Statistical mechanics of axisymmetric flows

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Outline

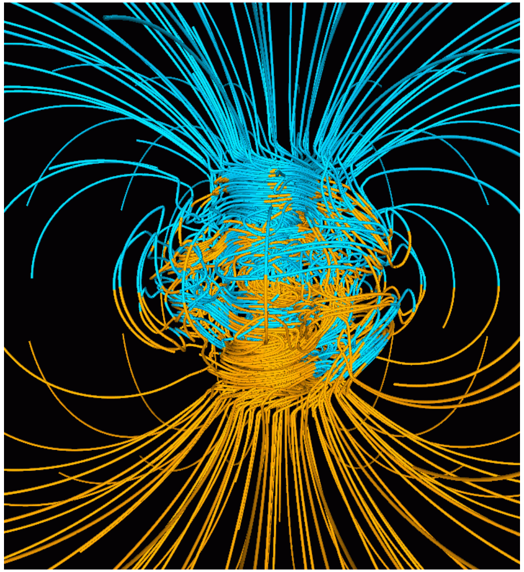
Problem : non-linearity and closure (I)

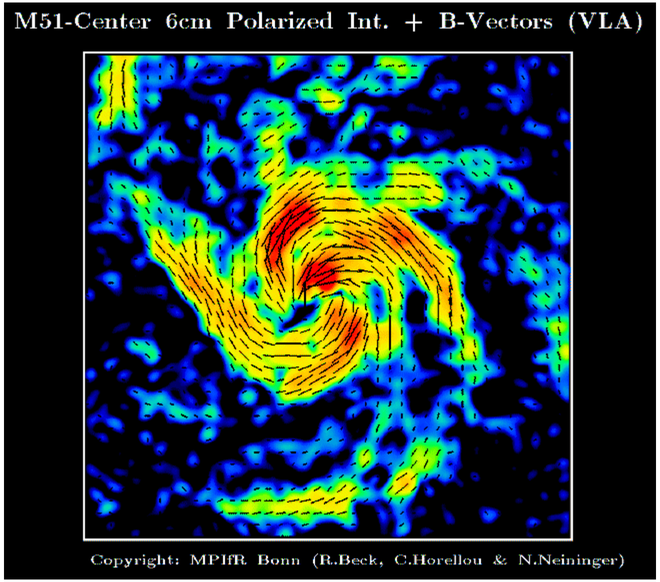
$$\frac{\partial \mathbf{V}}{\partial t} = -\mathbf{V} \cdot \nabla \mathbf{V} + \mathbf{J} \times \mathbf{B} - \nabla p + \nu \nabla^2 \mathbf{V}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}$$

Behaviour of a large scale field

$$\bar{V}(x,t) \quad \text{and} \quad \bar{B}(x,t)$$





Problem : non-linearity and closure (II)

Real space formalism :

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} = -\nabla p + \nu \nabla^2 \mathbf{V}$$

Behaviour of $\bar{V}$? Problem with non-linear terms

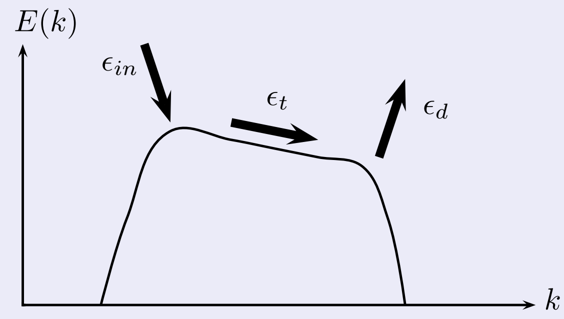
$$\frac{\partial \overline{V}}{\partial t} = L(\overline{V}) + N(\overline{v'v'})$$

$$\frac{\partial \overline{v'v'}}{\partial t} = L(\overline{v'v'}, \overline{V}) + N(\overline{v'v'v'})$$

...

