INHOMOGENEOUS PASSIVE SCALARS: THE POINT-SOURCE PROBLEM

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Warwick @ Coventry, 17\7\2006

PASSIVE SCALAR TRANSPORT

Passive scalar field $\theta(\mathbf{x}, t)$

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Basic advection-diffusion forced equation:

$$\partial_t \theta(\mathbf{x}, t) + \mathbf{v}(\mathbf{x}, t) \cdot \boldsymbol{\partial} \theta(\mathbf{x}, t) = \kappa_0 \partial^2 \theta(\mathbf{x}, t) + f(\mathbf{x}, t)$$

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two-point equal-time
CORRELATION FUNCTION:

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2. Expansion on spherical harmonics for $\mathbf{r}$:

$$\hat{C}(\mathbf{r}, \mathbf{q}) \mapsto \hat{C}_{l,m}(r, \mathbf{q})$$

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Point source:

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INHOMOGENEOUS SOLUTION

Inertial–convective range $r < L_f$:

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presence of Bessel functions $\mathcal{I}$

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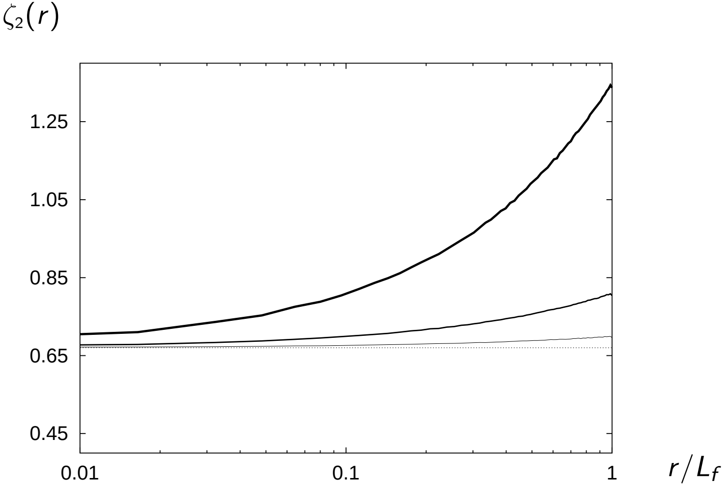
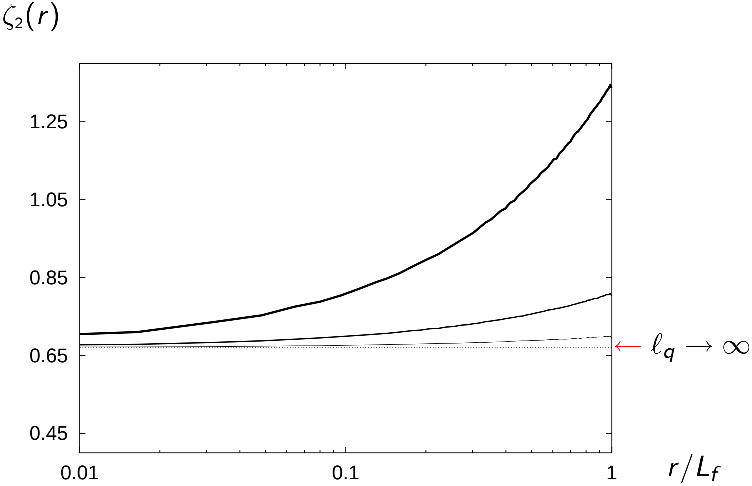
presence of Bessel functions $\mathcal{I}$

