Mori-Zwanzig models for the Euler equations

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Statement of the problem

Fluids exhibit a very wide range of behaviors.

Navier-Stokes and Euler equations are believed to be good candidates for the description of fluids.

However, their analytical and numerical investigation has proved to be a very challenging problem.

Analytical studies are hampered by the lack of understanding of the geometrical properties of the flow.

Direct numerical simulation is beyond present capabilities due to the very large range of active scales (for flows of practical interest).

What can one do?

1) Dimensional reduction (e.g. LES, RNG, α-model, Spectral vanishing viscosity, Mori-Zwanzig).

2) Construct algorithms that focus on specific aspects of the problem. E.g. construct algorithm that focuses on the point with largest vorticity (joint work with A.J. Chorin and J.B. Bell, LBL).

Properties of the Mori-Zwanzig reduced models for the Euler equations

1) The models come directly from the equations. They do not involve terms added by hand.

2) They are incompressible by construction.

3) The different terms appearing can be computed efficiently (in the case of periodic boundary conditions) by the use of FFT on appropriate arrays. Thus, incorporation in existing codes is straightforward.

4) The terms that effect the drain of energy to the unresolved modes are adaptive and kick in only when there is significant transfer of energy.

Outline

1) The Euler equations and the problem of underresolved computations

2) The Mori-Zwanzig formalism (Zwanzig 1961, Mori 1965, Chorin, Hald & Kupferman 2000)

3) The t-model (analysis and numerical results for the Taylor-Green vortex)

4) Current and future work

The Euler equations and the problem of underresolved computations

$$v_t + v \cdot \nabla v = -\nabla p, \ \nabla \cdot v = 0, \tag{1}$$

where $v(x,t) = (v_1(x_1,x_2,x_3,t), v_2(x_1,x_2,x_3,t), v_3(x_1,x_2,x_3,t))$ is the velocity, p is the pressure and $\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$.

Solve in the periodic box $[0,2\pi] \times [0,2\pi] \times [0,2\pi]$.

The system in (1) is supplemented with the initial condition $v(x,0) = v_0(x)$ which is also periodic and incompressible and $x = (x_1,x_2,x_3)$.

Expand in Fourier series keeping M modes in each direction,

$$v_M(x,t) = \sum_{k \in F \cup G} u_k(t) e^{ikx},$$

where $F \cup G = [-\frac{M}{2}, \frac{M}{2} - 1] \times [-\frac{M}{2}, \frac{M}{2} - 1] \times [-\frac{M}{2}, \frac{M}{2} - 1]$. Also $k = (k_1, k_2, k_3)$ and $u_k(t) = (u_k^1(t), u_k^2(t), u_k^3(t))$.

The equation of motion for the Fourier mode u_k becomes

$$\frac{du_k}{dt} = -i \sum_{\substack{p+q=k \\ p,q \in F \cup G}} k \cdot u_p A_k u_q, \tag{2}$$

where $A_k = I - \frac{kk^T}{|k|^2}$ is the incompressibility projection matrix and I is the 3×3 identity matrix.

The system (2) is supplemented by the initial condition $u_0 = \{u_k(0)\} = \{u_{0k}\},\ k \in F \cup G$, where u_{0k} are the Fourier coefficients of the initial condition $v_0(x)$.

Even if we start from a very smooth initial condition and M is of the order of 10^3 in each direction (the state of the art in massively parallel computers), the solution of the system of ordinary differential equations (2) can create significant activity in the highest modes of our allowed resolution.

Construct reduced model for the modes in $F = [-\frac{N}{2}, \frac{N}{2} - 1] \times [-\frac{N}{2}, \frac{N}{2} - 1] \times [-\frac{N}{2}, \frac{N}{2} - 1]$, where $N < M$.

The Mori-Zwanzig formalism

Suppose we are given an M-dimensional system of ordinary differential equations

$$\frac{du(t)}{dt} = R(u(t)) \tag{3}$$

with initial condition $u(0) = u_0$.

Transform into the linear partial differential equation

$$\rho_t = L\rho, \qquad \rho(u_0, 0) = g(u_0) \tag{4}$$

where $L = \sum_{i=1}^{M} R_i(u_0)\frac{\partial}{\partial u_{0i}}$ and the solution of (4) is given by $\rho(u_0, t) = g(\rho(u_0, t))$. Consider the following initial condition for the PDE

$$g(u_0) = u_{0k} \Rightarrow \rho(u_0, t) = u_k(u_0, t)$$

Rewrite (4) as

$$\frac{\partial}{\partial t} e^{tL} u_{0k} = L e^{tL} u_{0k}$$

Suppose that the vector of initial conditions can be divided as $u_0 = (\hat{u}_0, \tilde{u}_0)$, where $\hat{u}_0$ is the N-dimensional vector of the resolved variables and $\tilde{u}_0$ is the $(M-N)$-dimensional vector of the unresolved variables.

