Growth and Decay of Coherent Structures in Disordered Wave Fields

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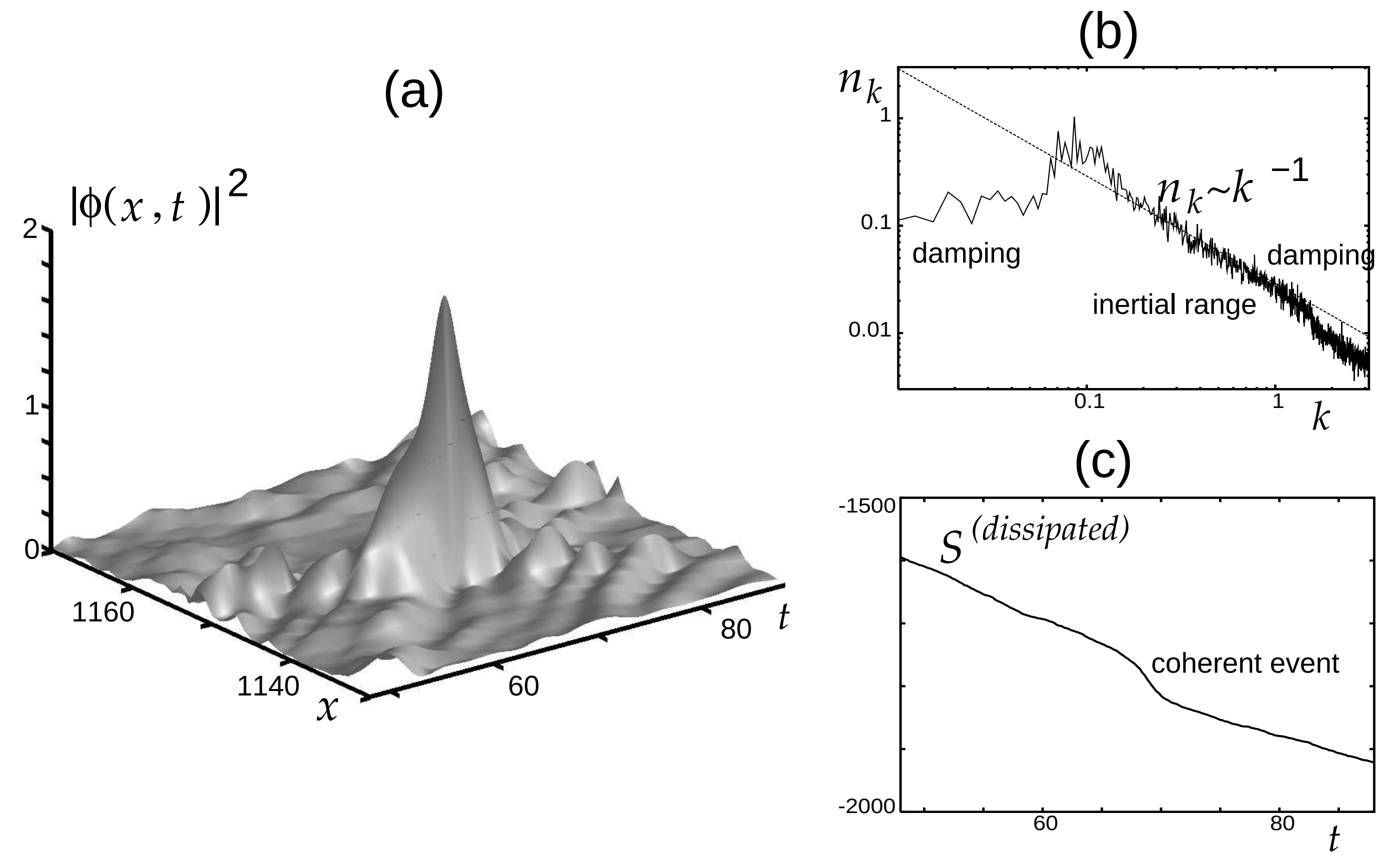
1. Coherent structures and low-amplitude disordered waves

2. Fluxes in amplitude space



(a) A coherent structure emerges spontaneously from a weakly turbulent wave field of the MMT equation. (b) Kolmogorov-Zakharov spectrum. (c) Increased entropy production by the coherent structure.