$\Rightarrow$ absence of a unique scaling exponent

$\Rightarrow$ superposition of different power laws


$\zeta_2(r)$
1.25
1.05
0.85
0.65
0.45
0.01
0.1
1
r/L_f



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1.25
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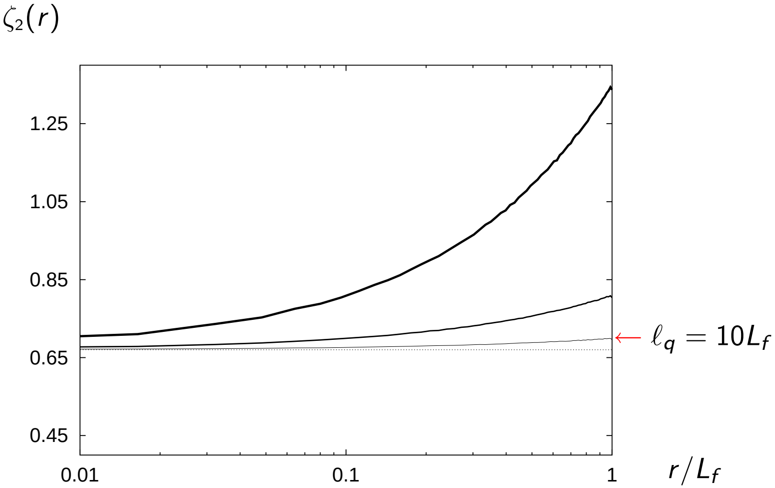
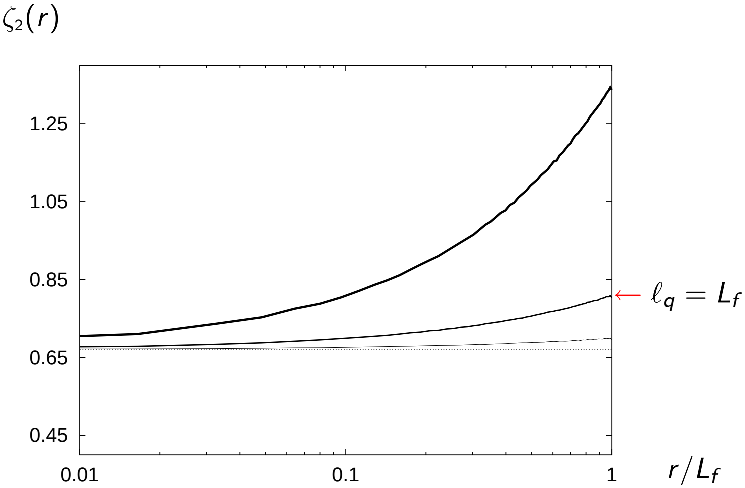
$\leftarrow \ell_q \to \infty$


$\zeta_2(r)$
1.25
1.05
0.85
0.65
0.45
0.01
0.1
1
r/L_f
$\leftarrow \ell_q = 10L_f$



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1.25
1.05
0.85
0.65
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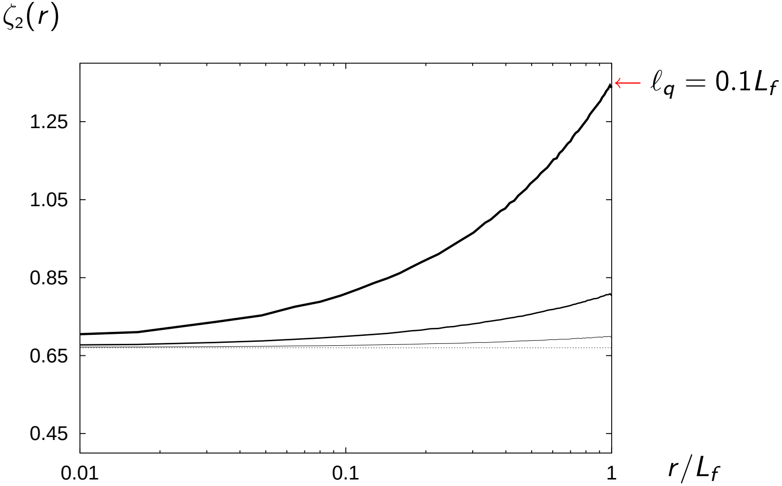
r/L_f
$\ell_q = L_f$



$\zeta_2(r)$
1.25
1.05
0.85
0.65
0.45
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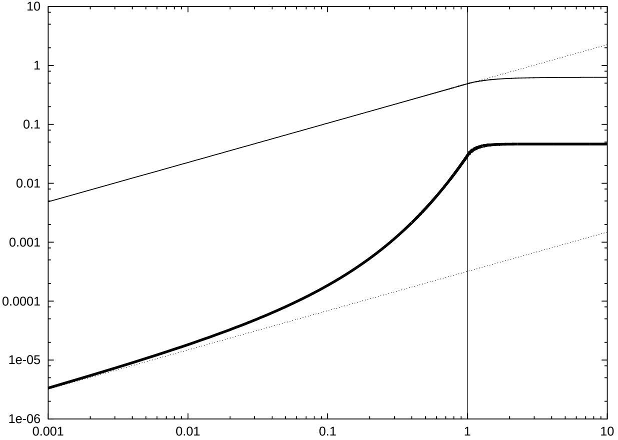
1
r/L_f
$\ell_q = 0.1L_f$