Let P be an orthogonal projection on the space of functions of $\hat{u}_0$ and $Q = I - P$. For a function $h(u_0)$ of all the variables, the projection operator we will use is defined by $P(h(\hat{u}_0, \tilde{u}_0)) = h(\hat{u}_0, 0)$, i.e. it replaces the value of the unresolved variables $\tilde{u}_0$ in any function $h(u_0)$ by zero. Similarly, the initial condition $u_0 = (\hat{u}_0, \tilde{u}_0)$ is replaced by $(\hat{u}_0, 0)$.

1) No joint density on u_0 used in the definition of P. Very expensive (if not impossible) to calculate a joint density analytically or experimentally.

2) Natural choice since the Euler equations can create significant activity in high wavenumbers even when we start with a very smooth initial condition (a few Fourier modes in each direction).

The equation (4) can be rewritten as

$$\frac{\partial}{\partial t} e^{tL} u_{0k} = e^{tL} PLu_{0k} + e^{tQL} QLu_{0k} + \int_0^t e^{(t-s)L} PLe^{sQL} QLu_{0k} ds, \tag{5}$$

for $k = 1, \dots, N$. We have used Dyson's formula (Duhamel's principle)

$$e^{tL} = e^{tQL} + \int_0^t e^{(t-s)L} PLe^{sQL} ds. \tag{6}$$

If we write

$$e^{tQL}QLu_{0k} = w_k,$$

$w_k(u_0, t)$ satisfies the equation

$$\begin{cases} \frac{\partial}{\partial t} w_k(u_0, t) = QLw_k(u_0, t) \\ w_k(u_0, 0) = QLu_{0k} = R_k(u_0) - (PR_k)(\hat{u}_0). \end{cases} \tag{7}$$

If we project (7) we get

$$P\frac{\partial}{\partial t} w_k(u_0, t) = PQLw_k(u_0, t) = 0,$$

since $PQ = 0$. Also for the initial condition

$$Pw_k(u_0, 0) = PQLu_{0k} = 0$$

by the same argument. Thus, the solution of (7) is at all times orthogonal to the range of P. We call (7) the orthogonal dynamics equation.

Since the solutions of the orthogonal dynamics equation remain orthogonal to the range of P, we can project the Mori-Zwanzig equation (5) and find

$$\frac{\partial}{\partial t} P e^{tL} u_{0k} = P e^{tL} P L u_{0k} + P \int_0^t e^{(t-s)L} P L e^{sQL} Q L u_{0k} ds. \qquad (8)$$

Use (8) as the starting point of approximations for the evolution of the quantity $Pe^{tL}u_{0k}$ for $k = 1, \ldots, N$ (note that the equation (8) involves the orthogonal dynamics operator e^{tQL}).

Construct reduced models based on physical and numerical observations.

These models come directly from the original equations and the terms appearing in them are not introduced by hand.

The t-model

We set

$$R_k(u) = -i \sum_{\substack{p+q=k \\ p,q \in F \cup G}} k \cdot u_p A_k u_q$$

and we have

$$\frac{du_k(t)}{dt} = R_k(u(t)) \tag{9}$$

for $k \in F \cup G$. In the above, $u_k(t) = e^{tL} u_{0k}$ and $L = \sum_{k \in F \cup G} R_k(u_0)$. The system (9) is supplemented by the initial condition $u_0 = (\hat{u}_0, \tilde{u}_0) = (\hat{u}_0, 0)$.

Expand the memory integrand $e^{(t-s)L}PLe^{sQL}$ around $s = 0$ and retain only the zeroth order term.

$$\frac{\partial}{\partial t}Pe^{tL}u_{0k} = Pe^{tL}\hat{R}_k(\hat{u}_0) + tPe^{tL}Z_k^0(\hat{u}_0),$$

where

$$Z_k^0(\hat{u}_0) = PLQLu_{0k} =$$
$$-i\Bigg(\sum_{\substack{p+q=k \\ p\in G,\, q\in F}} k \cdot \hat{R}_p(\hat{u}_0)A_k u_{0q} + \sum_{\substack{p+q=k \\ p\in F,\, q\in G}} k \cdot u_{0p}A_k\hat{R}_q(\hat{u}_0)\Bigg)$$

and

$$\hat{R}_k(\hat{u}_0) = R_k(\hat{u}_0, 0) = -i \sum_{\substack{p+q=k \\ p,q\in F}} k \cdot u_{0p}A_k u_{0q}.$$

The equations are *not* closed in the quantities $Pe^{tL}u_{0k}$. Commute projection with nonlinear functions and obtain a closed system.

