Dynamic Multiscaling in Turbulence

Dhrubaditya Mitra[1] Rahul Pandit[2]
Samriddhi Shankar Ray[2]

[1]Département Cassiopée,
Observatoire de la Côte d'Azur,
Nice, France

[2]Department of Physics
Indian Institute of Science
Bangalore, India.

July 20, 2006

Dynamics of Critical phenomenon

- Near a critical point the spin-spin correlation function shows *scaling* and the characteristic correlation length, ξ, diverges.

$$\begin{aligned} \langle s(\mathbf{x}) s(\mathbf{x}+\mathbf{r}) \rangle &= \frac{1}{r^{d-2+\eta}} G(|r|/\xi) \\ \xi &\sim |T - T_c|^{-\gamma} \qquad \text{as} \qquad T \to T_c \end{aligned}$$

where η, γ are critical exponents.

- The time-dependent correlation function:

$$\begin{aligned} \langle s(\mathbf{x}, 0) s(\mathbf{x}+\mathbf{r}, t) \rangle &= \langle s(\mathbf{x}) s(\mathbf{x}+\mathbf{r}) \rangle f(t/\tau) \\ \tau &\sim \xi^{z} \end{aligned}$$

where ξ is the dynamic critical exponent.

Turbulence

- Structure functions:

$$\begin{aligned} \delta\phi(\mathbf{x},\mathbf{r},t) &\equiv \phi(\mathbf{x}+\mathbf{r},t)-\phi(\mathbf{x},t) \\ S_p(r) &\equiv \langle[\delta\phi]^p\rangle \end{aligned}$$

- In fully developed turbulence, the structure functions show *multiscaling*

$$S_p(r) \sim r^{\zeta_p}$$

  where ζ_p is a nonlinear convex function of p.

- K41 implies $\zeta_p = p/3$.
- How does the dynamic structure functions behave ?

$$\begin{aligned} \mathcal{F}(r,t) &\equiv \langle\delta\phi(r,t)\delta\phi(r,0)\rangle \\ &= S_p(r)G[t/\tau(r)] \\ \tau(r) &\sim r^z \end{aligned}$$

Fig. 1.10. Wake behind two identical cylinders at $R = 1800$. Courtesy R. Dumas.

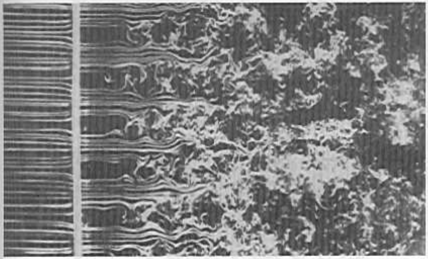


Fig. 1.11. Homogeneous turbulence behind a grid. Photograph T. Corke and H. Nagib.

Measurements are typically done by a single probe. The time-series of velocity is converted to spatial field by Taylor's hypothesis.

Eulerian, Lagrangian and quasi-Lagrangian

- For Eulerian variables : typical time and length scales are linearly related. Hence dynamic scaling exponent $z = 1$. A phenomenon called *sweeping*.
- For Lagrangian quantities, K41 predicts:

$$\tau(r) \sim \frac{r}{\delta u(r)} \sim r^{2/3}$$
$$z = 2/3$$

- Quasi-Lagrangian velocity is the Eulerian velocity relative to a fluid particle. This is expected to show no sweeping. The equal-time behaviour is similar to Eulerian and dynamic behaviour is similar to the Lagrangian velocities.

Dynamics of Lagrangian quantities

- Kraichnan model of passive-scalar
- Multifractal model for fluid turbulence.
- Multifractal model for passive-scalar turbulence.

Kraichnan model of passive scalar

$$\partial_t \theta + (\mathbf{u} \cdot \nabla)\theta = \kappa \nabla^2 \theta + f^\theta,$$

Velocity is random, Gaussian with co-variance

$$\begin{aligned} \langle u_i(\mathbf{x}, t) u_j(\mathbf{x} + \ell, t') \rangle &= 2D_{ij}(\ell)\delta(t - t') \\ D_{ij}(\ell) &= D_0 \delta_{ij} - \frac{1}{2} d_{ij}(\ell) \end{aligned}$$

- $D_0 \sim L^\xi$
- $L \to \infty$ and $\eta \to 0$,
  $d_{ij} = D_1 \ell^\xi \left[(d - 1 + \xi)\delta_{ij} - \xi \frac{\ell_i \ell_j}{\ell^2} \right] \sim \ell^\xi$

Equal-time statistics

- Multiscaling can be analytically (but perturbative) demonstrated.
- $S_p(\ell) \sim \ell^{\zeta_p^\theta}$.
- $0 < \xi < 2$.
- Structure functions have good limits, not correlation functions.