Equation of motion in Fourier-modes

$$i\dot{a}_k = \frac{\partial E}{\partial a_k^*}$$

Hamiltonian

$$\begin{aligned} E &= E_2 + E_4 + \ldots \\ &= \sum \omega_k |a_k|^2 \\ &+ \sum T_{k_1,k_2,k_3,k_4} a_{k_1} a_{k_2} a_{k_3}^* a_{k_4}^* \delta_{k_1+k_2,k_3+k_4} \\ &+ \ldots \end{aligned}$$

Conserved wave-action

$$A = \sum |a_k|^2$$

Examples

Nonlinear Schrödinger equation

$\omega = k^2$, $T_{k_1,k_2,k_3,k_4} = T$

$$i\dot{\phi} + \frac{\partial^2}{\partial x^2}\phi \pm \phi|\phi|^2 = 0$$

Discrete Nonlinear Schrödinger equation

$$i\dot{\phi}_n + \phi_{n+1} + \phi_{n-1} - 2\phi_n \pm \phi_n|\phi_n|^2 = 0$$

Majda-McLaughlin-Tabak equation

$\omega = |k|^{1/2}$, $T_{k_1,k_2,k_3,k_4} = T$

$$i\dot{\phi} + |\frac{\partial^{1/2}}{\partial x^{1/2}}|\phi \pm \phi|\phi|^2 = 0$$

Mixed state $a_k = c_k + d_k$

- coherent waves c_k
  for coherent structure
- random waves d_k
  -Gaussian amplitude statistics
  -in wavenumber space Rayleigh-Jeans or
  Kolmogorov-Zakharov distributed
- small amplitudes $< |a_k|^2 > \ll 1$

Wave action $A = A^{(c)} + A^{(d)}$

$$A^{(c)} = \sum_{\mathbf{k}} A_{\mathbf{k}}^{(c)} \quad = \quad \sum_{\mathbf{k}} < c_{\mathbf{k}} c_{\mathbf{k}}^* >$$

$$A^{(d)} = \sum_{\mathbf{k}} A_{\mathbf{k}}^{(d)} \quad = \quad \sum_{\mathbf{k}} < d_{\mathbf{k}} d_{\mathbf{k}}^* >$$

Energy $E \approx E_2^{(c)} + E_4^{(c)} + E_2^{(d)}$

$$E_2^{(c)} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} A_{\mathbf{k}}^{(c)}$$

$$E_4^{(c)} = \sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3+\mathbf{k}_4} T_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4} c_{\mathbf{k}_1} c_{\mathbf{k}_2} c_{\mathbf{k}_3}^* c_{\mathbf{k}_4}^*$$

$$E_2^{(d)} \approx \sum_{\mathbf{k}} \omega_{\mathbf{k}} A_{\mathbf{k}}^{(d)}$$

Wave entropy

$$S = \sum_{\mathbf{k}} \ln A_{\mathbf{k}}^{(d)}$$

Equilibrium with vanishing coherent waves $c_{\mathbf{k}} = 0$

Wave entropy

$$S = \sum_{\mathbf{k}} \ln A_{\mathbf{k}}^{(d)}$$

extremal for given $A^{(d)}$ and $E^{(d)}$

Rayleigh-Jeans distribution

$$A_{\mathbf{k}}^{(d)} = (\beta(\omega_{\mathbf{k}} - \mu))^{-1} > 0$$

$$\partial S(E^{(d)}, A^{(d)})/\partial A^{(d)} = -\beta\mu > 0$$

$\longrightarrow$ entropy grows with wave action

$$\partial S(E^{(d)}, A^{(d)})/\partial E^{(d)} = \beta > 0$$

$\longrightarrow$ entropy grows with energy

Coherent structure with extremal energy per wave action

$$dE^{(c)} - \Omega dA^{(c)} = 0$$

$$i\dot{c}_k = \partial E^{(c)}/\partial c_k^* = \Omega c_k$$

one phase frequency

$$\Omega = dE^{(c)}/dA^{(c)}$$

for all modes

$$c_k \sim \exp(-i\Omega t)$$

Rayleigh-Jeans waves plus coherent structure:

Growth of coherent structure Transfer of wave action

$$\begin{aligned} A^{(d)} &\rightarrow A^{(d)} - \Delta A \\ A^{(c)} &\rightarrow A^{(c)} + \Delta A \end{aligned}$$

Transfer of energy

$$\begin{aligned} E^{(d)} &\rightarrow E^{(d)} + \Delta E \\ E^{(c)} &\rightarrow E^{(c)} - \Delta E \end{aligned}$$

