Constant flux relation for interacting particle systems

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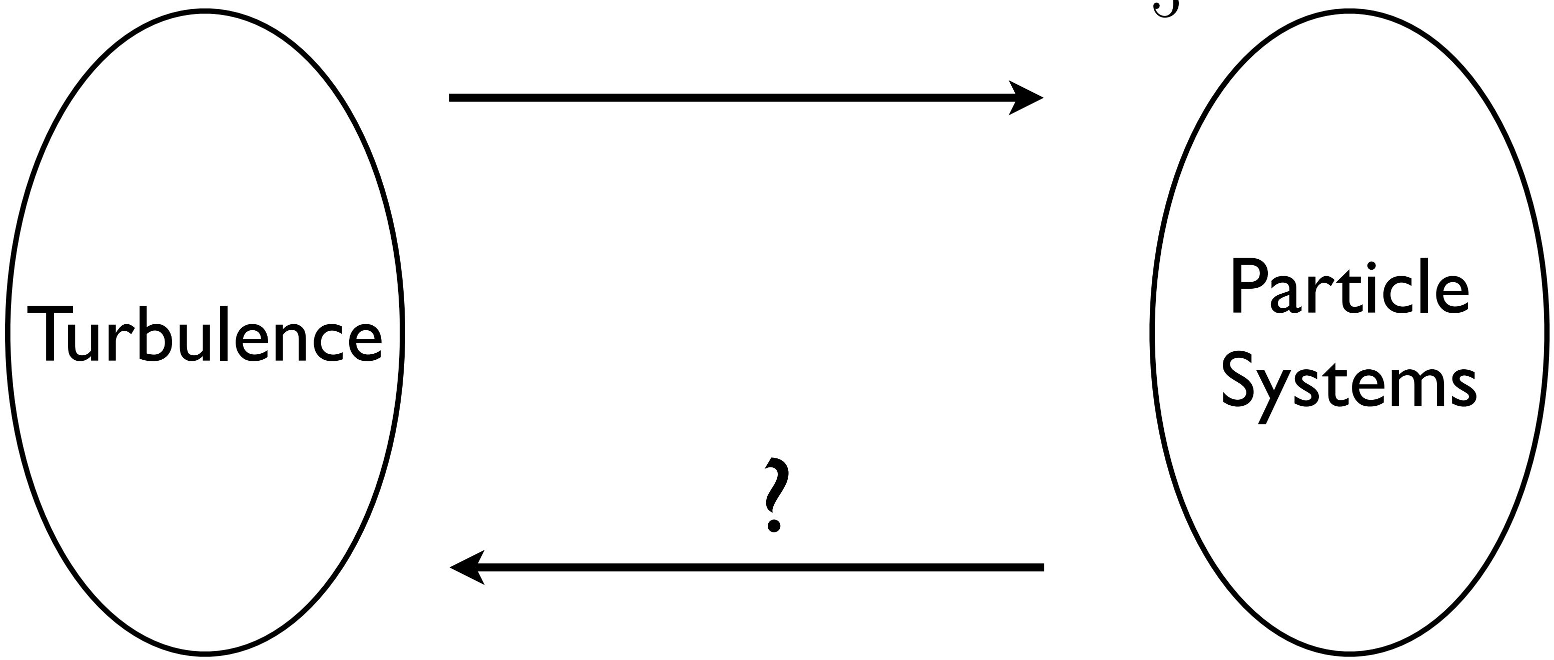
Introduction

- Turbulence
  - Driving and dissipation
  - Existence of an inertial range
  - Steady states characterised by constant flux
- Interacting particle systems
  - Coagulating particles
  - Sandpile model
  - Granular systems

Questions

$$S_3(r) = \langle (v_l(0) - v_l(r))^3 \rangle = -\frac{4}{5}\epsilon r$$

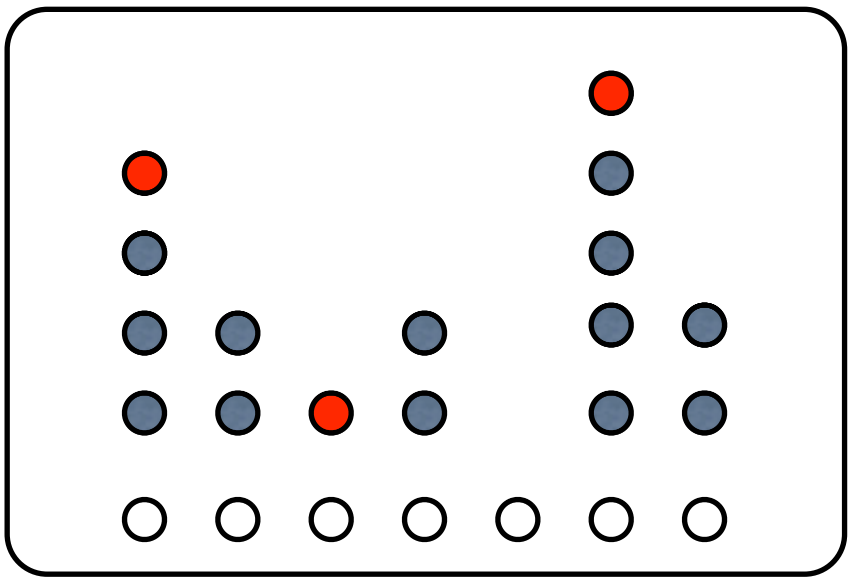


$$S_n(r) = \langle (v_l(0) - v_l(r))^n \rangle$$

Plan

- Aggregation model
- Constant flux condition (ZT)
- Uniqueness
- Locality
- Numerical results
- Other models

Aggregation Model



Particles carry mass, m=1,2,...