This is not the usual mean-field approximation because we account for the fluctuations by keeping the memory term.

$$\frac{\partial}{\partial t}Pe^{tL}u_{0k} = \hat{R}_k(Pe^{tL}\hat{u}_0) + tZ_k^0(Pe^{tL}\hat{u}_0)$$

How accurate is such an approximation?

$$\int_0^t e^{(t-s)L} PLe^{sQL} QLu_{0k} ds - te^{tL} PLQLu_{0k} = \int_0^t [e^{(t-s)L} PLe^{sQL} - e^{tL} PL] QLu_{0k} ds.$$

Adding and subtracting equal quantities we find

$$e^{(t-s)L} PLe^{sQL} = e^{tL} PL + e^{tL} [e^{-sL} PLe^{sQL} - PL],$$

and a Taylor series around $s = 0$ gives

$$e^{-sL} PLe^{sQL} - PL = (I - sL + \ldots) PL (I + sQL + \ldots) - PL = O(s).$$

This implies

$$\int_0^t e^{(t-s)L} PLe^{sQL} QLu_{0k} ds = te^{tL} PLQLu_{0k} + O(t^2).$$

If we retain only the leading term, we do not keep any information about the time evolution of the integrand, which in turn means no information about the coupling of the resolved components to the evolution of the orthogonal dynamics.

Such an approximation is expected to be appropriate in cases where the memory term integrand is slowly decaying, so that information about its initial value is sufficient to make predictions.

In essence, we approximate the response of the unresolved modes to the "field" of the resolved modes by a constant (analogous to the zero order sum rule).

Remarks about the t-model for the Euler equations

1) Motivation: Turbulent flow develops vortical structures that exhibit long temporal correlations (e.g. Alder and Wainwright 1967,1970). Temporal correlations appear as memory integrands in the Mori-Zwanzig formalism, thus, a very long memory approximation is natural.

2) Such a modeling approach is different from the usual modeling approximation of very short or no memory at all.

Theorems (due to O.H. Hald)

1) If the solution of the full system is smooth, the solution of the t-model converges to this solution in the limit of infinite number of modes.

2) For the evolution of the energy of the resolved components $E = \frac{1}{2}\sum_{k\in F}|Pe^{tL}u_{0k}|^2$ we have

$$\frac{dE}{dt} = -t\sum_{p\in G}|\hat{R}_p(Pe^{tL}\hat{u}_0)|^2$$

3) The above result can be generalized to the t-model of *any* system of ODEs that conserves the L^2 norm of the solution.

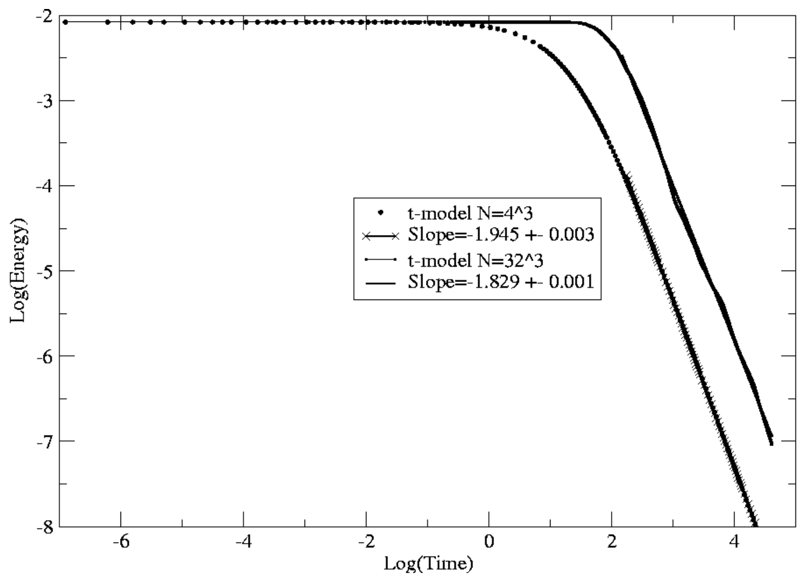
Numerical results of the t-model for the Taylor-Green vortex

$$v_1(x,0) = \sin(x_1)\cos(x_2)\cos(x_3),$$
$$v_2(x,0) = -\cos(x_1)\sin(x_2)\cos(x_3),$$
$$v_3(x,0) = 0$$