Fourier space translation :

Energy Cascade



Huge number of scales : N $\propto R_e^{9/4} \sim 10^{13}$

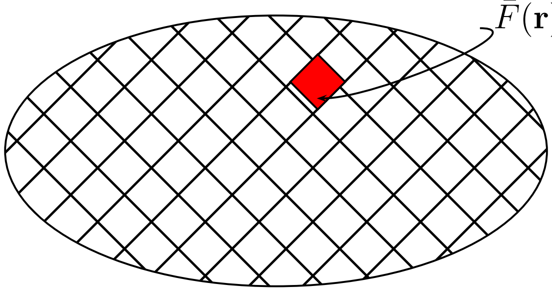
Necessity for a closure assumption :

$$\frac{\partial \bar{\mathbf{V}}}{\partial t} = -\bar{\mathbf{V}} \cdot \nabla \bar{\mathbf{V}} + \bar{\mathbf{J}} \times \bar{\mathbf{B}} - \nabla \bar{p} + \nu \nabla^2 \bar{\mathbf{V}} - \nabla \cdot (\mathcal{R} - \mathcal{M})$$

$$\frac{\partial \bar{\mathbf{B}}}{\partial t} = \nabla \times (\bar{\mathbf{V}} \times \bar{\mathbf{B}}) + \eta \nabla^2 \bar{\mathbf{B}} + \nabla \times \mathcal{E}$$

- Reynolds stress : $\mathcal{R} = \overline{\mathbf{u}'\mathbf{u}'}$
- Maxwell stress : $\mathcal{M} = \overline{\mathbf{b}'\mathbf{b}'}$
- Electromotive force : $\mathcal{E} = \overline{\mathbf{u}' \times \mathbf{b}'}$

Statistical mechanics



Micro-states : $(\mathbf{V},\mathbf{B})(\mathbf{r},t)$

Macro-states : $\bar{F}(r,t) = \int F(r,u,b)\rho(r,u,b)\,dudb$

$$S[\rho] = -\int \rho(r,u,b)\ln[\rho(r,u,b)]\,drdudb$$

Self organisation

- Maximum entropy principle under constraints : Integrals of motion must keep their initial value ($t=0$).
- A majority of the possible micro-states will be close to this macroscopic state.
- Dynamical equations (NS + Ind.) are supposed to induce a mixing in phase space (ergodicity)

Mathematical procedure :

$$\delta S + \delta C = 0$$

$$\Downarrow$$

Gibbs state : $\rho^*(u,b)$

$\Rightarrow$ mean (coarse-grained) fields

(Short) Historical overview :

- Kraichnan and Montgomery (1980) : Self organisation of (MHD) turbulence

$$
\begin{aligned}
E &= \frac{1}{2}\int (v^2 + B^2)\, d^3x \\
H_m &= \frac{1}{2}\int \mathbf{A}\cdot\mathbf{B}\, d^3x \\
H_c &= \frac{1}{2}\int \mathbf{v}\cdot\mathbf{B}\, d^3x
\end{aligned}
$$

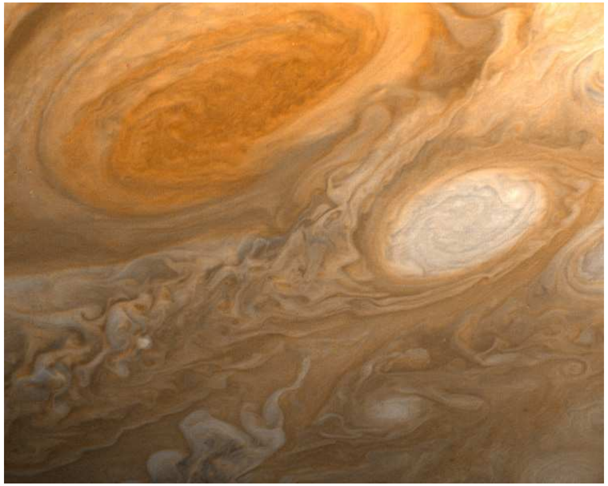
Gibbs state : $\rho = Z^{-1}\exp[-\beta E - \alpha H_m - \gamma H_c]$

obtained after truncation in Fourier space (Liouville's theorem and entropy definition)

$\Longrightarrow$ Only quadratic invariants are retained

- Miller *et al* (1991), Robert and Sommeria (1991)
  - Statistical mechanics of continous field
  - Coherent structures in 2D turbulence
- Jordan and Turkington (1997)

2D MHD



But, in 2D, no vorticity stretching by velocity gradients !