Increase of wave entropy

$$\begin{aligned} \Delta S &= \sum_k \Delta A_k^{(d)} / A_k^{(d)} \\ &= \beta(\mu - \Omega)\Delta A \\ &> 0 \end{aligned}$$

A coherent structure with a phase frequency

$$\Omega = \frac{dE^{(c)}}{dA^{(c)}}$$

surrounded by Rayleigh-Jeans distributed waves

$$\boxed{A_k = \frac{T}{\omega_k - \mu}}$$

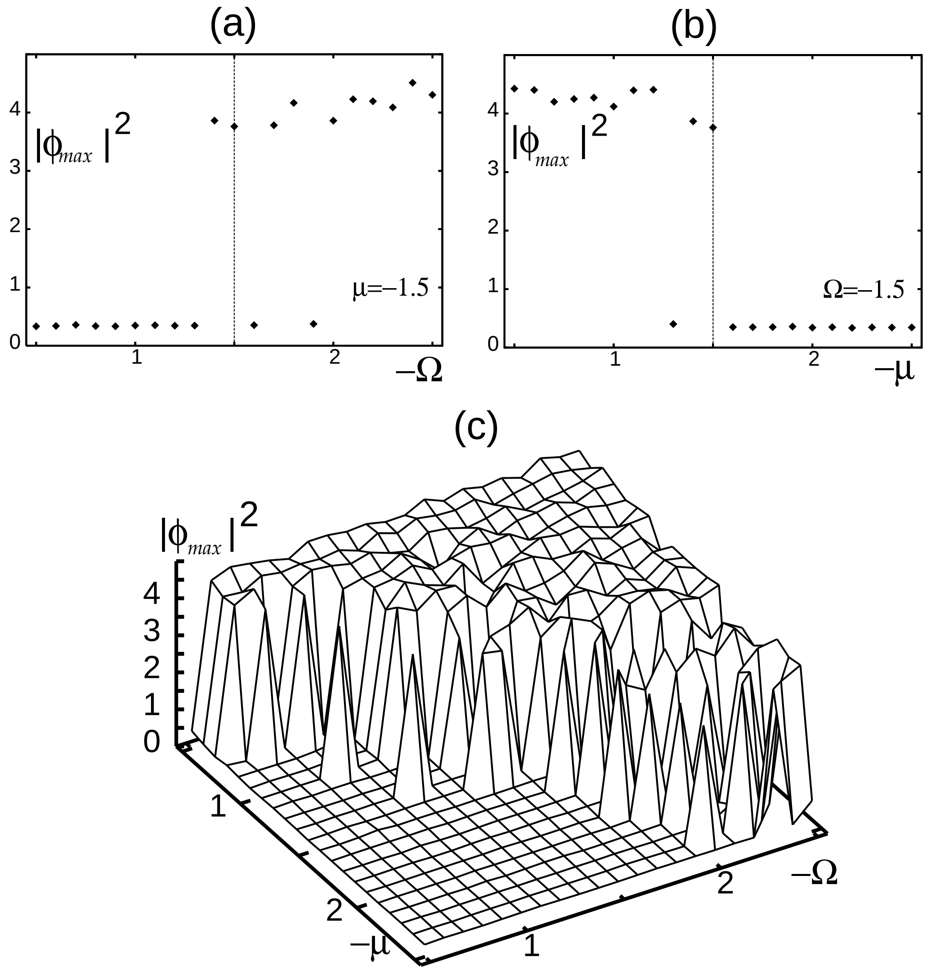
grows if

$$\boxed{-\Omega > -\mu}$$

Discrete nonlinear Schrödinger equation