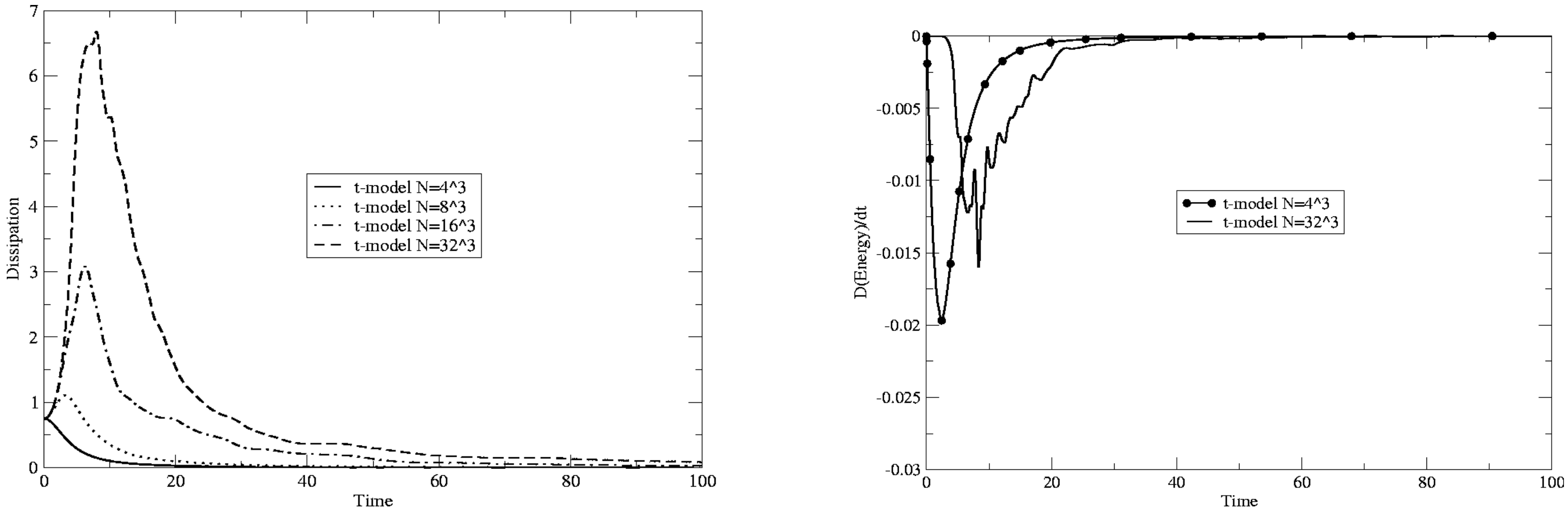
All the expressions in the RHS can be computed using FFTs of appropriate arrays.

The pseudospectral calculations for the t-model term are dealiased by construction.

Runge-Kutta-Fehlberg method with tolerance set to 10^{-15}.



Energy evolution for the t-model for different resolutions.



(a) Evolution of dissipation and (b) energy decay rate for the t-model for different resolutions.

Current and future work

1) Reduced model constructed directly from the Euler equations without the introduction of terms by hand

2) Construct a collection of models based on the Taylor expansion of the orthogonal dynamics operator e^{tQL}, *not* of the whole memory integrand as in the t-model

3) Use terms in the Taylor expansion to construct more elaborate approximations e.g. Padé approximants

4) Parallel implementation

More details can be found in the papers:

Chorin, Hald, Shvets, S. "Long memory Mori-Zwanzig models for the Euler equations", preprint UCB (2006)

Chorin, S. "Problem reduction, renormalization and memory" Comm. App. Math. Comp. Sci. 1 (2005) 1-27

S. "Higher order Mori-Zwanzig models for the Euler equations" (2006) math.NA/0607108 (submitted to Multi. Mod. Sim.)

The papers can be downloaded at http://math.lbl.gov/~stinis/

Models based on the Taylor expansion of the orthogonal dynamics operator. Part I: Zeroth, first and second order models

Proceed by expanding the orthogonal dynamics operator around $s = 0$. Depending on how many terms we keep $(1, 2, 3, \ldots)$, we obtain zeroth, first, second, ... order approximations respectively,

$$e^{sQL} = I + sQL + \frac{s^2}{2}QLQL + O(s^3), \tag{10}$$

$$PLe^{sQL} = PL + sPLQL + \frac{s^2}{2}PLQLQL + O(s^3). \tag{11}$$

Every term in the expansion has one more factor of QL than the previous term.

Zeroth order model (Cubic in $\hat{u}_0$)

$$\frac{\partial}{\partial t} P e^{tL} u_{0k} = P e^{tL} \hat{R}_k(\hat{u}_0) + P \int_0^t e^{(t-s)L} Z_k^0(\hat{u}_0) ds, ,$$

where

$$Z_k^0(\hat{u}_0) = PLQLu_{0k} =$$

$$-i \Bigg(\sum_{\substack{p+q=k \\ p \in G,\, q \in F}} k \cdot \hat{R}_p(\hat{u}_0) A_k u_{0q} + \sum_{\substack{p+q=k \\ p \in F,\, q \in G}} k \cdot u_{0p} A_k \hat{R}_q(\hat{u}_0) \Bigg)$$

and

$$\hat{R}_k(\hat{u}_0) = R_k(\hat{u}_0, 0) = -i \sum_{\substack{p+q=k \\ p,q \in F}} k \cdot u_{0p} A_k u_{0q}.$$