Axisymmetric MHD

(Phys. Rev. E **71**, 036311, 2005)

Basic equations

$$\begin{aligned}\frac{\partial \mathbf{V}}{\partial t} &= -\mathbf{V}\cdot\nabla\mathbf{V} + \mathbf{J}\times\mathbf{B} - \nabla p\\ \frac{\partial \mathbf{B}}{\partial t} &= \nabla\times(\mathbf{V}\times\mathbf{B})\\ 0 &= \nabla\cdot\mathbf{V}\\ 0 &= \nabla\cdot\mathbf{B}\end{aligned}$$

Toroidal/Poloidal decomposition :

$$\begin{aligned}\mathbf{U} &= \mathbf{U_p} + \mathbf{U_t} = \nabla\times(\frac{\psi}{r}\,\mathbf{e}_\theta) + U\,\mathbf{e}_\theta\\ \mathbf{B} &= \mathbf{B_p} + \mathbf{B_t} = \nabla\times(A\,\mathbf{e}_\theta) + B\,\mathbf{e}_\theta\end{aligned}$$

$\Rightarrow$ **4 Scalar Equations**

$$\sigma_u = rU \qquad \xi_u = \frac{\omega}{r} = -\Delta_*\psi$$

$$\sigma_b = rA \qquad \xi_b = \frac{B}{r}$$

$$\begin{aligned}\partial_t\sigma_u + \{\psi,\sigma_u\} &= \{\sigma_b, 2y\xi_b\}\\ \partial_t\xi_u + \{\psi,\xi_u\} &= \partial_z(\frac{\sigma_u^2}{4y^2} - \xi_b^2) - \{\sigma_b, \Delta_*\sigma_b\}\\ \partial_t\xi_b + \{\psi,\xi_b\} &= \{\sigma_b, \frac{\sigma_u}{2y}\}\\ \partial_t\sigma_b + \{\psi,\sigma_b\} &= 0\end{aligned}$$

Definitions :

$$\begin{aligned}y &= \frac{r^2}{2}\\ \{f,g\} &= \partial_y f\,\partial_z g - \partial_z f\,\partial_y g\\ \Delta_* &= \frac{\partial^2}{\partial y^2} + \frac{1}{2y}\frac{\partial^2}{\partial z^2}\end{aligned}$$

Conserved quantity :

$$\partial_t(rA) + \mathbf{U_p}\cdot\nabla(rA) = 0$$

$\Longrightarrow$ An infinite set of constraints

Integrals of motion (I)

Complete set : Woltjer (1959)

- Energy :

$$E = \frac{1}{2}\int (\mathbf{V}^2 + \mathbf{B}^2) rdrdz$$

- Casimirs :

$$I = \int C(\sigma_b) dydz$$

- Generalised magnetic Helicity :

$$H_m = \int \mathbf{A}\cdot\mathbf{B} rdrdz = 2\int ABrdrdz = 2\int \xi_b \sigma_b dydz \quad \Longrightarrow \quad H_m = 2\int \xi_b N(\sigma_b) dydz$$

- Generalised cross Helicity :

$$H_c = \int \mathbf{V}\cdot\mathbf{B} rdrdz = 2\int (\xi_u \sigma_b + \sigma_u \xi_b) dydz \quad \Longrightarrow \quad H_c = 2\int (\xi_u F(\sigma_b) + \sigma_u \xi_b F'(\sigma_b)) dydz$$

- Generalised Angular momentum :

$$L = \int (rU) rdrdz = \int \sigma_u dydz \quad \Longrightarrow \quad L = \int \sigma_u F(\sigma_b) dydz$$

Integrals of motion (II)

Linear and quadratic invariants : Chandrasekhar (1958)