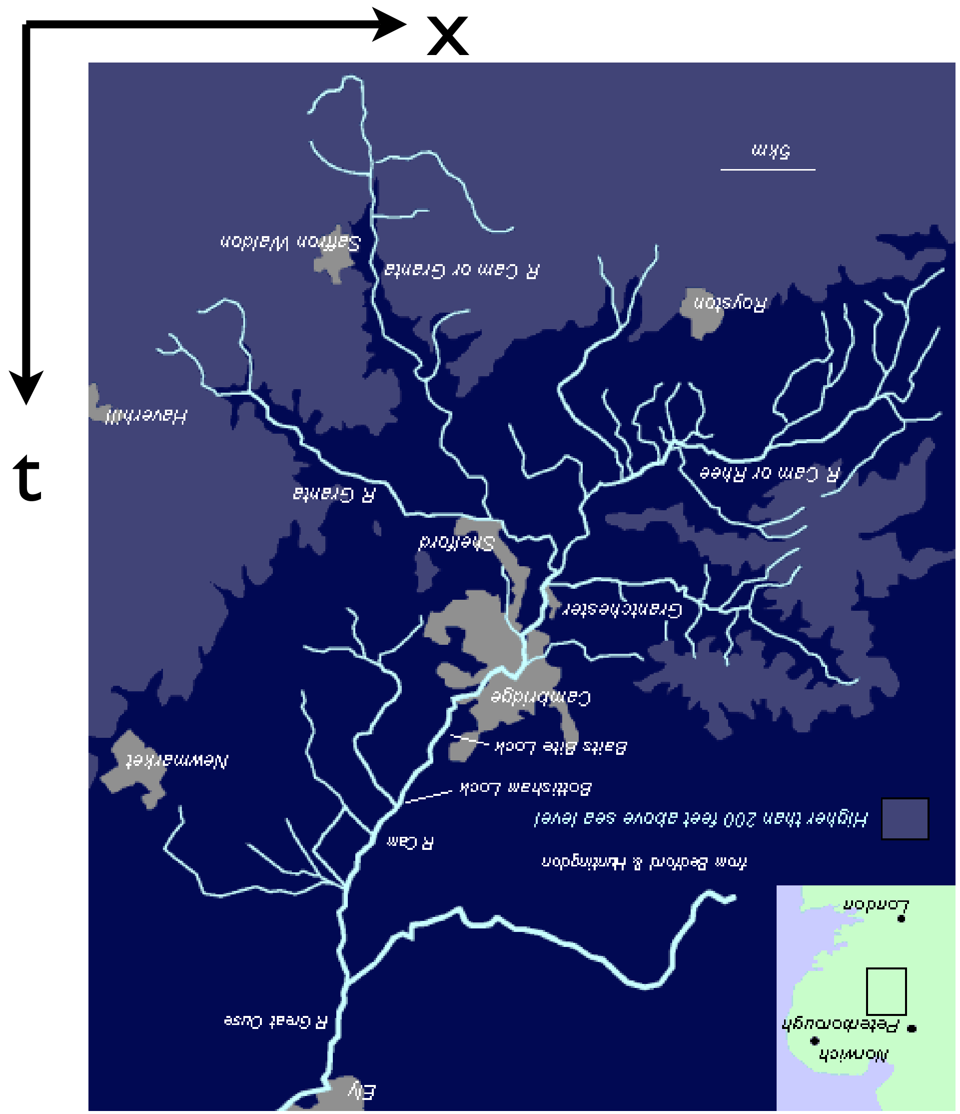
Diffusion: $D(m) \propto m^{-\mu}$

Aggregation: $\lambda(m_1, m_2)$

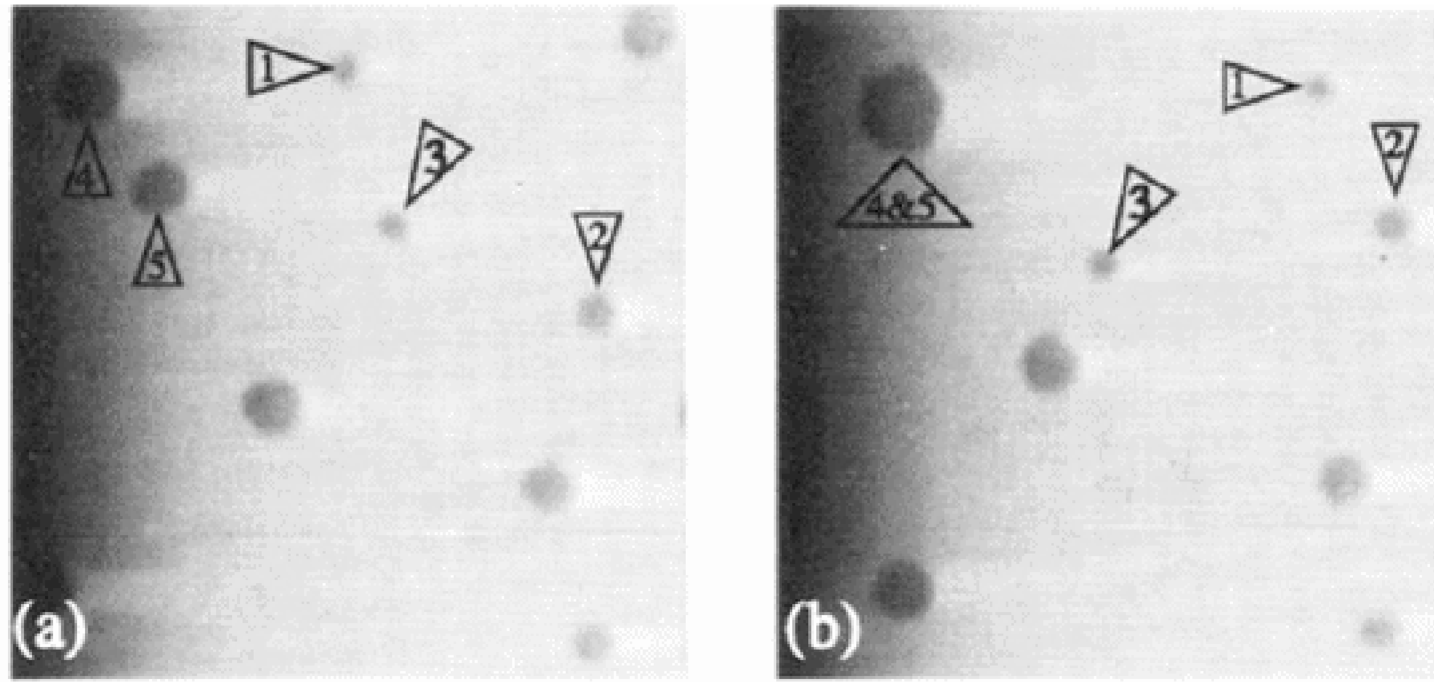
Input: J

$$\lambda(\Lambda m_1, \Lambda m_2) = \Lambda^{\beta} \lambda(m_1, m_2)$$

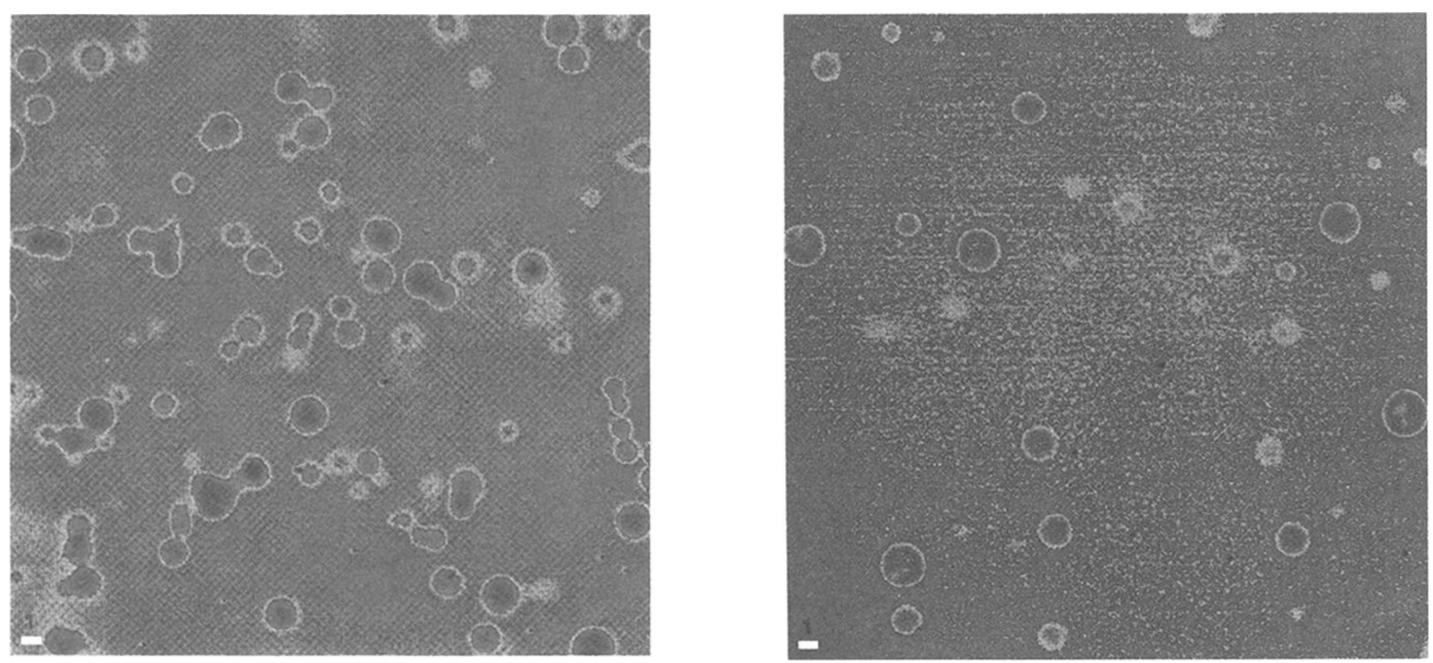
River Networks



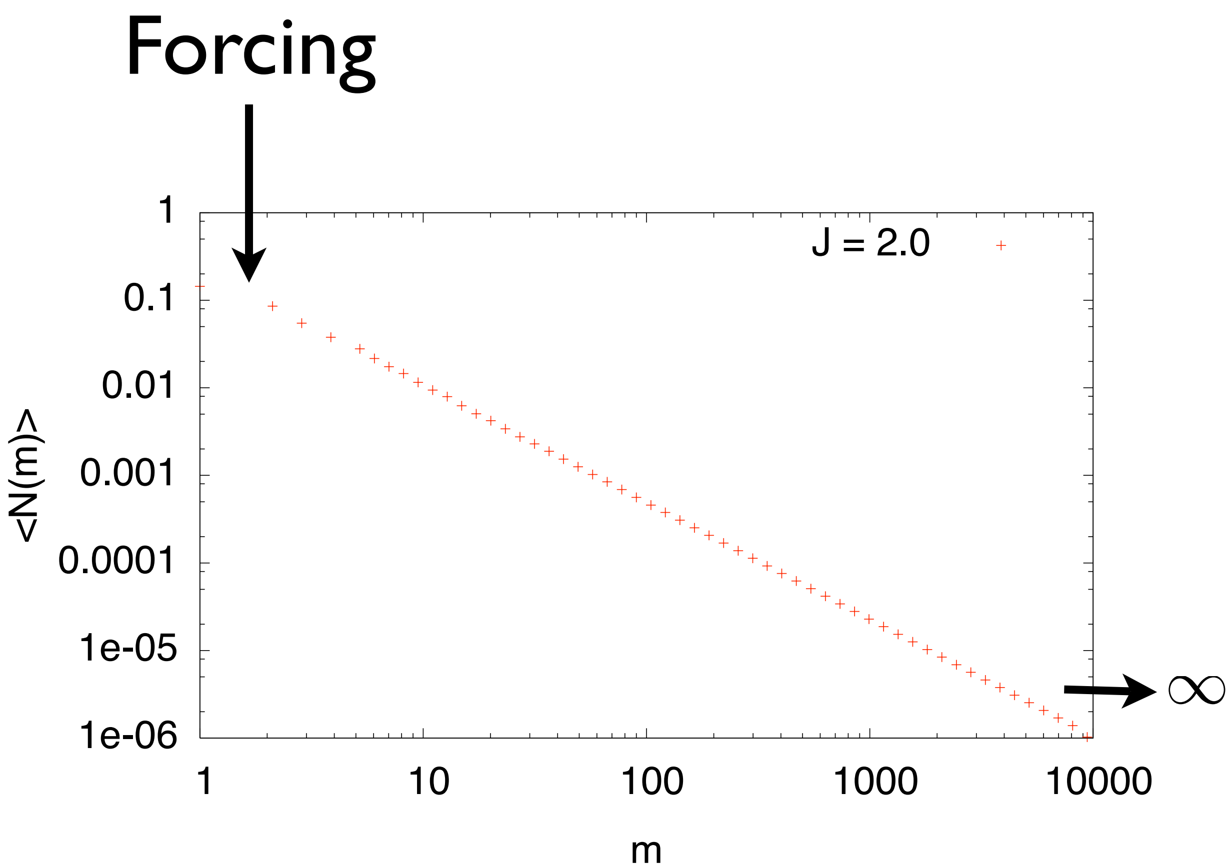
Surface growth



Coarsening



Constant flux



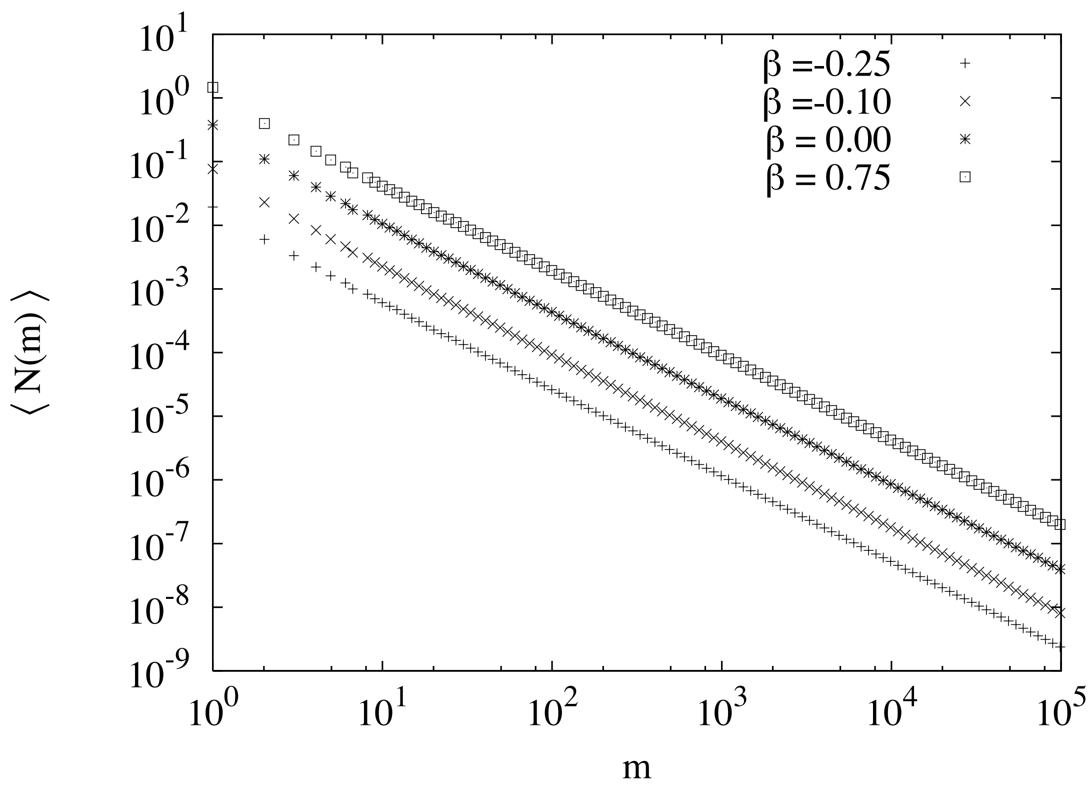
Flux from small to big masses

$\langle N(m) \rangle \equiv$ density

$\langle N(m_1) \ldots N(m_n) \rangle \leftrightarrow S_n$

Diffusion limited

$d < d_c$: Diffusion limited



$$
\begin{aligned}
\frac{d\langle N(m)\rangle}{dt} &= -\int_0^\infty dm_1 \int_0^\infty dm_2\, \lambda(m_1, m)\langle N(m_1)N(m)\rangle \delta(m_1 + m - m_2) \\
&- \int_0^\infty dm_1 \int_0^\infty dm_2\, \lambda(m, m_2)\langle N(m)N(m_2)\rangle \delta(m + m_2 - m_1) \\
&+ \int_0^\infty dm_1 \int_0^\infty dm_2\, \lambda(m_1, m_2)\langle N(m_1)N(m_2)\rangle \delta(m_1 + m_2 - m) \\