Dynamic structure functions

$$\begin{aligned}
\delta\phi(\mathbf{x},t,\mathbf{r}) &\equiv \phi(\mathbf{x}+\mathbf{r},t)-\phi(\mathbf{x},t)\\
\mathcal{F}_2^{\phi}(r,t) &= \langle[\delta\phi(\mathbf{x},0,\mathbf{r})\delta\phi(\mathbf{x},t,\mathbf{r})]\rangle\\
&= 2C^{\phi}(\mathbf{0},t)-2C^{\phi}(\mathbf{r},t)\\
C^{\phi}(\mathbf{r},t) &\equiv \langle\phi(\mathbf{x}+\mathbf{r},0)\phi(\mathbf{x},t)\rangle
\end{aligned}$$

- t strictly positive.

Dynamic scaling

$$\begin{aligned} \partial_t C^{\phi}(\mathbf{r},t) &= \langle \phi(\mathbf{x}+\mathbf{r},0)\partial_t[\phi(\mathbf{x},t)]\rangle \\ &= -\langle \phi(0)(\mathbf{u}\cdot\nabla)\theta\rangle + \kappa\nabla^2\langle\phi(0)\phi\rangle + \left\langle \phi(0)f^{\phi}\right\rangle \end{aligned}$$

$\partial_t C^{\theta}(\mathbf{r},t) = D^0(L)\partial_{ii}C^{\theta} \sim L^{\xi}\partial_{ii}C^{\theta}$;
$\partial_t C^{\hat{\theta}}(\mathbf{r},t) = \left(D^0\delta_{ij} - D_{ij}\right)\partial_{ij}C^{\hat{\theta}} \sim d_{ij}(\mathbf{r})\partial_{ij}C^{\hat{\theta}}$.
$C^{\phi}(r,t) \sim \exp[-t/\tau^{\phi}(r)]$

$\tau^{\hat{\theta}}(r,t) = r^{1-\xi}$; $\tau^{\theta}(r,t) = r^2$
In the limit of $L \to \infty$, $C^{\theta}(r,t)$ diverges for all r.
$z^{ql} = 2 - \xi$, a bridge relationship.

Dynamic scaling

$$\begin{aligned}\partial_t C^\phi(\mathbf{r},t) &= \langle \phi(\mathbf{x}+\mathbf{r},0)\partial_t[\phi(\mathbf{x},t)]\rangle \\ &= -\langle \phi(0)(\mathbf{u}\cdot\nabla)\theta\rangle + \kappa\nabla^2\langle\phi(0)\phi\rangle + \left\langle \phi(0)f^\phi\right\rangle\end{aligned}$$

$\partial_t C^\theta(\mathbf{r},t) = D^0(L)\partial_{ii}C^\theta \sim L^\xi \partial_{ii}C^\theta$;
$\partial_t C^{\hat\theta}(\mathbf{r},t) = \left(D^0\delta_{ij} - D_{ij}\right)\partial_{ij}C^{\hat\theta} \sim d_{ij}(\mathbf{r})\partial_{ij}C^{\hat\theta}$.
$C^\phi(r,t) \sim \exp[-t/\tau^\phi(r)]$

$\tau^{\hat\theta}(r,t) = r^{1-\xi}$; $\tau^\theta(r,t) = r^2$
In the limit of $L \to \infty$, $C^\theta(r,t)$ diverges for all r.
$z^{ql} = 2 - \xi$, a bridge relationship.

Dynamic scaling

$$\partial_t C^\phi(\mathbf{r},t) = \langle \phi(\mathbf{x}+\mathbf{r},0)\partial_t[\phi(\mathbf{x},t)]\rangle$$
$$= -\langle \phi(0)(\mathbf{u}\cdot\nabla)\theta\rangle + \kappa\nabla^2\langle\phi(0)\phi\rangle + \left\langle \phi(0) f^\phi \right\rangle$$

$\partial_t C^\theta(\mathbf{r},t) = D^0(L)\partial_{ii}C^\theta \sim L^\xi \partial_{ii}C^\theta;$
$\partial_t C^{\hat\theta}(\mathbf{r},t) = \left(D^0\delta_{ij} - D_{ij}\right)\partial_{ij}C^{\hat\theta} \sim d_{ij}(\mathbf{r})\partial_{ij}C^{\hat\theta}.$
$C^\phi(r,t) \sim exp[-t/\tau^\phi(r)]$

$\tau^{\hat\theta}(r,t) = r^{1-\xi}$; $\tau^\theta(r,t) = r^2$
In the limit of $L \to \infty$, $C^\theta(r,t)$ diverges for all r.
$z^{ql} = 2 - \xi$, a bridge relationship.

