$

If turbulent structures follow the Bernoulli-like equation, then

$$\delta_r\partial_x p = O(\partial_r(\delta_r u)^2)$$

This means that the moments are found from homogeneous differential equations and the scaling exponents are determined by the coefficients leading to anomalous scaling exponents

$$\xi_n = \frac{0.383n}{1 + 0.05n}$$

One free parameter.

0.367- (0.366); 0.696- (0.699); 1.277-(1.279)...2.55-(2.45)

Let us check assumptions (Bernoulli-like pressure, homogeneous equations, disappearance of dissipation.

Gotoh-Nakano.

For the odd orders dissipation terms are very important and the closure problem still exists.

Dissipation anomaly (Polyakov, Duchon, Robert, Eyink, VY, Sreenivasan). If $S_3(y) \approx y^3$ **then** $\partial_y S_3(y) \to 0$**. This is in contradiction with the Kolmogorov's relation. This means that in this limit the velocity field is singular leading to the so called dissipation anomaly.**

$$1.\ \nu \to 0;\ 2.\ y \to 0$$

$$x_{\pm} = \mathbf{x} \pm \mathbf{y/2}$$

$$\frac{1}{2}\frac{\partial u^2}{\partial t} + \frac{1}{2}\mathbf{u} \cdot \nabla u^2 = -\nabla p \cdot \mathbf{u} + \nu \mathbf{u} \cdot \frac{\partial^2 \mathbf{u}}{\partial x_i^2}$$

$$\mathbf{u}(\mathbf{x} + \frac{\mathbf{y}}{\mathbf{2}}) \cdot \mathbf{u}(\mathbf{x} - \frac{\mathbf{y}}{\mathbf{2}}) \equiv \mathbf{u}(+) \cdot \mathbf{u}(-)$$

:

$$\frac{\partial \mathbf{u}(+)\cdot\mathbf{u}(-)}{\partial t} + \mathbf{u}(+)\cdot\frac{\partial}{\partial \mathbf{x}_+}\mathbf{u}(+)\cdot\mathbf{u}(-) + \mathbf{u}(-)\cdot\frac{\partial}{\partial \mathbf{x}_-}\mathbf{u}(-)\cdot\mathbf{u}(+) =$$
$$-\frac{\partial p(+)}{\partial x_{+,i}}u_i(-) - \frac{\partial p(-)}{\partial x_{-,i}}u_i(+) + \nu[\mathbf{u}(-)\cdot\frac{\partial^2}{\partial x_{+,j}^2}\mathbf{u}(+) + \mathbf{u}(+)\cdot\frac{\partial^2}{\partial x_{-,j}^2}\mathbf{u}(-)]$$

In the limit $y \to 0$, so that $\mathbf{x}_\pm \to \mathbf{x}$, this equation gives the energy balance. Following Polyakov, let us consider two identities:

$$\frac{\partial}{\partial y_i}(u_i(+) - u_i(-))(u_j(+) - u_j(-))^2 =$$
$$\frac{1}{2}\frac{\partial}{\partial x_{+,i}}u_i(+)u_j^2(+) + \frac{1}{2}\frac{\partial}{\partial x_{+,i}}u_i(+)u_j^2(-) - \frac{\partial}{\partial x_{+,i}}u_i(+)u_j(+)u_j(-) +$$
$$\frac{1}{2}\frac{\partial}{\partial x_{-,i}}u_i(-)u_j^2(-) + \frac{1}{2}\frac{\partial}{\partial x_{-,i}}u_i(-)u_j^2(+) - \frac{\partial}{\partial x_{-,i}}u_i(+)u_j(-)u_j(+)$$

and

$$u_i(+)\frac{\partial^2}{\partial x_{-,j}^2}u_i(-) + u_i(-)\frac{\partial^2}{\partial x_{-,j}^2}u_i(+) =$$
$$-4(u_i(+) - u_i(-))\frac{\partial^2}{\partial y_j^2}(u_i(+) - u_i(-)) + u_i(+)\frac{\partial^2}{\partial x_{+,j}^2}u_i(+) +$$
$$u_i(-)\frac{\partial^2}{\partial x_{-,j}^2}u_i(-)$$