&+ \frac{J}{m_0}\delta(m - m_0)
\end{aligned}
$$

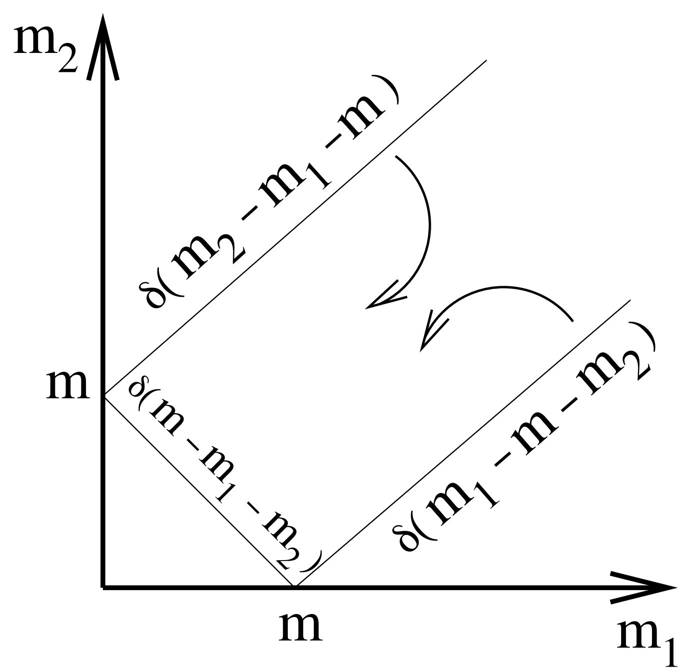
$$
\begin{aligned}
\frac{dN(m)}{dt} &= D\nabla^2 N(m) + \frac{J}{m_0}\delta(m - m_0) \\
&- 2\int_0^\infty dm_1 \int_0^\infty dm_2\, \lambda(m_1, m) N(m_1)N(m)\delta(m_1 + m - m_2) \\
&+ \int_0^\infty dm_1 \int_0^\infty dm_2\, \lambda(m_1, m_2) N(m_1)N(m_2)\delta(m_1 + m_2 - m) + n
\end{aligned}
$$

Constant Flux

$$
\frac{d}{dt}\int_0^\infty dm\; m\langle N(m)\rangle = J
$$

$$
\begin{aligned}
\frac{d\langle N(m)\rangle}{dt} &= -\int_0^\infty dm_1 \int_0^\infty dm_2 \lambda(m_1, m)\langle N(m_1)N(m)\rangle \delta(m_1 + m - m_2) \\
&- \int_0^\infty dm_1 \int_0^\infty dm_2 \lambda(m, m_2)\langle N(m)N(m_2)\rangle \delta(m + m_2 - m_1) \\
&+ \int_0^\infty dm_1 \int_0^\infty dm_2 \lambda(m_1, m_2)\langle N(m_1)N(m_2)\rangle \delta(m_1 + m_2 - m) \\
&+ \frac{J}{m_0}\delta(m - m_0)
\end{aligned}
$$