First order model (Quartic in $\hat{u}_0$)

$$\frac{\partial}{\partial t} P e^{tL} u_{0k} = \hat{R}_k(P e^{tL} \hat{u}_0) + \int_0^t Z_k^0(P e^{(t-s)L} \hat{u}_0) ds$$
$$+ \int_0^t s Z_k^1(P e^{(t-s)L} \hat{u}_0) ds,$$

where

$$Z_k^1(\hat{u}_0) = PLQLQLu_{0k} =$$
$$-i\Bigg(\sum_{\substack{p+q=k \\ p\in F\cup G,\, q\in G}} k \cdot \hat{R}_p(\hat{u}_0) A_k \hat{R}_q(\hat{u}_0) + \sum_{\substack{p+q=k \\ p\in G,\, q\in F\cup G}} k \cdot \hat{R}_p(\hat{u}_0) A_k \hat{R}_q(\hat{u}_0) +$$
$$\sum_{\substack{p+q=k \\ p\in G,\, q\in F}} k \cdot Z_p^0(\hat{u}_0) A_k u_{0q} + \sum_{\substack{p+q=k \\ p\in F,\, q\in G}} k \cdot u_{0p} A_k Z_q^0(\hat{u}_0) \Bigg)$$

Second order model (Quintic in $\widehat{u}_0$)

$$\frac{\partial}{\partial t} Pe^{tL} u_{0k} = \widehat{R}_k(Pe^{tL}\widehat{u}_0) + \int_0^t Z_k^0(Pe^{(t-s)L}\widehat{u}_0)ds$$

$$+ \int_0^t sZ_k^1(Pe^{(t-s)L}\widehat{u}_0)ds + \int_0^t \frac{s^2}{2} Z_k^2(Pe^{(t-s)L}\widehat{u}_0)ds,$$

$$
\begin{aligned}
&Z_k^2(\hat{u}_0) = PLQLQLQLu_{0k} = \\
&-i\Bigg(\sum_{\substack{p+q=k\\ p\in F\cup G,\ q\in G}} k\cdot Z_p^0(\hat{u}_0)A_k\hat{R}_q(\hat{u}_0) + \sum_{\substack{p+q=k\\ p\in G,\ q\in F\cup G}} k\cdot \hat{R}_p(\hat{u}_0)A_kZ_q^0(\hat{u}_0) + \\
&\sum_{\substack{p+q=k\\ p\in F\cup G,\ q\in G}} k\cdot B_p(\hat{u}_0)A_k\hat{R}_q(\hat{u}_0) + \sum_{\substack{p+q=k\\ p\in G,\ q\in F\cup G}} k\cdot \hat{R}_p(\hat{u}_0)A_kB_q(\hat{u}_0) + \\
&\sum_{\substack{p+q=k\\ p\in G,\ q\in F}} k\cdot Z_p^0(\hat{u}_0)A_k\hat{R}_q(\hat{u}_0) + \sum_{\substack{p+q=k\\ p\in F,\ q\in G}} k\cdot \hat{R}_p(\hat{u}_0)A_kZ_q^0(\hat{u}_0) + \\
&\sum_{\substack{p+q=k\\ p\in F\cup G,\ q\in G}} k\cdot Z_p^0(\hat{u}_0)A_k\hat{R}_q(\hat{u}_0) + \sum_{\substack{p+q=k\\ p\in G,\ q\in F\cup G}} k\cdot \hat{R}_p(\hat{u}_0)A_kZ_q^0(\hat{u}_0) + \\
&\sum_{\substack{p+q=k\\ p\in G,\ q\in F\cup G}} k\cdot Z_p^0(\hat{u}_0)A_k\hat{R}_q(\hat{u}_0) + \sum_{\substack{p+q=k\\ p\in F\cup G,\ q\in G}} k\cdot \hat{R}_p(\hat{u}_0)A_kZ_q^0(\hat{u}_0) + \\
&\sum_{\substack{p+q=k\\ p\in G,\ q\in F}} k\cdot Z_p^1(\hat{u}_0)A_ku_{0q} + \sum_{\substack{p+q=k\\ p\in F,\ q\in G}} k\cdot u_{0p}A_kZ_q^1(\hat{u}_0) \Bigg)
\end{aligned}
$$

where

$$B_k(\widehat{u}_0) = -i \sum_{\substack{p+q=k \\ p,q \in F}} k \cdot \widehat{R}_p(\widehat{u}_0) A_k u_{0q}.$$

1) All models are incompressible by construction due to the incompressibility projection operator A_k.

2) All models of order 1 and up involve convolution type integrals. These can be transformed into sums of ordinary integrals by a simple change of variables $s' = t - s$. Ordinary integrals can be evaluated very efficiently by adding at each step the contribution to the integral from this step.

Models based on the Taylor expansion of the orthogonal dynamics operator. Part II: Higher order models

1) The form of the nonlinearity,

2)The specific form of the projection,

3)The general property of any projection $PQ = P(I-P) = 0$.

The nth order term involves powers of $n+3$ in the Fourier modes. It involves expressions of the form $\sum_{\substack{p+q=k \\ p\in\Lambda,\ q\in\Theta}} k \cdot H_p A_k C_q$, where H_p is of order m and C_q is of order l with $n+3 = m+l$.

The nth order term has the form $Z^n(\hat{u}_0) = PL(QL)^n QLu_{0k}$, i.e. n applications of the operator QL and then application of the operator PL.