....

$$-\frac{\partial}{\partial y_i}\delta_y u_i|\delta_{\mathbf{y}}\mathbf{u}|^2+\frac{1}{2}(\frac{\partial}{\partial x_{+,i}}u_i(+)u_j(-)^2+\frac{\partial}{\partial x_{-,i}}u_i(-)u_j(+)^2) = -2\delta_{\mathbf{y}}\mathbf{u}\cdot\delta_{\mathbf{y}}\mathbf{a}$$

This equation is locally exact. We have an estimate:

$$y \to \eta$$

$$\nu \approx \eta\delta_\eta u$$

The dissipative structures are those with the LOCAL $Re = 1$. The characteristic dissipation scale:

$$\eta \approx \nu/\delta_\eta u$$

THE DISSIPATION SCALE IS A RANDOM FIELD AND NOT A NUMBER AS IN K41!

Multifracal theory has similar feature with the dissipation scale depending on codimention h. Here the situation is somewhat different.

Extrapolating the expression for dissipation anomaly to $y \to r$ where r is in the IR, we have:

$$\frac{\partial}{\partial r}\delta_r u|\delta_r \mathbf{u}|^2 \approx \delta_r \mathbf{a} \cdot \delta_r \mathbf{u} = O(1)$$

This is the mechanism of viscosity disappearance from the equations for the IR structure functions. However, we see that the magnitudes of anomalous exponents are strongly influenced by the O(1) viscous terms. Thus, barring some very special singularities, the Euler equation cannot correctly describe anomalous scaling exponents.

Acceleration:

$$a = \frac{du}{dt} \approx \frac{\delta_\eta u}{\tau} \approx (\delta_\eta u)^\eta \approx (\delta_\eta u)^3/\nu$$

$$\frac{\partial S_{2n,0}}{\partial r} + \frac{d-1}{r} S_{2n,0} = \frac{(2n-1)(d-1)}{r} S_{2n-2,2} + (2n-1) Re \overline{(\delta_\eta u)^3 (\delta_r u)^{2n-2}} \quad (2)$$

This equation is valid everywhere including $r \to \eta_{2n}$**. Thus,**

$$\eta_{2n} \propto L \; Re^{\frac{1}{\xi_{2n} - \xi_{2n+1} - 1}}$$

DISSIPATIONS SCALES OF MOMENTS S_{2n} **DECREASE WITH BOTH MOMENT-ORDER** n **AND RE.**

CALCULUS.

$$\frac{\partial u}{\partial x} = \lim_{y\to 0}\frac{u(x+y)-u(x)}{y} \approx \delta_\eta u/\eta$$

This is correct due to analyticity of velocity at $y \to 0$.

$$\overline{(\frac{\partial u}{\partial x})^n} \approx \overline{(\frac{\delta_\eta u}{\eta})^n} \approx \nu^{-n}\overline{(\delta_\eta u)^{2n}} \propto Re^n \eta_{2n}^{\xi_{2n}} \propto Re^{\rho_n}$$

$$\rho_n = n + \frac{\xi_{2n}}{\xi_{2n}-\xi_{2n+1}-1}$$

$$\overline{\mathcal{E}^n} \propto Re^n\overline{(\delta_\eta u)^{4n}} \propto Re^{d_n}$$