$\hat{S}(r; \ell_q)$



r/L_f

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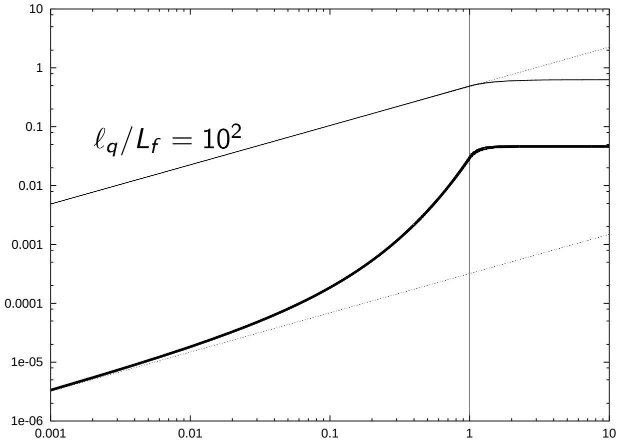
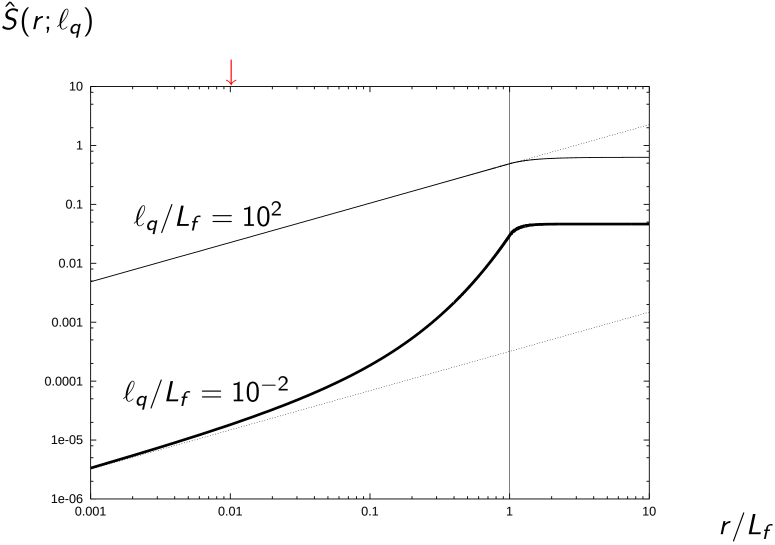


r/L_f


$\hat{S}(r;\ell_q)$
10
1
0.1
0.01
0.001
0.0001
1e-05
1e-06
0.001
0.01
0.1
1
10
$\ell_q/L_f = 10^2$
$\ell_q/L_f = 10^{-2}$
r/L_f

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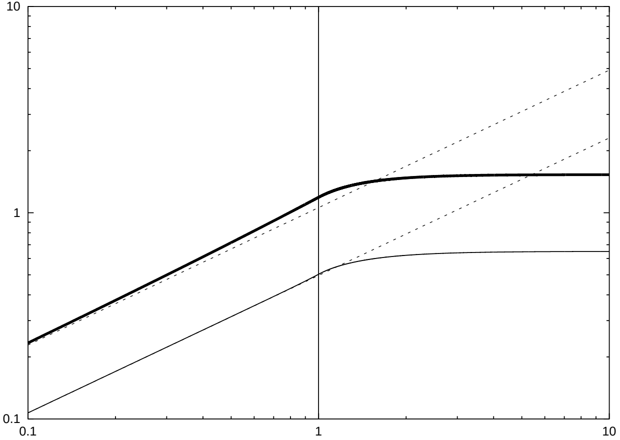