Zakharov Transform



$$
\begin{aligned}
(m_1, m_2) &\to \left(\frac{m m_1'}{m_2'}, \frac{m^2}{m_2'}\right) \\
(m_1, m_2) &\to \left(\frac{m^2}{m_1'}, \frac{m m_2'}{m_1'}\right)
\end{aligned}
$$

$$\langle N(Km_1)N(Km_2)\rangle = K^{-h}\langle N(m_1)N(m_2)\rangle$$

$$0 \quad = \quad \int_0^\infty dm_1 \int_0^\infty dm_2 \lambda(m_1, m_2)\langle N(m_1)N(m_2)\rangle$$
$$(m^{2h-2-\beta} - m_1^{2h-2-\beta} - m_2^{2h-2-\beta})\delta(m_1 + m_2 - m)$$

$$2h - 2 - \beta = 1$$

$$h = \frac{3+\beta}{2}$$

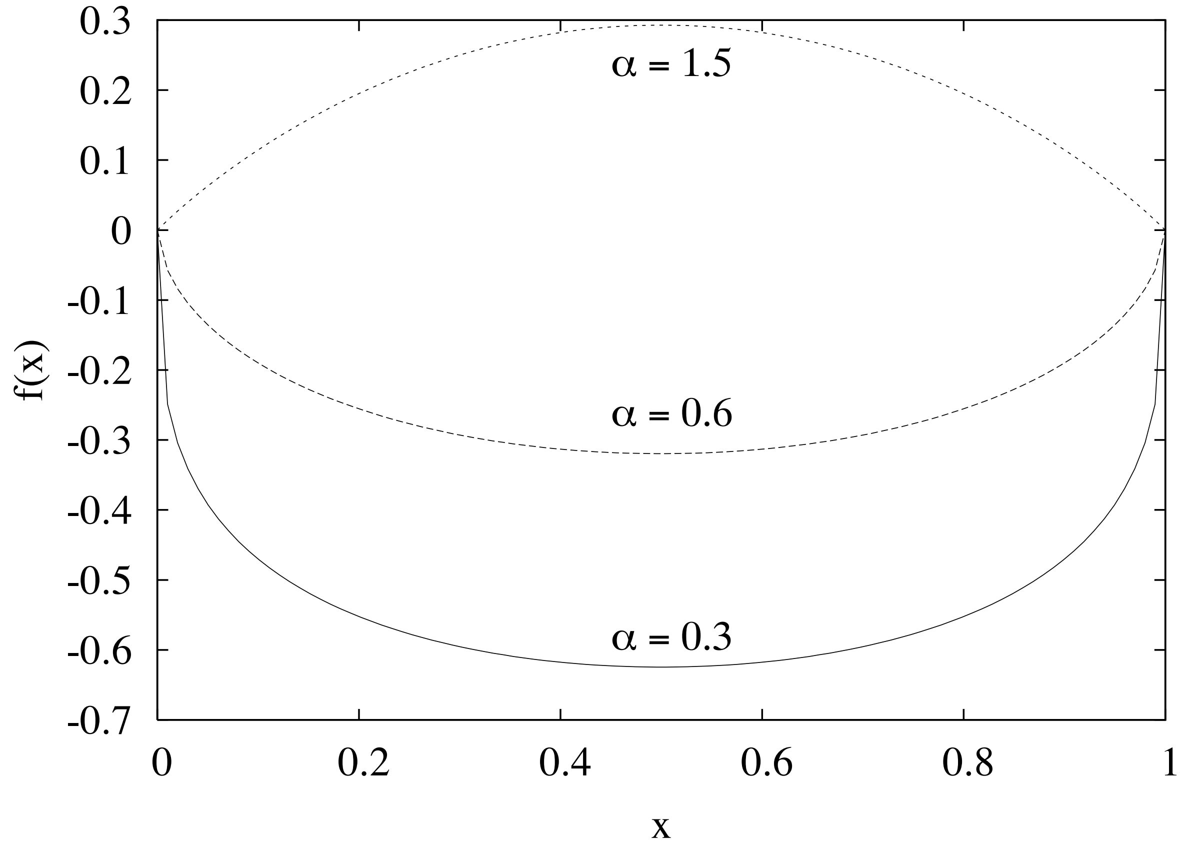
$$\langle N(m)N(m)\rangle \sim \frac{1}{m^{3+\beta}} \quad \text{in all d}$$

Uniqueness

$$f(x) = 1 - x^\alpha - (1-x)^\alpha$$

$$\begin{aligned} 0 &= \int_0^\infty dm_1 \int_0^\infty dm_2 \lambda(m_1, m_2) \langle N(m_1) N(m_2) \rangle \\ &\quad m^\alpha f\left(\frac{m_1}{m}\right) \delta(m_1 + m_2 - m) \end{aligned}$$

$$\alpha = 2h - 2 - \beta$$



$f(x)$ sign definite for $\alpha \neq 1$

Locality

$$
\begin{aligned}
0 &= -\int_0^\infty dm_1 \int_0^\infty dm_2 \lambda(m_1, m) \langle N(m_1) N(m) \rangle \delta(m_1 + m - m_2) \\
&- \int_0^\infty dm_1 \int_0^\infty dm_2 \lambda(m, m_2) \langle N(m) N(m_2) \rangle \delta(m + m_2 - m_1) \\
&+ \int_0^\infty dm_1 \int_0^\infty dm_2 \lambda(m_1, m_2) \langle N(m_1) N(m_2) \rangle \delta(m_1 + m_2 - m)
\end{aligned}
$$

$$\lambda(m_1, m_2) \sim m_1^\mu m_2^\nu, \quad m_2 \gg m_1$$

$$\langle N(m_1) N(m_2) \rangle = (m_1 m_2)^{-h/2} \Phi\left(\frac{m_1}{m_2}\right)$$

$$\Phi(x) \sim x^\theta, \quad x \to 0$$

$$\frac{h}{2} \ \epsilon \ [\nu + 1 - \theta, \mu + 2 + \theta]$$

Locality ..