$$d_n = n + \frac{\xi_{4n}}{\xi_{4n}-\xi_{4n+1}-1}$$

$$\overline{a^n} = Re^{\alpha_n}$$

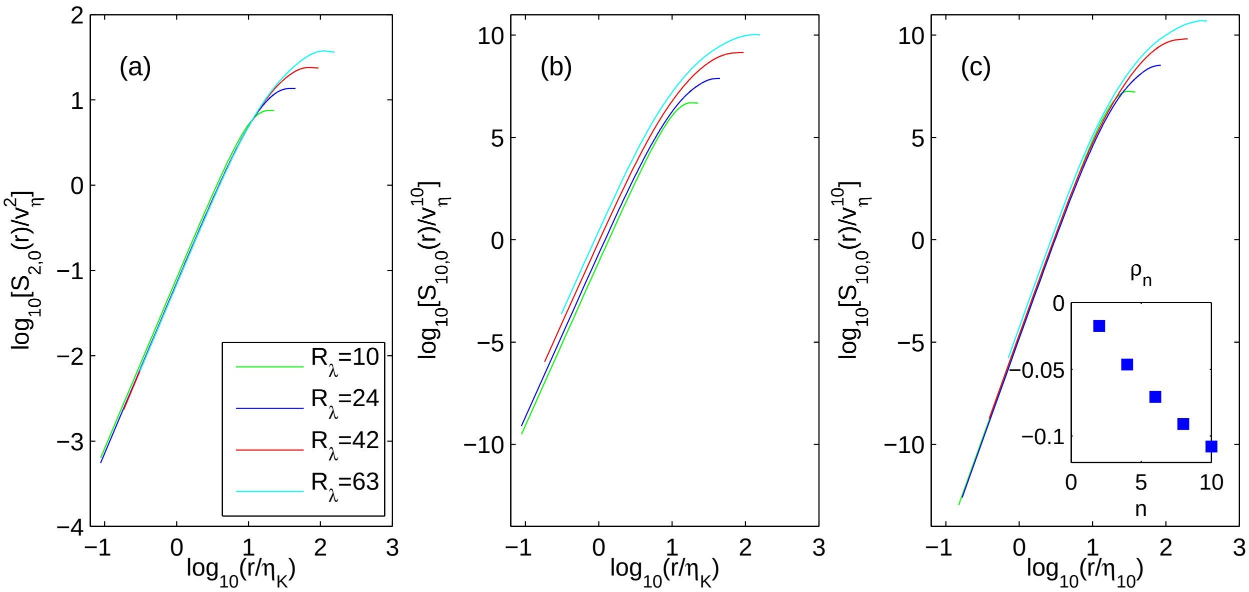
$$\alpha_n = n + \frac{\xi_{3n}}{\xi_{4n} - \xi_{3n+1} - 1}$$

$$< \mathcal{E}^n > \propto \eta_{4n}^{\xi_{4n}}$$

To evaluate moments of velocity derivatives, the following constraint must be satisfies: the resolution must be high enough to resolve at least past of the analytic range, i.e.

$$S_{4n}(\Delta) \propto \Delta^{4n}$$

It is almost never satisfied.