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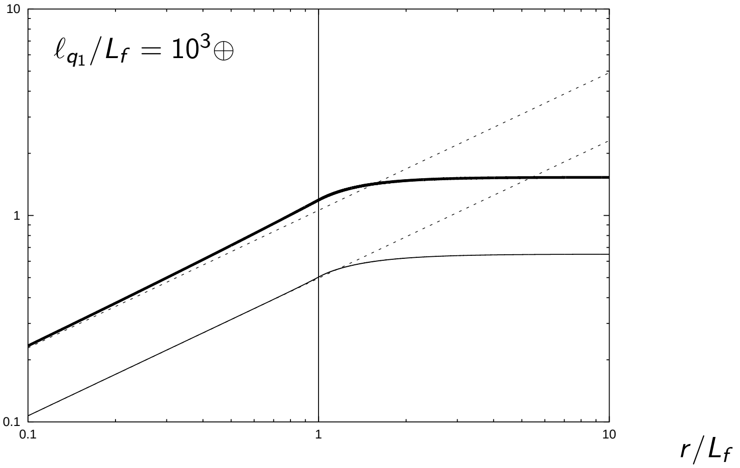

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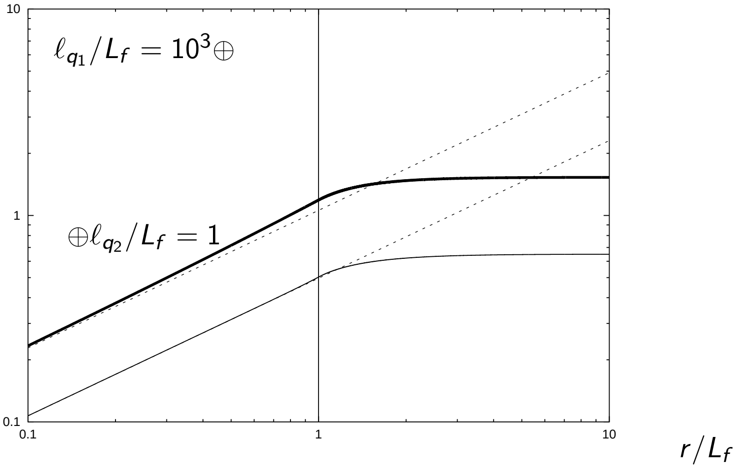

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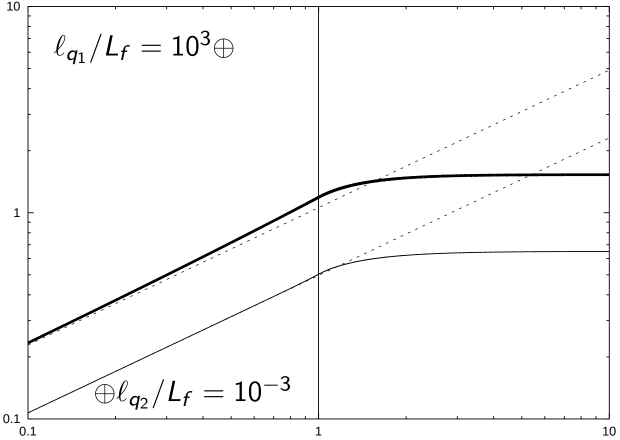

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- forcing with continuum spectrum

RANDOM POINT SOURCE

Random emission or absorption in the origin:

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$$\begin{cases} \langle f(\mathbf{x}, t)\rangle = 0 \\ \langle f(\mathbf{x}, t) f(\mathbf{x}', t')\rangle = \delta(t - t')\delta(\mathbf{x})\delta(\mathbf{x}')F_0 \end{cases}$$

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$$s = \left(\frac{z}{r}\right)^2 \left(\frac{r}{\mathcal{L}_v}\right)^\xi$$

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RANDOM POINT SOURCE

Random emission or absorption in the origin:

$$\begin{cases} \langle f(\mathbf{x},t)\rangle = 0 \\ \langle f(\mathbf{x},t)f(\mathbf{x}',t')\rangle = \delta(t-t')\underbrace{\delta(\mathbf{x})\delta(\mathbf{x}')}_{\delta(\mathbf{r})\delta(\mathbf{z})\,\mapsto\,\delta(\mathbf{r})\,\mapsto\,\Theta}F_0 \end{cases}$$