$$i\dot{\phi}_n + \phi_{n+1} + \phi_{n-1} - 2\phi_n + \phi_n|\phi_n|^2 = 0$$

Initial condition: Discrete breather plus Rayleigh-Jeans waves



Kolmogorov-Zakharov ($const \geq 1$) distributed waves

$$\boxed{A_k \sim \omega_k^{-const}}$$

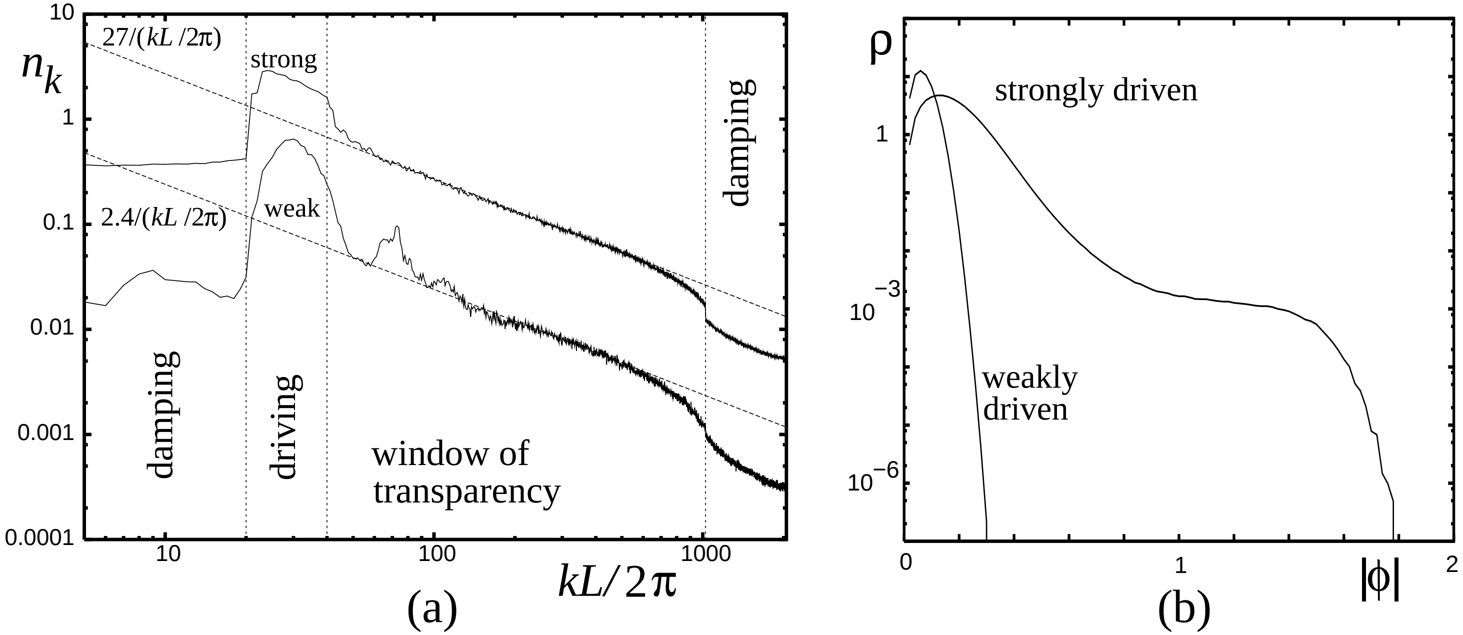
are unstable against coherent structures with

$$\boxed{\frac{\Delta E^{(c)}}{\Delta A^{(c)}} \leq 0}$$

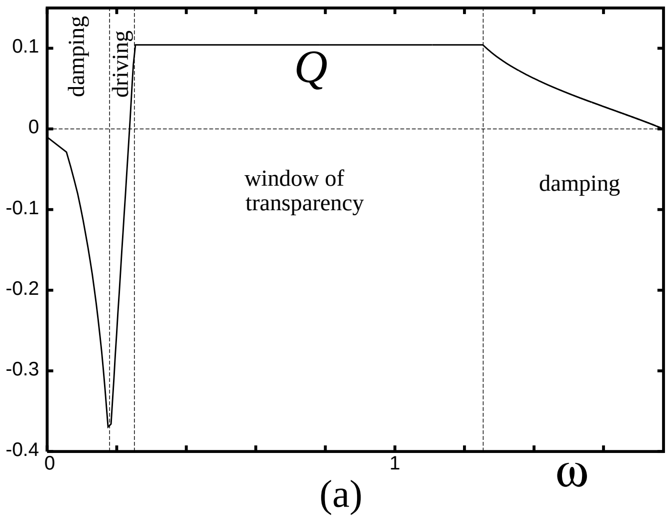
Majda-McLaughlin-Tabak-equation

B.R.,L.Biven, Physica D 204, 188 (2005)