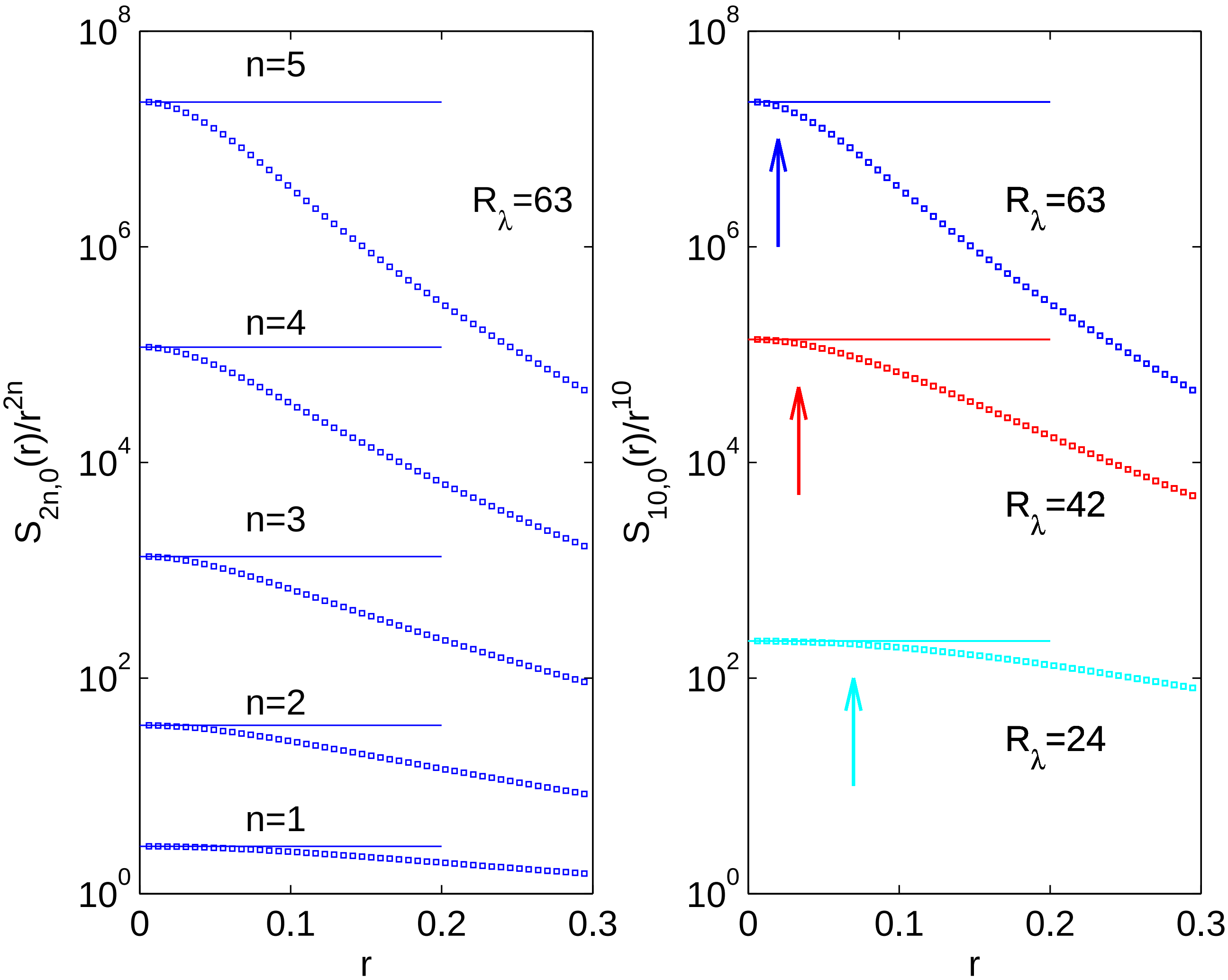


(a)

FIG. 1: Structure functions S_n; a. $n = 2$; $R_\lambda = 10$; 24; 42; 63; b and c S_n for $R_\lambda = 63$. Different rescalings.

Schumacher, Sreenivasan, VY (2006)

1024^3, $R_\lambda = 10$; 24. 42. 63. Trying to resolve analytic ranges.