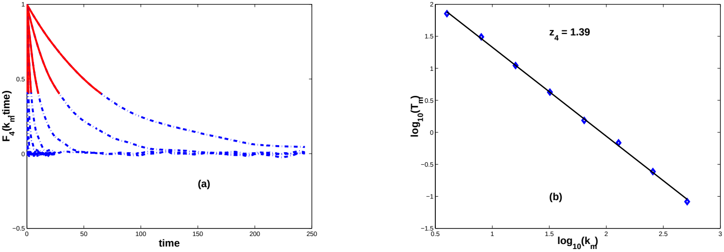
Dynamics of Kraichnan model

- Generalization to higher order is possible but messy.
- Eulerian time-dependent structure functions have no good limit due to sweeping, diverges as L^{ξ}.
- Quasi-Lagrangian time-dependent structure functions remain finite, shows exponential decay in time, and simple dynamic scaling:

$$z^{ql} = 2 - \xi$$

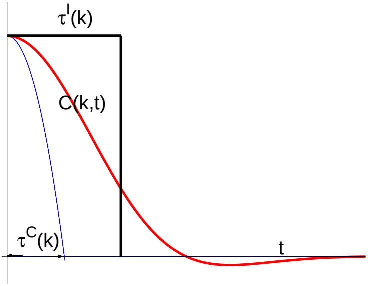
- Same dynamic scaling can be recovered by simple dimensional analysis.
- The dynamic scaling exponent bears no relationship with zero-modes.

Numerics of Kraichnan-shell model



- Shell model analog of Kraichnan model.
- Dynamic exponent of 4-th order.

Varieties of time-scales



- In dynamic critical phenomenon the integral and curvature time has the same scaling properties. More generally the dynamic scaling property does not depend on how you define the time scale.

$$\tau^C \sim \tau^I \sim k^{-z} \sim r^z$$

Varieties of dynamic multiscaling

- Integral time scale

$$\mathcal{T}_p^I(\ell) \equiv \frac{1}{\mathcal{S}_p(\ell)} \int_0^\infty \mathcal{F}_p(\ell, t) dt \sim \ell^{z_{p,1}^I}$$

- Curvature time scale

$$\mathcal{T}_{p,2}^D(\ell) \equiv \left[\frac{1}{\mathcal{S}_p(\ell)} \frac{\partial^2}{\partial t^2} \mathcal{F}_p(\ell, t) dt\right]^{(-1/2)} \sim \ell^{z_{p,2}^D}$$

- Using the multifractal model we derive the bridge relations

$$z_{p,1}^I = 1 + [\zeta_{p-1} - \zeta_p],$$

$$z_{p,2}^D = 1 + [\zeta_p - \zeta_{p+2}]/2.$$

Multifractal Derivation for Integral Scale

$$\mathcal{F}_p(\ell, t) \propto \int_{\mathcal{I}} d\mu(h) (\frac{\ell}{L})^{\mathcal{Z}(h)} \mathcal{G}^{p,h}(\frac{t}{\tau_{p,h}}),$$

where $\mathcal{G}^{p,h}(\frac{t}{\tau_{p,h}})$ has a characteristic decay time $\tau_{p,h} \sim \ell/\delta v(\ell) \sim \ell^{1-h}$, and $\mathcal{G}^{p,h}(0) = 1$. If $\int_0^\infty t^{(M-1)} \mathcal{G}^{p,h} dt$ exists,