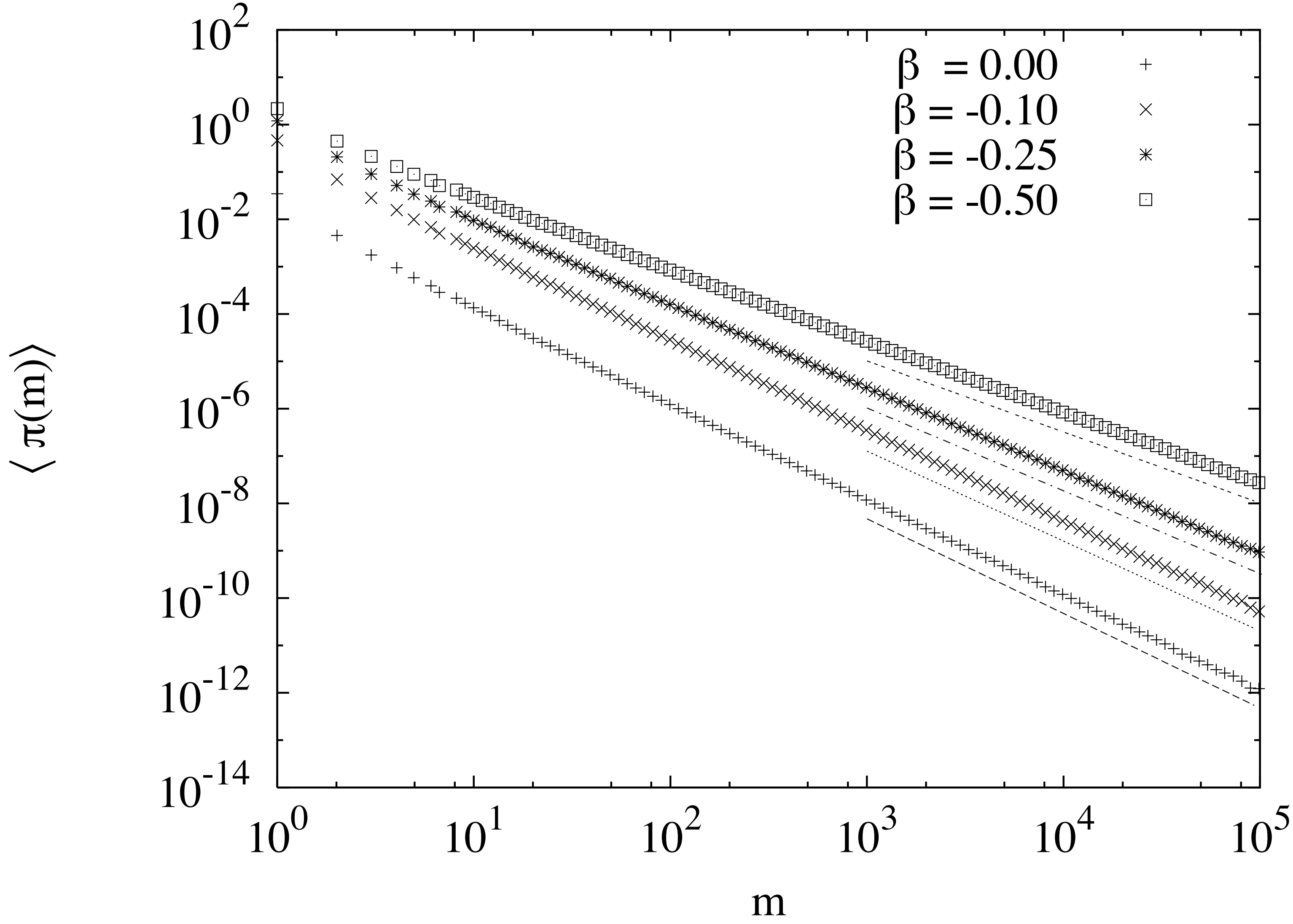
$$\theta > \frac{1}{2}(\nu - \mu - 1)$$

$$\theta = 0 \text{ in meanfield}$$

$$\theta = c\epsilon \text{ when } \epsilon = d_c - d$$

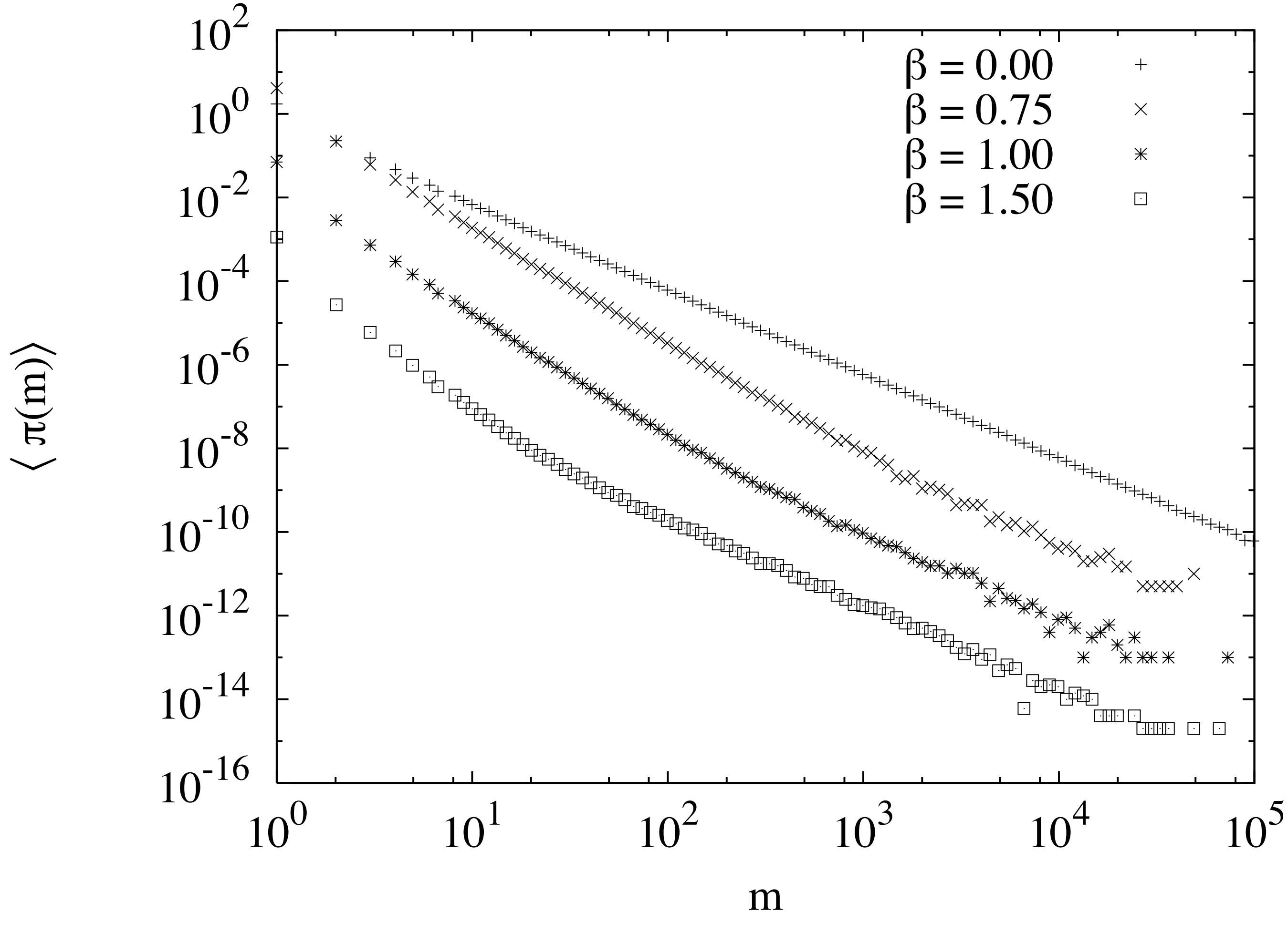
Numerical simulations

$\beta \leq 0$

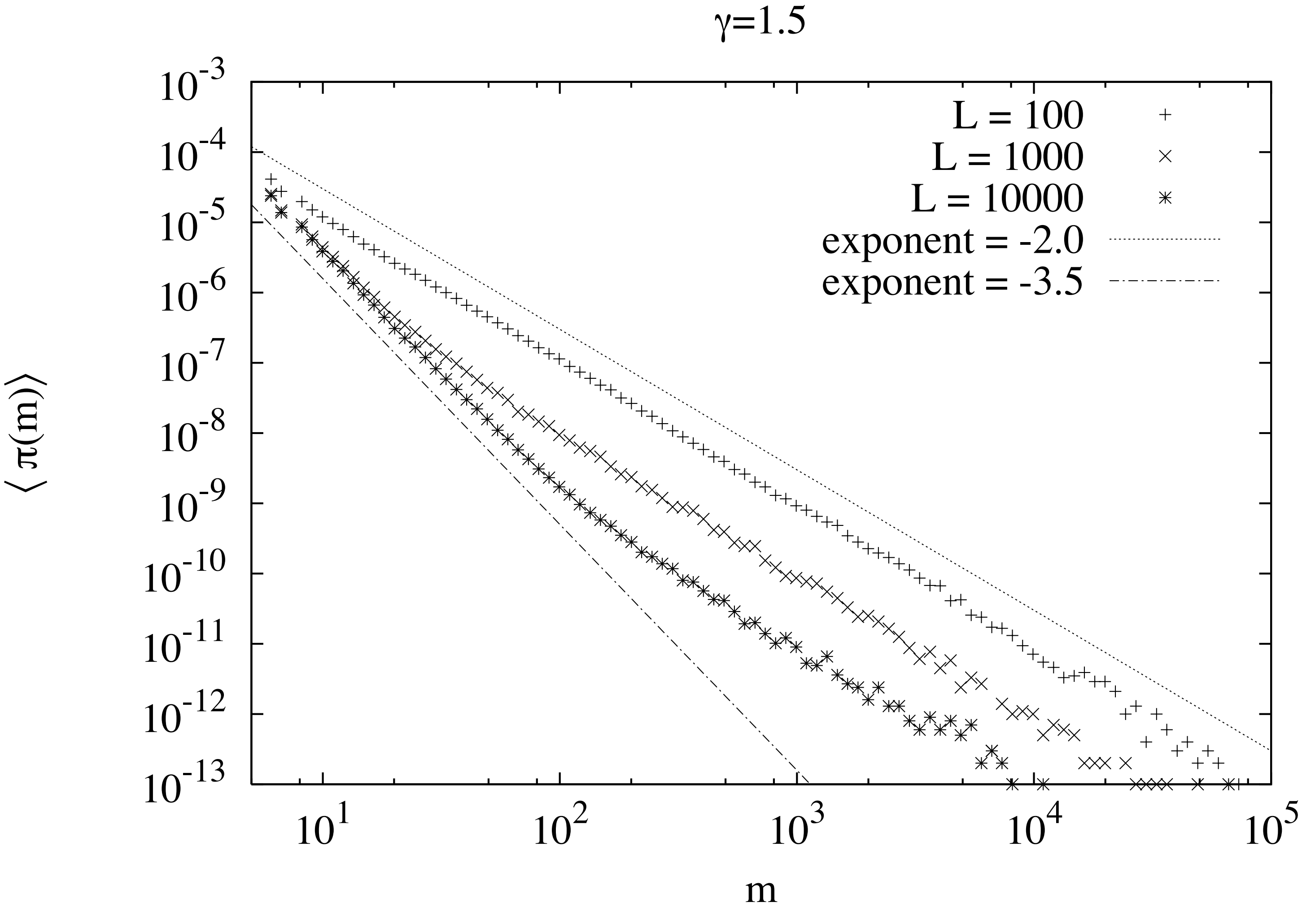


β = 0.00
β = -0.10
β = -0.25
β = -0.50
⟨ π(m) ⟩
m
10^2
10^0
10^-2
10^-4
10^-6
10^-8
10^-10
10^-12
10^-14
10^0
10^1
10^2
10^3
10^4
10^5

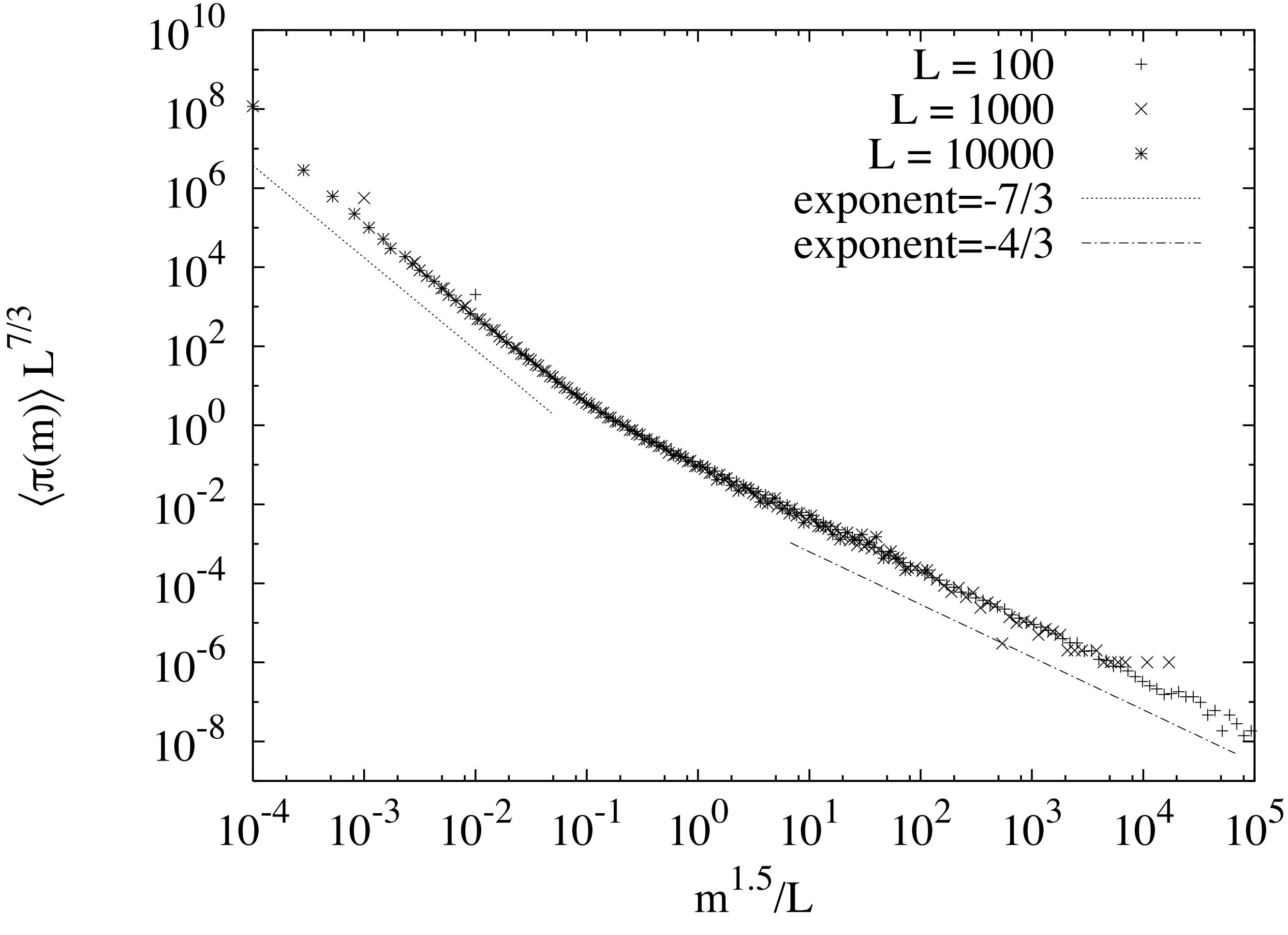
$$\beta \geq 0$$



β = 0.00
β = 0.75
β = 1.00