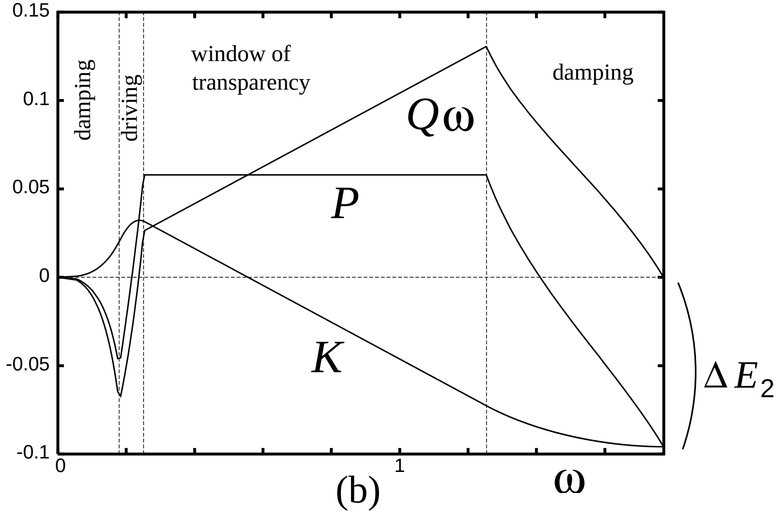
$$i\dot{\phi} + |\frac{\partial^{1/2}}{\partial x^{1/2}}|\phi - \phi|\phi|^2 = \mathcal{F} + \mathcal{D}$$



Flux Q of wave action and P energy E_2 in wavenumber space

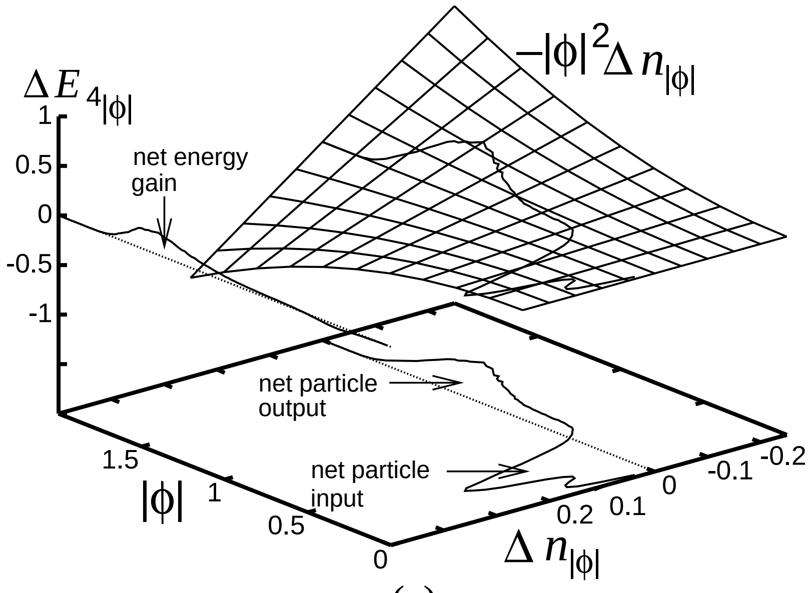


(a)

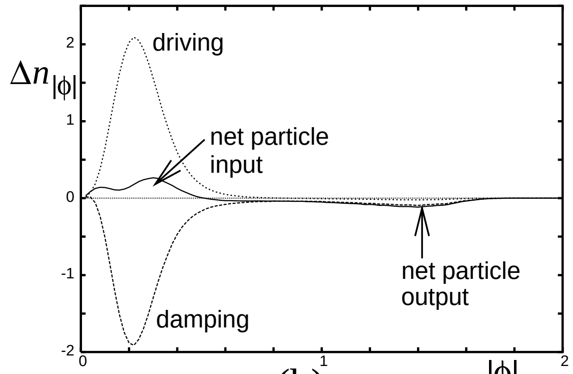


(b)

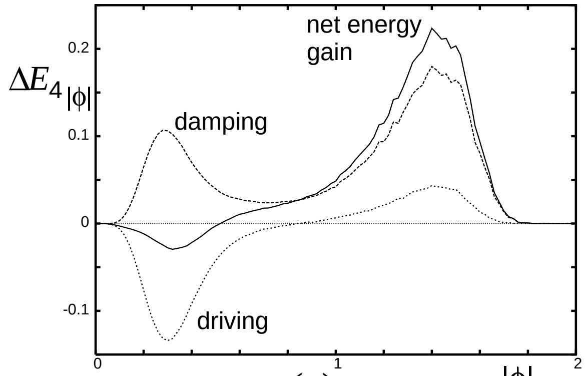
Input and output of wave action and energy E_4 in amplitude space $|\phi|$