We have $PL(QL)^n QLu_{0k} = PLQL(QL)^{n-1} QLu_{0k}$.

The part $(QL)^{n-1} QLu_{0k}$ is common with the $n-1$st term expression $Z^{n-1}(\hat{u}_0) = PL(QL)^{n-1} QLu_{0k}$.

When we act on $(QL)^{n-1} QLu_{0k}$ with the extra factor $QL = L - PL$, we get $QL(QL)^{n-1} QLu_{0k} = L(QL)^{n-1} QLu_{0k} - Z^{n-1}(\hat{u}_0)$.

The expression for $QL(QL)^{n-1}QLu_{0k}$ contains 3 types of terms:

i) Terms that could not appear in Z^{n-1} because of the special property of the projection which sets to zero expressions linear in u_{0k} for $k \in G$.

ii) Terms that could not appear in Z^{n-1} due to the general property of any projection that $PQ = 0$.

iii) Terms of the form $h(u_0) - (Ph)(\hat{u}_0)$, where $(Ph)(\hat{u}_0)$ is any expression appearing in the term Z^{n-1}.

We can assemble the expressions appearing in $QL(QL)^{n-1}QLu_{0k}$ into three groups according to i),ii) and iii). Then we can apply the operator PL once and we are done.

1) Write down the expression $(QL)^{n-1}QLu_{0k}$.

2) Apply the operator QL and assemble the terms in the expression $QL(QL)^{n-1}QLu_{0k}$ in 3 groups: i) Terms that could not appear in Z^{n-1} because of the special property of the projection which sets to zero expressions linear in u_{0k} for $k \in G$; ii) Terms that could not appear in Z^{n-1} due to the general property of any projection that $PQ = 0$ and iii) Terms of the form $h(u_0) - (Ph)(\hat{u}_0)$, where $(Ph)(\hat{u}_0)$ is any expression appearing in the term Z^{n-1}.

3) Apply the operator PL to the type i) terms. An $m+1$ term in the expression $QL(QL)^{n-1}QLu_{0k}$ arose from an $(m-1)+1$ term and will give rise to an $m+2$ term. The symmetric term $2+m$ should also appear (the symmetric terms appear due to the rule of differentiating a product.)

4) Apply the operator PL to the type ii) terms. An $m+l$ term in the expression $QL(QL)^{n-1}QLu_{0k}$ arose from an $m+(l-1)$ term where the m term is of the form $(Qh)(u_0)$ for some function $h(u_0)$. It will give rise to an $(m+1)+l$ term. The symmetric $l+(m+1)$ term should also appear.

5) Apply the operator PL to the type iii) terms. There are two cases: a) An $(n-2)+1$ term will give rise to an $(n-1)+1$ term with the $n-1$st part being equal to Z^{n-1}. The symmetric term $1+(n-1)$ should also appear; b) An $m+l$ term with $l \neq 1$ will give rise to an $(m+1)+l$ and an $m+(l+1)$ term. The symmetric terms $l+(m+1)$ and $(l+1)+m$ should also appear.

6) Make sure that in the final expressions *all* possible decompositions of $n+3$ into sums of two positive integers appear. All the expressions for the nth term in the series should be $n+3$ powers of Fourier modes.

7) As a last resort, forget about the rules and proceed with straightforward differentiation.

Numerical results for the Taylor-Green vortex

The models need to have the range of integration for the memory term reduced from $[0, t]$ to $[t_0, t]$, otherwise the calculation becomes unstable.

A Taylor series around the current instant cannot be expected to be accurate for long times in the past and this is the reason for the need to truncate the memory term's range of integration.

There is no tuning needed. The results become better, the longer the range of integration, until the value of the range that leads to instability. Thus, trial and error is needed not to fit the results to some prescribed curve, but just to find when does the calculation becomes unstable.

Let us assume that we want to calculate the integral

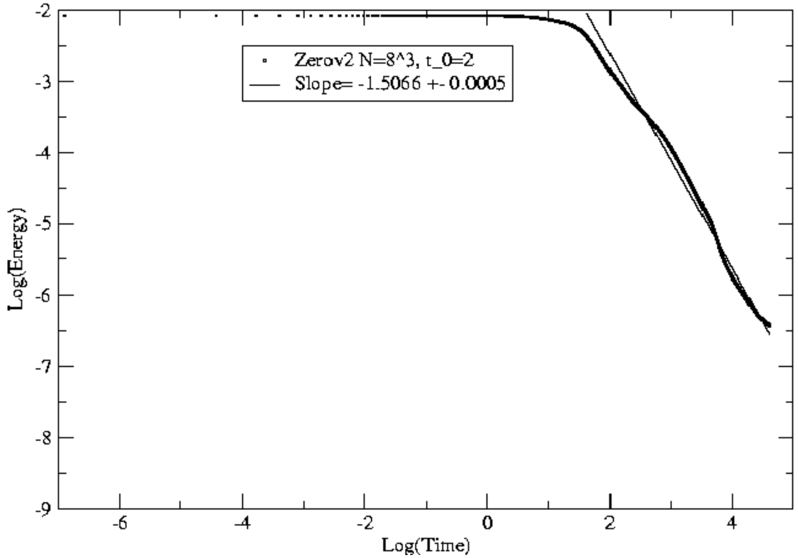
$$\int_{t_k+\Delta t-t_0}^{t_k+\Delta t} f(s)ds,$$