- Energy :

$$E = \frac{1}{2}\int (\mathbf{V}^2 + \mathbf{B}^2) r dr dz$$

- Casimirs :

$$I = \int C(\sigma_b) dy dz$$

- Generalised magnetic Helicity :

$$H_m = \int \mathbf{A} \cdot \mathbf{B} r dr dz = 2\int \xi_b \sigma_b dy dz \qquad H_m' = 2\int \xi_b dy dz = 2\int \frac{B}{r} r dr dz$$

- Generalised cross Helicity :

$$H_c = \int \mathbf{V} \cdot \mathbf{B} r dr dz = 2\int (\xi_u \sigma_b + \sigma_u \xi_b) dy dz$$

- Generalised Angular momentum :

$$L' = \int (rU) r dr dz = \int \sigma_u dy dz \qquad L = \int \sigma_u \sigma_b dy dz$$

Maximum Entropy state (I)

Local distribution of velocity and magnetic field : $\rho(\mathbf{x},\mathbf{u},\mathbf{b}) \quad \Rightarrow \quad \begin{cases} \bar{V} = \int \mathbf{u}\rho(\mathbf{x},\mathbf{u},\mathbf{b})d\mathbf{u}d\mathbf{b} \\ \bar{B} = \int \mathbf{b}\rho(\mathbf{x},\mathbf{u},\mathbf{b})d\mathbf{u}d\mathbf{b} \end{cases}$

Mixing entropy : $S[\rho] = -\int \rho(\mathbf{x},\mathbf{u},\mathbf{b}) \ln[\rho(\mathbf{x},\mathbf{u},\mathbf{b})]d\mathbf{x}d\mathbf{u}d\mathbf{b}$

Maximisation under constraints : $\bar{C}[u,b] = C_0$

Robust constraints :

$$\bar{C}[U,B] = C[\bar{U},\bar{B}]$$

$\Rightarrow$ Integrals calculated with coarsed grained-fields are conserved

Fragile constraints :

$$\bar{C}[U,B] \neq C[\bar{U},\bar{B}]$$

$\Rightarrow$ Part of Integrals of motions can go into fluctuations

ψ, A are integrated from $\mathbf{U},\mathbf{B}$ (Jordan & Turkington, 1997) $\Rightarrow$ Fluctuations are negligible in the thermodynamics limit $A = \bar{A}$ and $\psi = \bar{\psi}$

$$\bar{I} = \int C(r\bar{A})dydz$$

$$\bar{H}_m = 2\int \bar{A}\bar{B}dydz\,, \qquad \bar{H}'_m = 2\int \frac{\bar{B}}{r}rdrdz$$

$$\bar{L}' = \int r\bar{U}rdrdz\,, \qquad \bar{L} = \int r^2\bar{U}\bar{A}dydz$$

$$\bar{E} = \frac{1}{2}\int(\mathbf{u}^2+\mathbf{b}^2)\rho(\mathbf{x},\mathbf{u},\mathbf{b})rdrdz$$

$$\bar{H}_c = \int \mathbf{u}\cdot\mathbf{b}\;\rho(\mathbf{x},\mathbf{u},\mathbf{b})rdrdz$$

Maximum Entropy state (II)

$$\delta\bar{S} - \beta\delta\bar{E} - \mu_m\delta\bar{H}_m - \mu_c\delta\bar{H}_c - \sum_{n=1}^{+\infty}\alpha^{(n)}\delta\bar{I}^{(n)} - \mu'_m\delta\bar{H}'_m - \gamma\delta\bar{L} - \gamma'\delta\bar{L}' = 0$$

Robust constraints :

$$\bar{I} = \int C(r\bar{A})dydz$$

$$\bar{H}_m = 2\int \bar{A}\bar{B}dydz\,, \qquad \bar{H}'_m = 2\int \frac{\bar{B}}{r}rdrdz$$

$$\bar{L}' = \int r\bar{U}rdrdz\,, \qquad \bar{L} = \int r^2\bar{U}\bar{A}dydz$$

$\Downarrow$

Plays a role only for the mean field

Fragile constraints :

$$\bar{E} = \frac{1}{2}\int(\mathbf{u}^2 + \mathbf{b}^2)\rho(\mathbf{x}, \mathbf{u}, \mathbf{b})rdrdz$$

$$\bar{H}_c = \int \mathbf{u}\cdot\mathbf{b}\ \rho(\mathbf{x}, \mathbf{u}, \mathbf{b})rdrdz$$