β = 1.50
⟨ π(m) ⟩
m
10^2
10^0
10^-2
10^-4
10^-6
10^-8
10^-10
10^-12
10^-14
10^-16
10^0
10^1
10^2
10^3
10^4
10^5



γ=1.5
L = 100
L = 1000
L = 10000
exponent = -2.0
exponent = -3.5
⟨ π(m)⟩
m
10-3
10-4
10-5
10-6
10-7
10-8
10-9
10-10
10-11
10-12
10-13
10^1
10^2
10^3
10^4
10^5

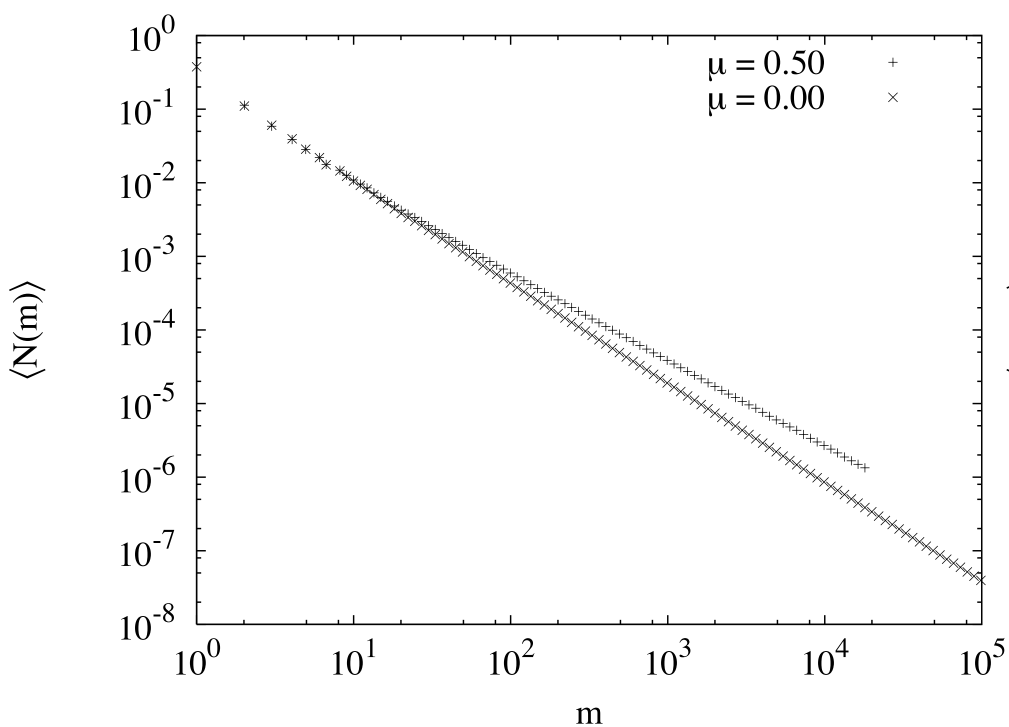


L = 100
L = 1000
L = 10000
exponent=-7/3
exponent=-4/3
$\langle \pi(m) \rangle L^{7/3}$
$m^{1.5}/L$
10^{10}
10^{8}
10^{6}
10^{4}
10^{2}
10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
10^{4}
10^{5}