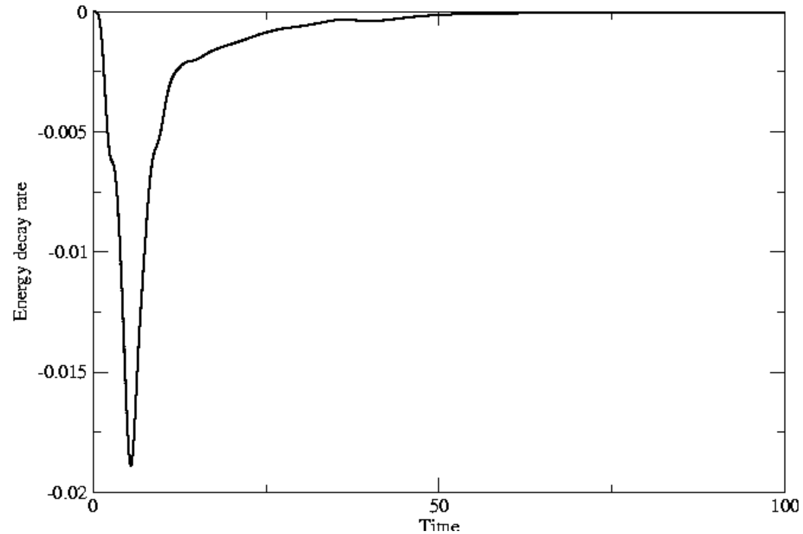
where $f(s)$ is any of the integrands appearing in the different order models' memory terms. Decompose the integral as

$$\int_{t_k-t_0}^{t_k} f(s)ds + \int_{t_k}^{t_k+\Delta t} f(s)ds - \int_{t_k-t_0}^{t_k+\Delta t-t_0} f(s)ds$$

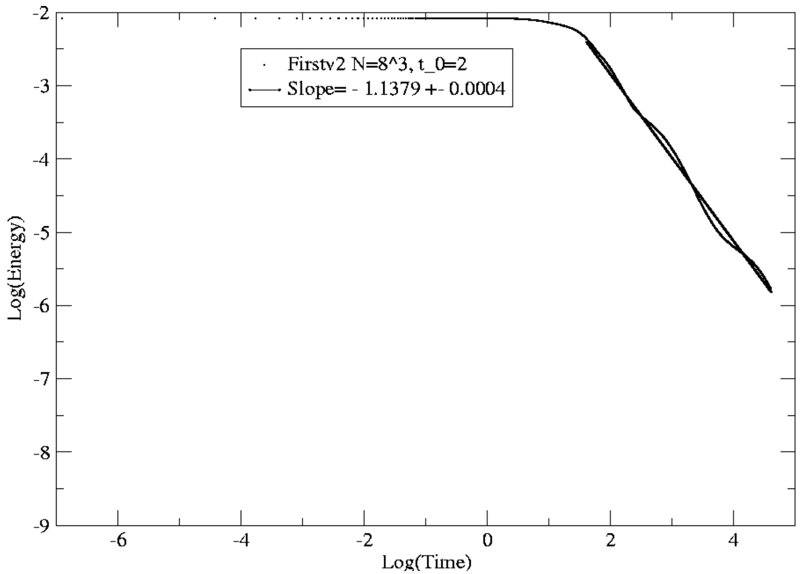
We need to keep an array of length $[t_0/\Delta t]$, where [] stands for integer part. This array needs to be updated at the end of every step so that it always keeps the values of the integrand for the last $[t_0/\Delta t]$ steps.

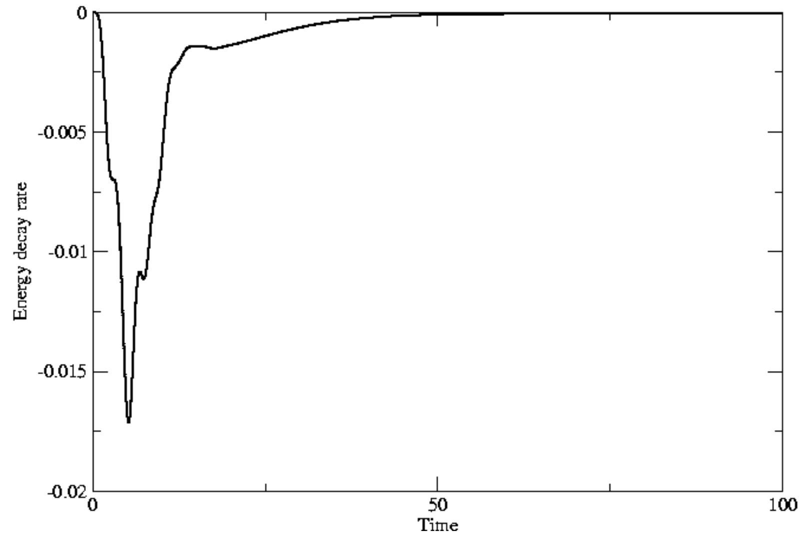
Use modified Euler method with the trapezoidal rule for the evaluation of the integrals and $\Delta t = 10^{-3}$.