$\Downarrow$

Plays a role for mean field and fluctuations

$$\begin{cases}\mathbf{u} = \bar{\mathbf{U}} + \mathbf{u}' \\ \mathbf{b} = \bar{\mathbf{B}} + \mathbf{b}'\end{cases} \Rightarrow \rho(\mathbf{u}, \mathbf{b}) = M(\bar{\mathbf{U}}, \bar{\mathbf{B}}) + F(\mathbf{u}', \mathbf{b}')$$

Mean Field Results

$$\delta\bar{S} - \beta\delta\bar{E} - \mu_m\delta\bar{H}_m - \mu_c\delta\bar{H}_c - \sum_{n=1}^{+\infty}\alpha^{(n)}\delta\bar{I}^{(n)} - \mu'_m\delta\bar{H}'_m - \gamma\delta\bar{L} - \gamma'\delta\bar{L}' = 0$$

$$\Rightarrow \begin{cases} \beta\mathbf{U}_P + \mu_c\mathbf{B}_P = \mathbf{0} \\ \beta U + \mu_c B + \gamma' r + \gamma A r^2 = 0 \\ \beta B + 2\mu_m A + \mu_c U + \frac{2\mu'_m}{r} = 0 \\ \beta\mathbf{B}_P + 2\mu_m\mathbf{A}_P + \mu_c\mathbf{U}_P + \mathbf{curl}^{-1}[rC'(rA)] + \gamma\,\mathbf{curl}^{-1}(r^2U) = \mathbf{0} \end{cases}$$

The mean fields are stationary solutions of the MHD equations

(No Reynolds stress)

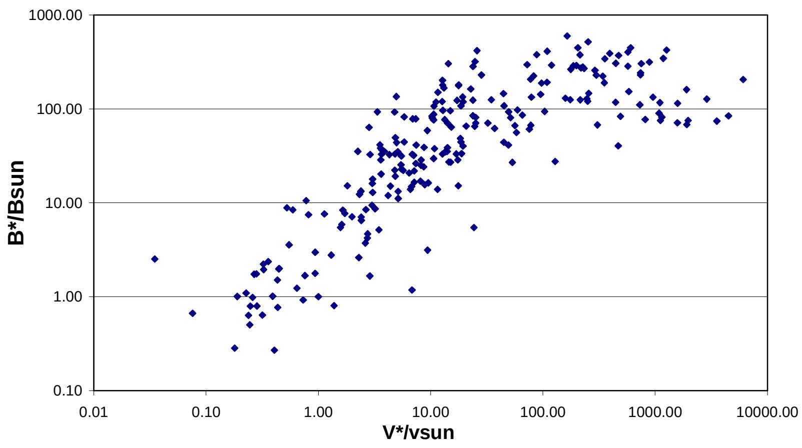
$\mathbf{B} - \mathbf{V}$ relation :