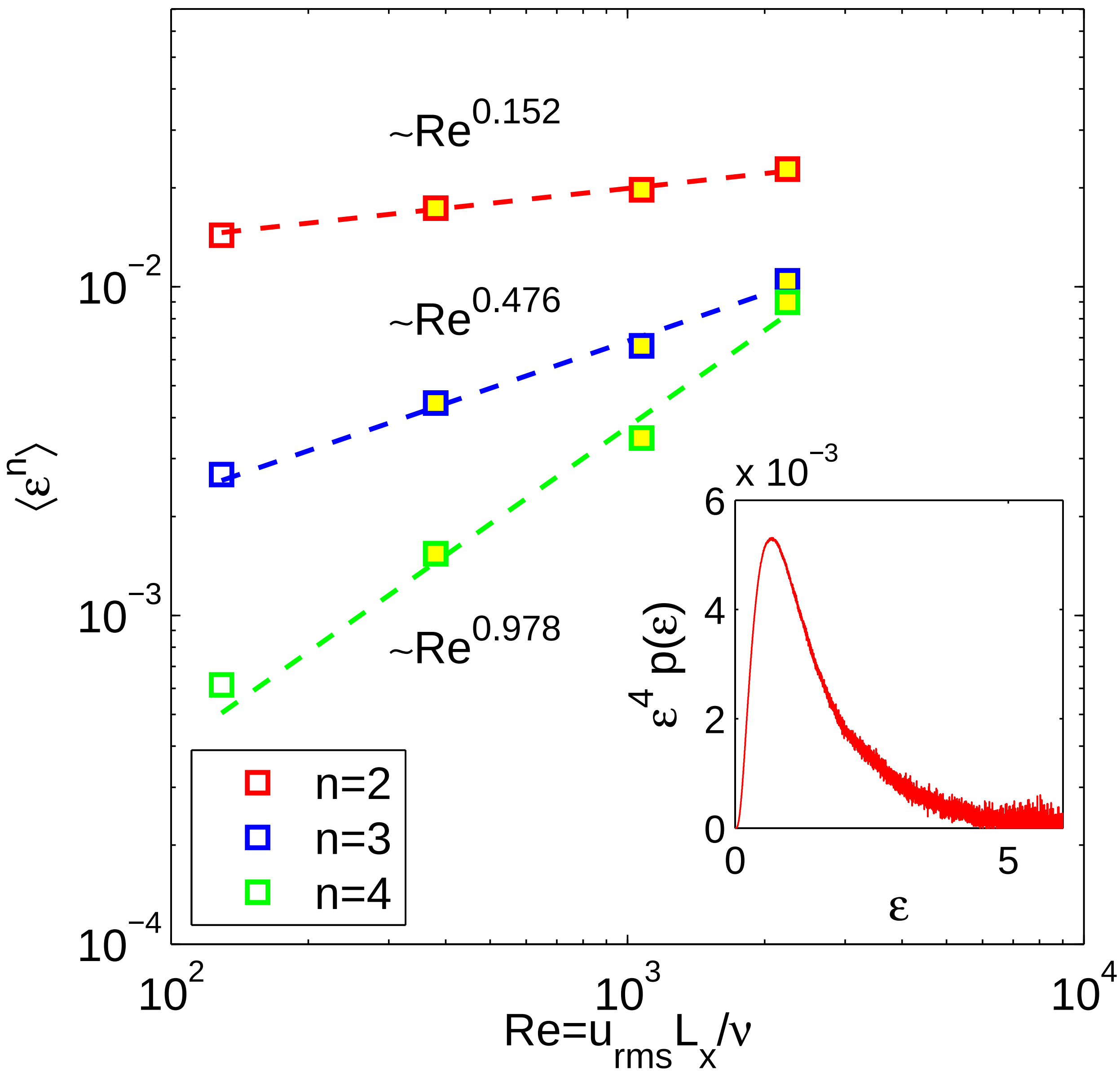
(a)

FIG. 2: **LogLog plots of compensated structure functions:** $S_n(r)/r^n$ **vs** r**; a.** $R_\lambda = 63$ **b.** $S_{10}(r)/r^{10}$**;** $.R_\lambda = 24;\ 42;\ 63;$

Schumacher, Sreenivasan, VY (2006)

1024^3**,** $R_\lambda = 10;\quad 24;\quad 42;\quad 63;$**. Trying to resolve analytic ranges.**