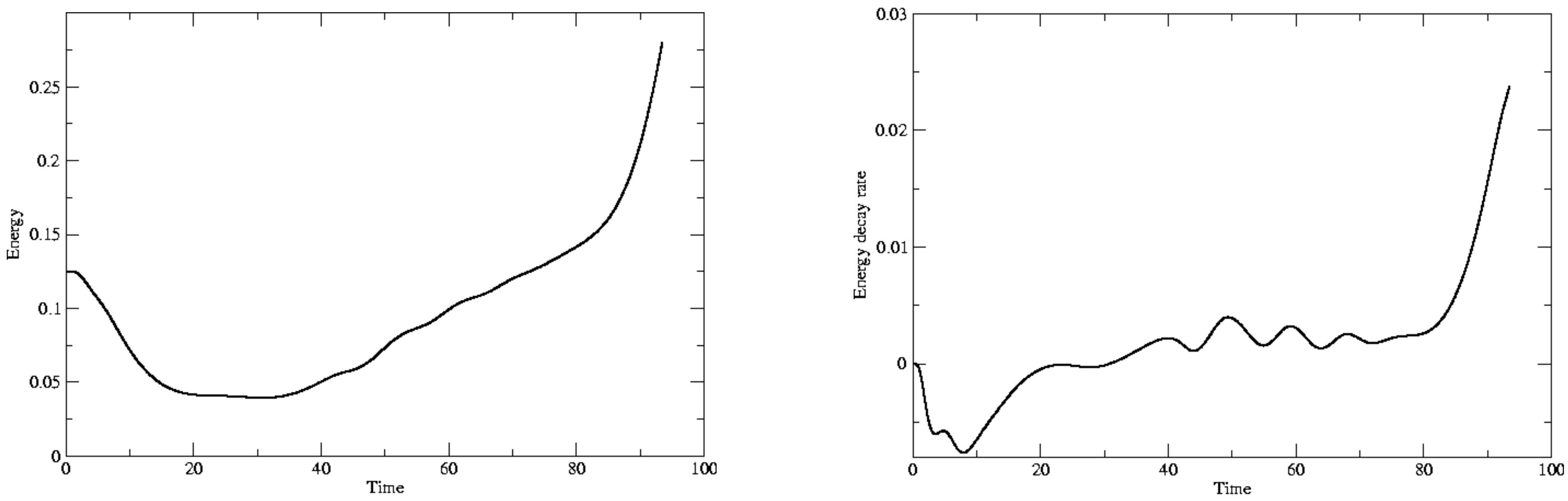


(a) Energy evolution for the zeroth order model with $N = 8^3$ modes and $t_0 = 1$. (b) Evolution of the energy decay rate.

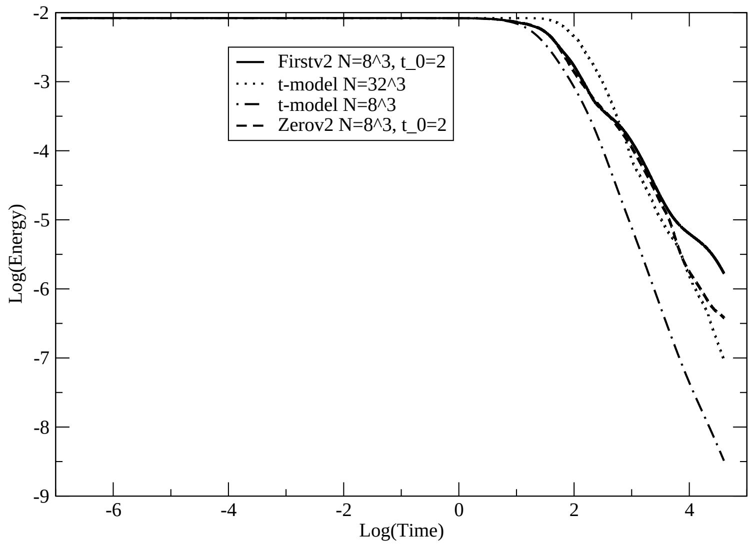




(a) Energy evolution for the first order model with $N = 8^3$ modes and $t_0 = 1$. (b) Evolution of the energy decay rate.



(a) Energy evolution for the second order model with $N = 8^3$ modes and $t_0 = 0.5$. (b) Evolution of the energy decay rate.



(a) Comparison of the energy evolution for the zeroth and first order models and the t-model with two different resolutions.