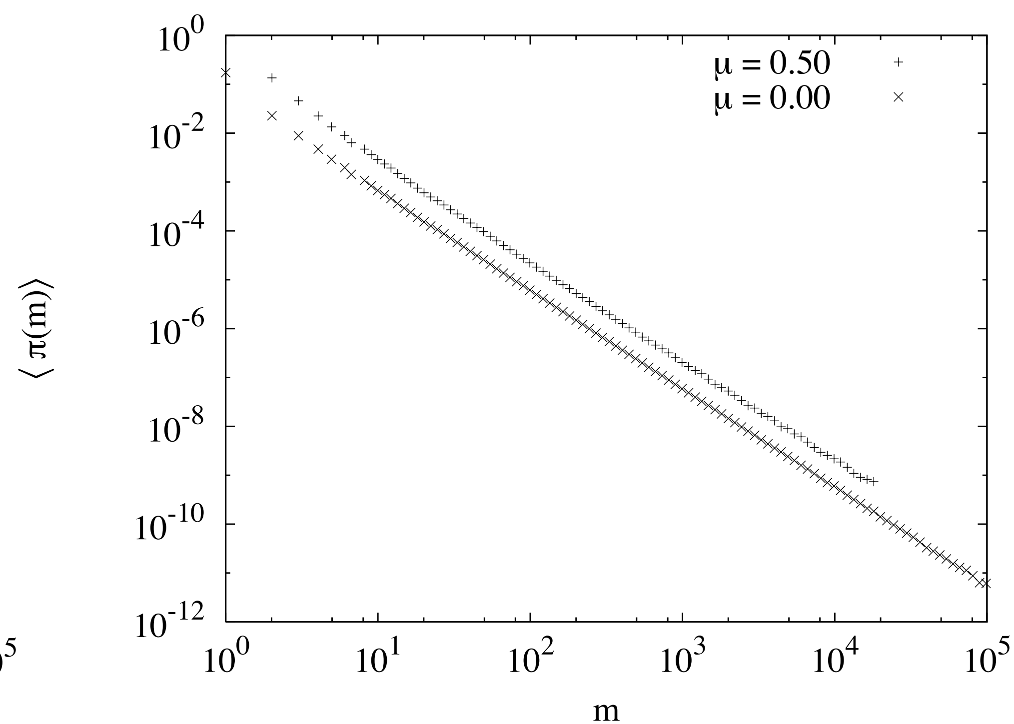
Mass dependent diffusion

$$D(m) \propto \frac{1}{m^{\mu}}$$

one-point



two-point

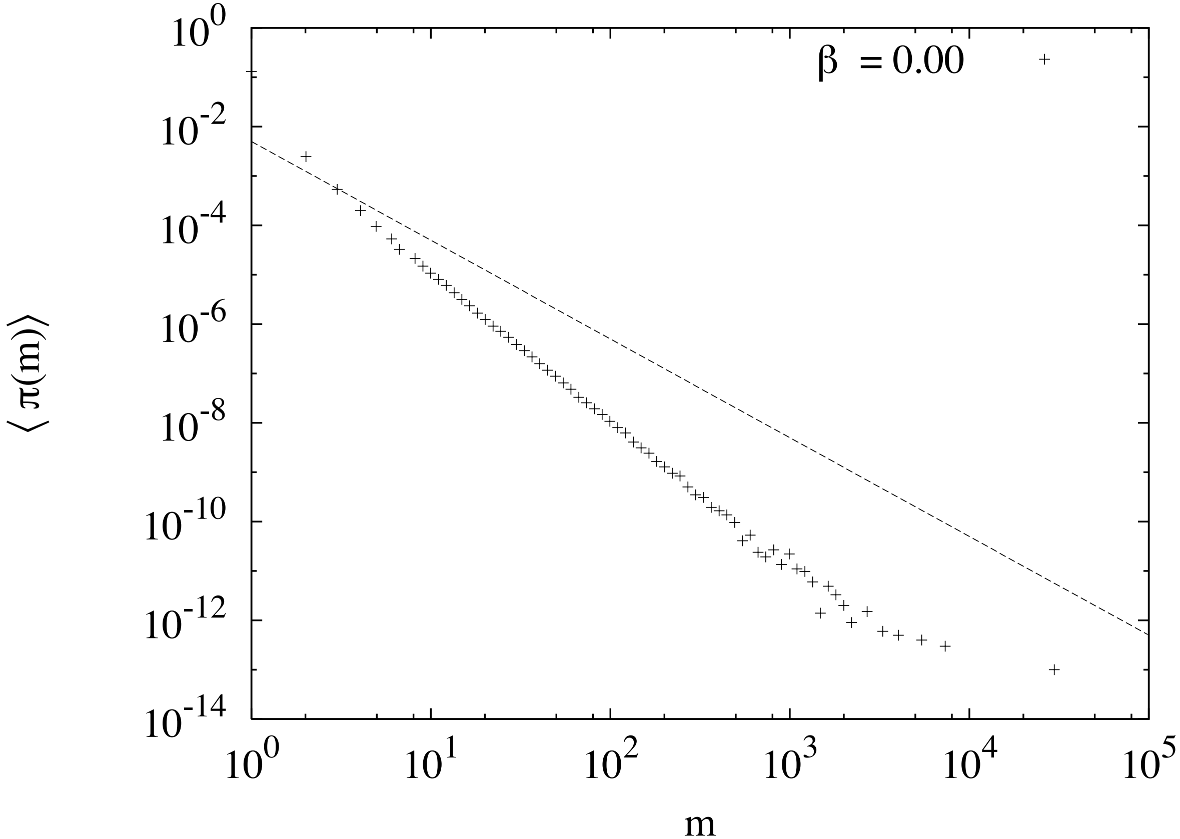


Constant flux relation

$$C(m_1, \ldots, m_{n-1}) = \langle N(m_1) \ldots N(m_{n-1}) \rangle$$

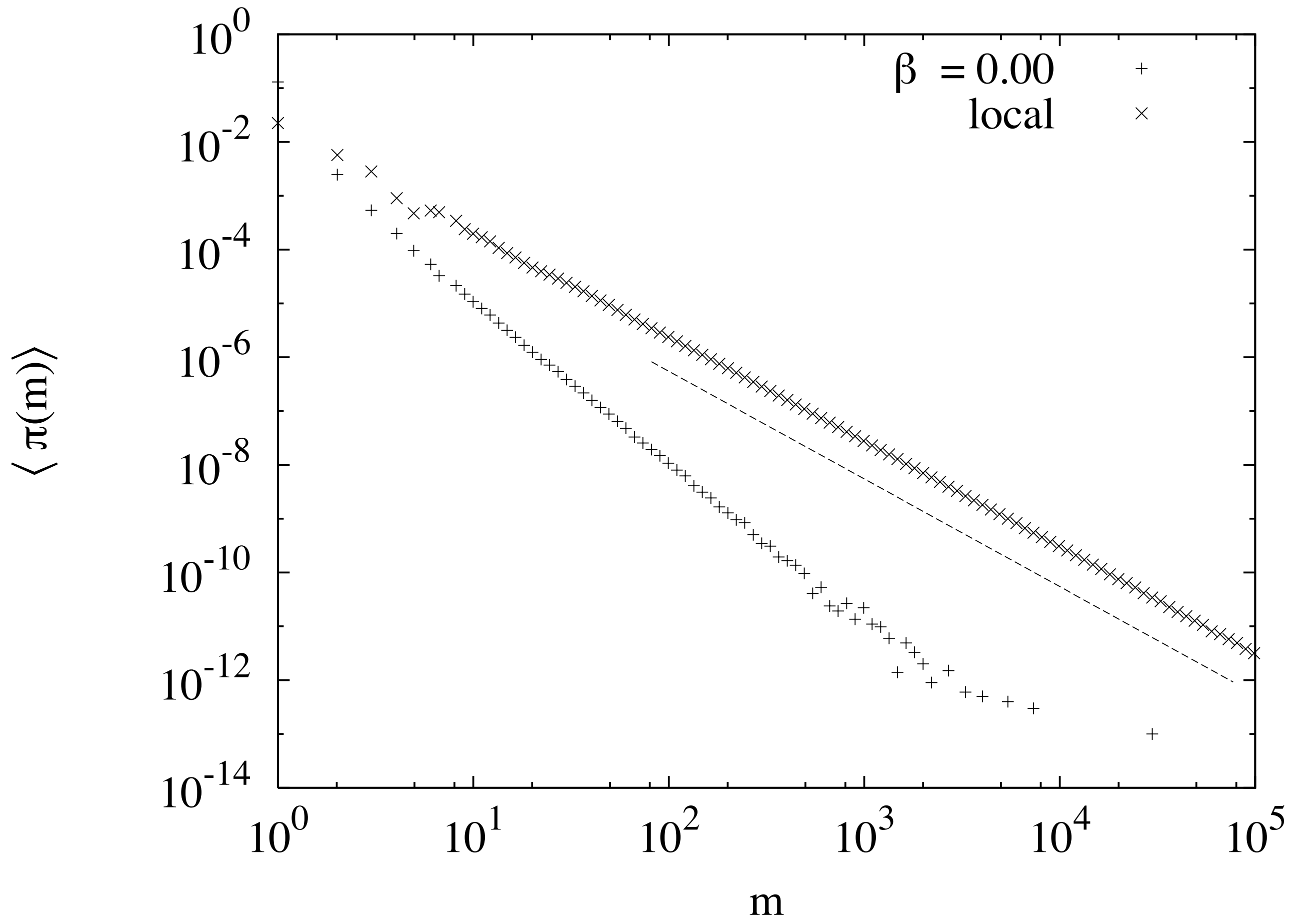
$$C(pm_1, \ldots, pm_{n-1}) = p^y C(m_1, \ldots, m_{n-1})$$