1. At $R_\lambda = 10$ **the dissipation field is gaussian !**

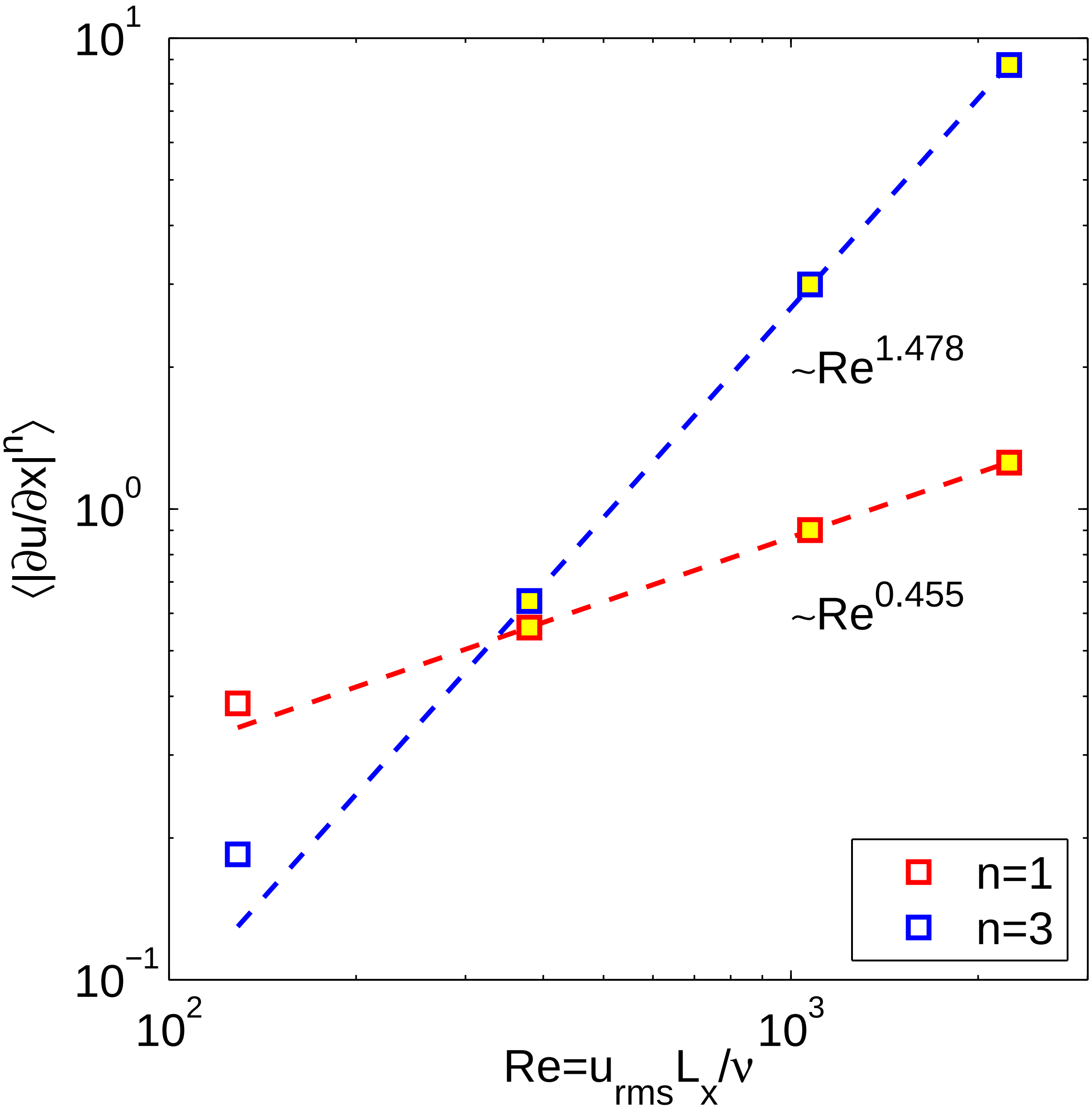


(a)

FIG. 3: **Moments of dissipation rate.**

2.

3. At $R_\lambda = 24$- it is anomalous with asymptotic exponents.



(a)

FIG. 4: **Moments of velocity derivatives.**

The measured numbers are to be compared with theoretical obtained from calculus:

$\rho_1. = 0.455.$ $\rho_3 = 1.478.$ $d_2 = 0.152.$ $d_3 = 0.476.$ $d_4 = 0.978.$

$\rho_1 = 0.465.$ $\rho_3 = 1.548.$ $d_2 = 0.157.$ $d_3 = 0.489.$ $d_4 = 0.944.$

$$\xi_2 = 0.706$$

Fifth moment of the dissipation rate did not converge.

THE EXPONENTS OF THE MOMENTS OF DERIVATIVES, OBTAINED IN THE LOW Re-FLOWS, LACKING EVEN TRACES OF INERTIAL RANGE ARE EXPRESSED IN TERMS OF THE IR EXPONENTS ξ_n DERIVED IN THE LIMIT $Re \to \infty$. THIS MEANS THAT THE IR EXPONENTS ARE PRESCRIBED BY THE MATCHING CONDITIONS ON UV CUT-OFFS.