(a)

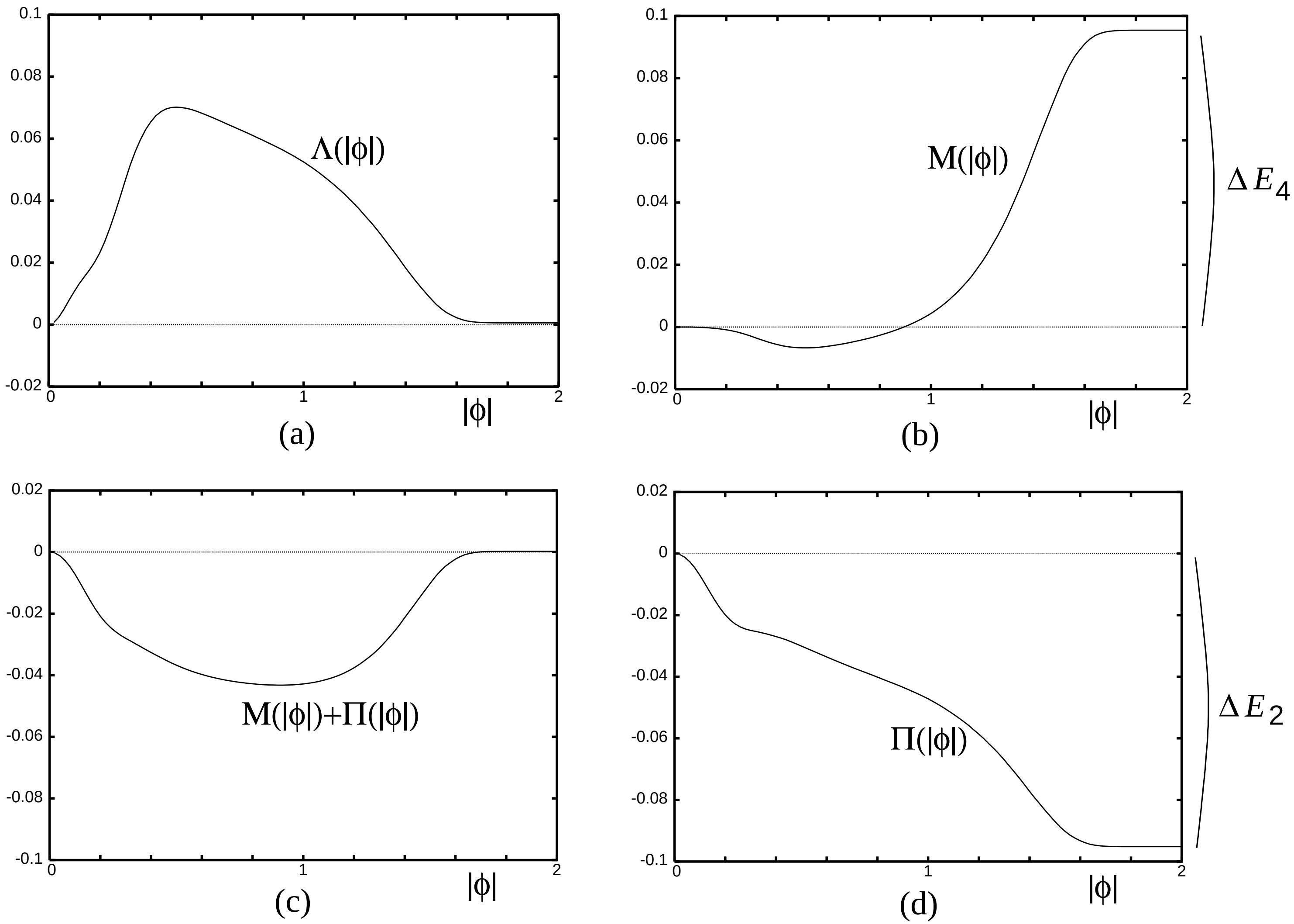


(b)



(c)

Flux Λ of wave action, M of E_4, and Π of E_2 in amplitude space $|\phi|$



Conclusions

- High-amplitude structures grow or decay when they interact with low-amplitude disordered waves depending on their phase fequency and on the statistical properties of the waves
- In nonequilibrium, fluxes of wave action and energy in amplitude space emerges