$$\boxed{y = -\zeta - n, \quad \text{in all } d}$$



β = 0.00
⟨ π(m) ⟩
m
10^0
10^-2
10^-4
10^-6
10^-8
10^-10
10^-12
10^-14
10^1
10^2
10^3
10^4
10^5

$$\lambda(m_1, m_2, m_3) = (m_1^\beta + m_2^\beta + m_3^\beta) f\left(\frac{m_1}{m_2}\right) f\left(\frac{m_2}{m_3}\right) f\left(\frac{m_3}{m_1}\right)$$



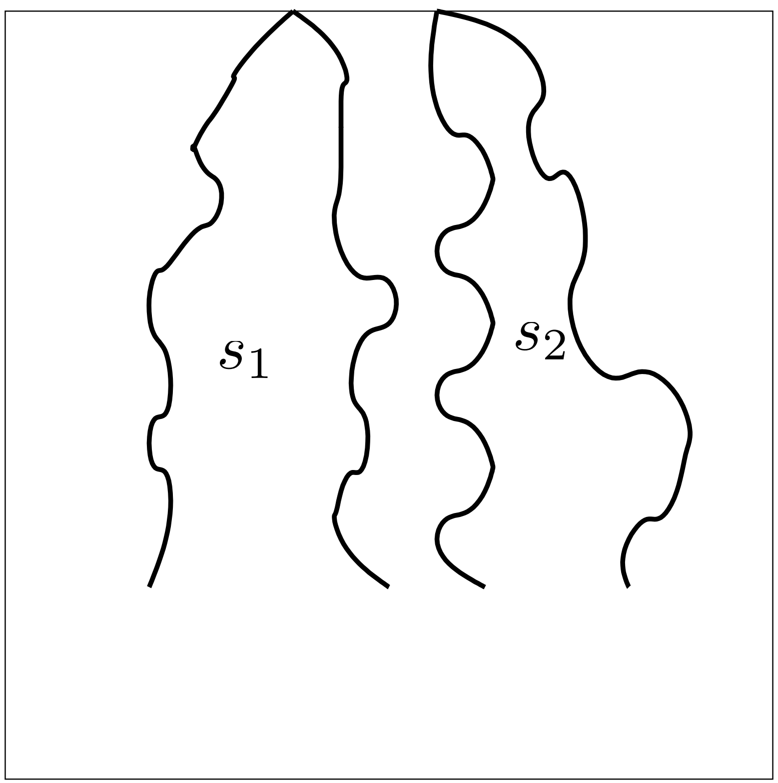
Other models

- Charge Model
  - ⋆ Negative masses
  - ⋆ $\pm m_0$ input
  - ⋆ no flux in mass
  - ⋆ constant flux in mass square
  - ⋆ CFR can be worked out

- Sandpile Models

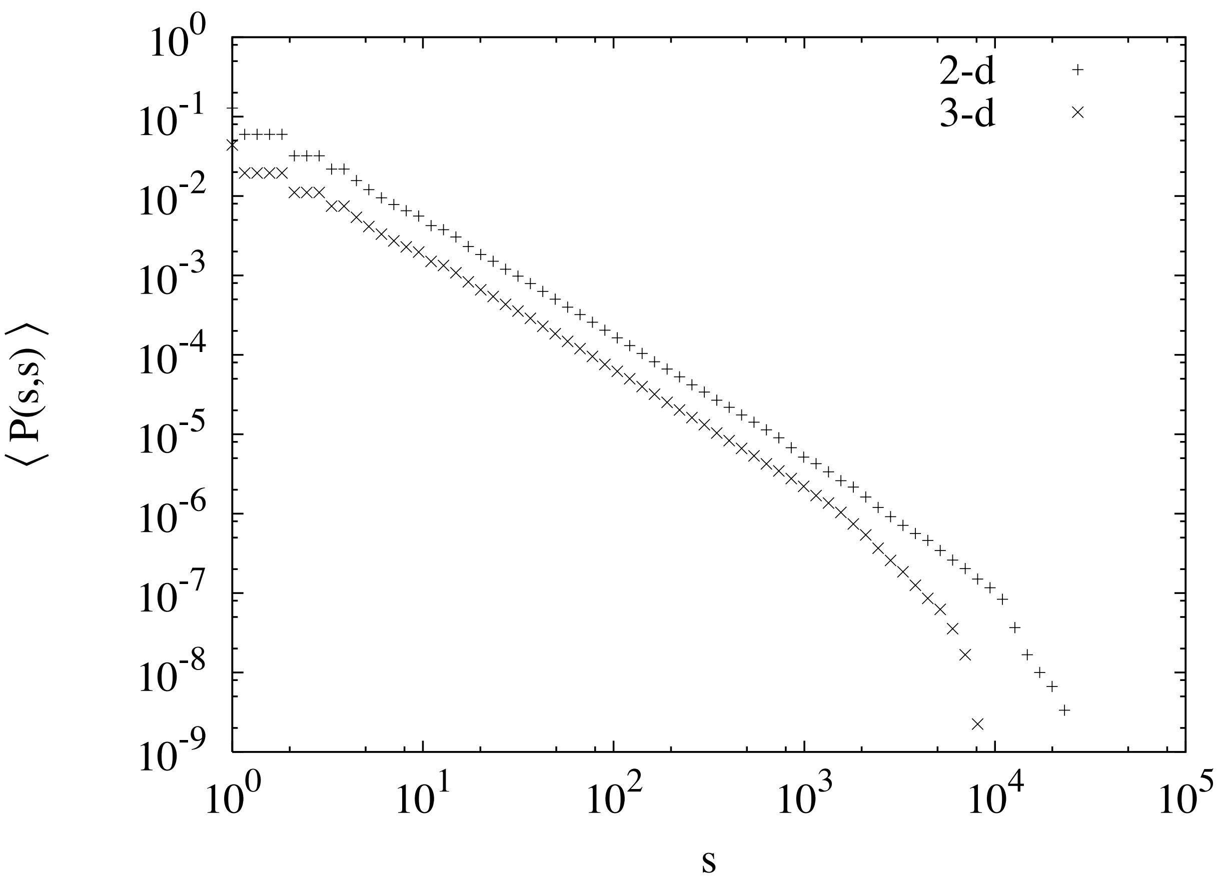
1 / 2	4 / 2	4 / 5 / 3	3 / 2
4	1 / 2	3 / 2	3
4	2	3	2
1	2	2	4

- Directed abelian sandpile model
  - ⋆ Maps onto aggregation model (β = 0)



$$\tilde{P}(s_1, s_2) \sim \langle N(s_1) N(s_2) \rangle$$

- Undirected abelian sandpile
  - ⋆ Edge avalanches



- Ballistic Aggregation
- Wave turbulence (Connaughton)

Summary

- Kolmogorov 4/5-th law
- No mean field assumption
- Locality