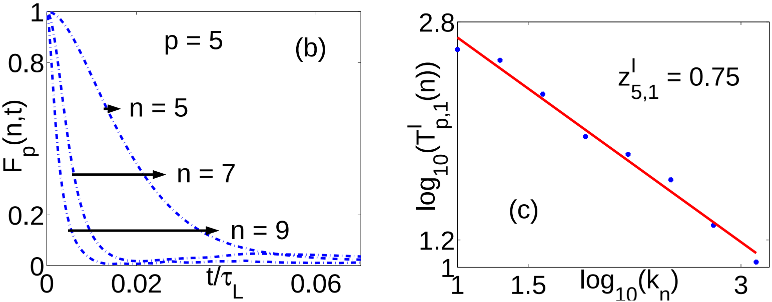
$$\begin{aligned} \mathcal{T}^{I}_{p,1}(\ell) &\equiv \frac{1}{\mathcal{S}_p(\ell)} \int_0^\infty \mathcal{F}_p(\ell, t) dt \\ &\propto \frac{1}{\mathcal{S}_p(\ell)} \int_{\mathcal{I}} d\mu(h) (\frac{\ell}{L})^{\mathcal{Z}(h)} \int_0^\infty dt \mathcal{G}^{p,h}(\frac{t}{\tau_{p,h}}) \\ &\propto \left[\frac{1}{\mathcal{S}_p(\ell)} \int_{\mathcal{I}} d\mu(h) (\frac{\ell}{L})^{ph+3-D(h)} \ell^{1-h} \right] \end{aligned}$$

In the last step, we have used :
$\tau_{p,h} \sim \ell/\delta v(\ell) \sim \ell^{1-h}$

Summary

- Dynamic multiscaling is expected to be observed to Lagrangian or quasi-Lagrangian dynamic structure functions.
- There is no scale invariance of the whole dynamic structure function, scaling of a particular time-scale depends on how you have defined that time-scale.
- Dynamic scaling exponents can be related to the equal-time exponents by bridge relations.

Evidence from shell models



- Needs very long avearaging.
- Several very serious difficulties with present computing power to do similar study in DNS of fluid turbulence.

Passive scalar model

- Using the multifractal model for passive-scalar (not white-in-time) we can derive bridge relations for the passive-scalar dynamic exponents.

$$
\begin{aligned}
z_{p,2}^{D,\theta} &= 1 - \zeta_2^u/2 \\
z_{p,2}^{I,\theta} &= 1 - \zeta_{-1}^u
\end{aligned}
$$

- But we need to assume certain correlation between the velocity and the passive scalar flux.
- These predictions are expected to work for Lagrangian or quasi-Lagrangian representation for the passive-scalar.

Dynamic multiscaling in decaying turbulence

- In critical phenomenon, dynamics can be monitored by:
  - Disturbing the system away from equilibrium and looking at its approach to equilibrium.
  - Measuring time-dependent correlation function in equilibrium.
  - And these two approaches gives the same dynamic exponent.
- The analog in turbulence would be to compare dynamic exponents in forced and decaying turbulence. There is no a-priori reason why they should be equal.
- There are two ways to study decay:
  - Get the system to a non-equilibrium stationary steady state and then turn off forcing at $t = t_0$ and observe the subsequent evolution.
  - Begin from an initial state with most of the energy in large-scales, then as the system decays the cascade is completed at time $t = t_0$, observe the subsequent evolution.
- We get the same exponents from both of these methods.

Decaying dynamic structure functions

- The simplest possible form (factorisation)

$$\mathcal{F}_p^d(\ell, t_0, t) = g(t_0)\mathcal{F}_p^f(\ell, t)$$

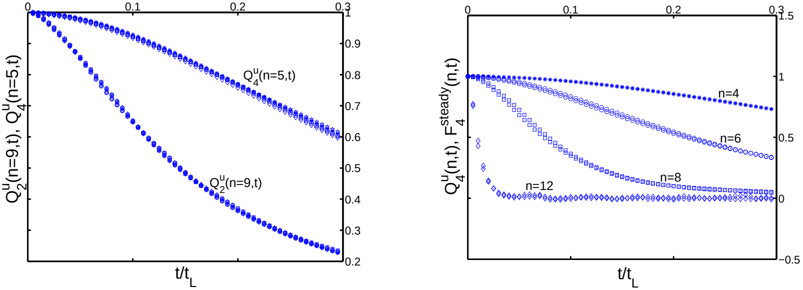
- We should then have

$$Q_p(\ell, t) \equiv \frac{\mathcal{F}_p^d(\ell, t_0, t)}{\mathcal{F}_p^d(\ell, t_0, 0)}$$

  to be independent of t_0.

- The dynamic multiscaling properties of $Q_p(\ell, t)$ is seen to be same as the dynamic multiscaling properties of $\mathcal{F}_p^f(\ell, t)$

Data from shell model



Summary

- There is strong dynamic univsersality in turbulence, at least in shell models of turbulence.
- Confirmed numerically in Kraichnan-shell model.
- Confirmed numerically in GOY shell model and passive-scalar shell model.
- The simplest factorisation ansatz seems to be correct.