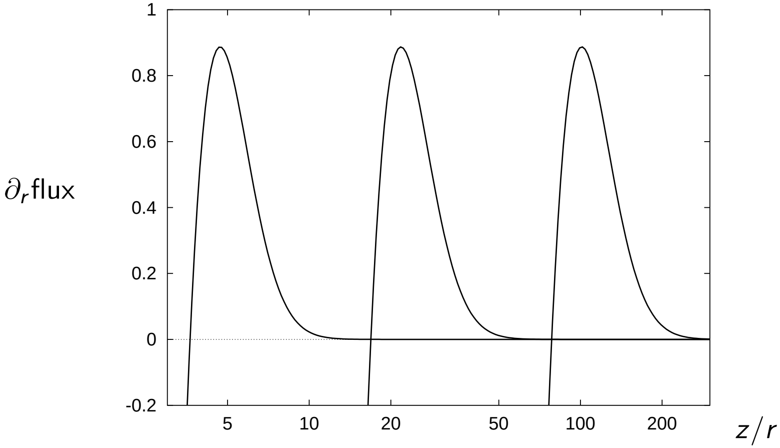
Unforced equation ($r > L_f \to 0$):

$$\hat{C}(r;\ell_q) \sim G(\ell_q)\mathcal{K}(r) \xrightarrow{(\ell_q \gg r)} \gamma r^{2-d-\xi}$$

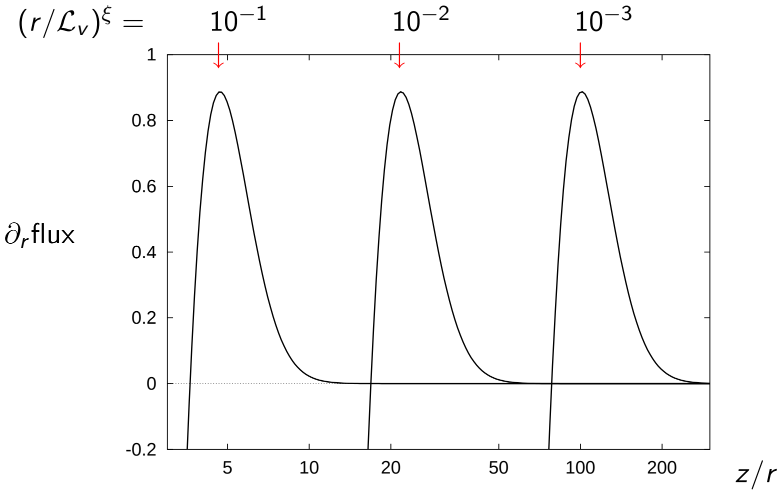
$$\overset{(d=3)}{\Longrightarrow} C(r,z) \sim \begin{cases} r^{-4+\xi/2} & \text{(small } s) \\ z^{-(8-\xi)/(2-\xi)}\left[1 + \mathrm{O}\left(r^{2-\xi}\right)\right] & \text{(large } s) \end{cases}$$

$$s = \left(\frac{z}{r}\right)^2 \left(\frac{r}{\mathcal{L}_v}\right)^\xi$$

$\Rightarrow$ PSEUDO-RECOVERY OF HOMOGENEITY AT SMALL SCALES



∂_rflux
1
0.8
0.6
0.4
0.2
0
-0.2
5
10
20
50
100
200
z/r



$(r/\mathcal{L}_v)^\xi =$
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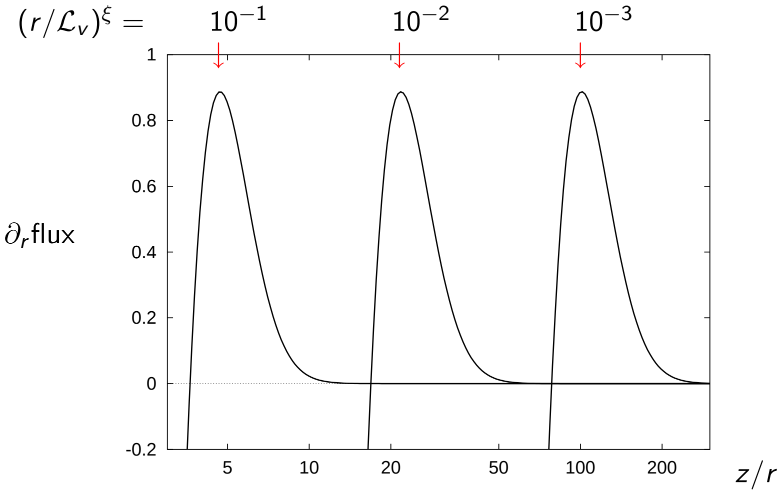

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