- **Classical case**

$$\beta\bar{\mathbf{U}} = -\mu_c\bar{\mathbf{B}}$$

- **Rotating case**

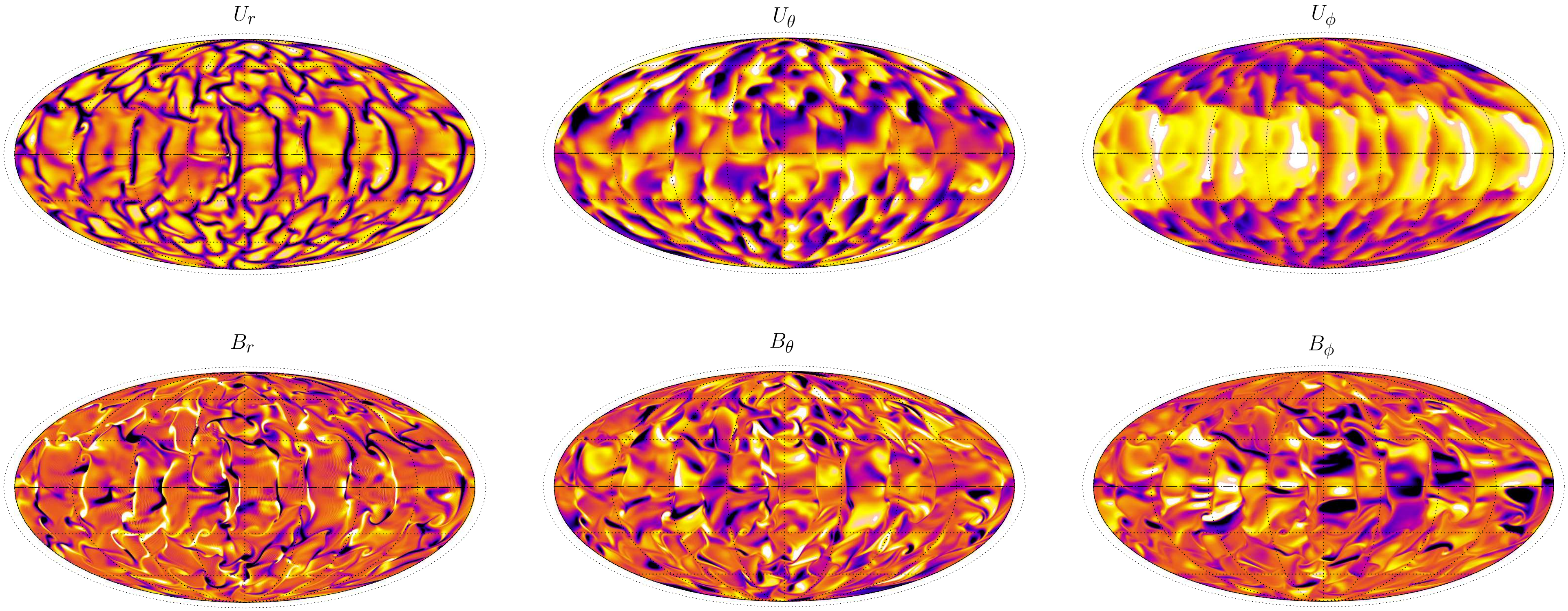
$$\begin{aligned} \beta\bar{\mathbf{U}}_p &= -\mu_c\bar{\mathbf{B}}_p \\ \beta(\bar{U} + \frac{\gamma'}{\beta}r) &= -\mu_c\bar{B} - \gamma\bar{A}r^2 \end{aligned}$$



Some pictures

Numerical simulation of the sun's interior by A. S. Brun

Radial Fields | Latitudinal Fields | Azimuthal Fields



Fluctuations

<u>**Distributions of fluctuations :**</u> Only affected by fragile constraints

$$\bar{E} = \frac{1}{2}\int (\mathbf{u}^2 + \mathbf{b}^2)\rho(\mathbf{x},\mathbf{u},\mathbf{b})rdrdz$$
$$\bar{H}_c = \int \mathbf{u}\cdot\mathbf{b}\ \rho(\mathbf{x},\mathbf{u},\mathbf{b})rdrdz$$

$$\Rightarrow \rho = \frac{1}{Z}\exp\left\{-\frac{\beta}{2}(u'^2+b'^2)-\mu_c\mathbf{u}'\cdot\mathbf{b}'\right\} \quad \left(\begin{cases}\mathbf{u} = \bar{\mathbf{U}} + \mathbf{u}' \\ \mathbf{B} = \bar{\mathbf{B}} + \mathbf{b}'\end{cases}\right)$$

Universal Gaussian Fluctuations (Jordan & Turkington, 1997)

<u>**Fluctuating part of the Fragile constraints**</u>

$$\bar{E} = \frac{1}{2}\int (\bar{\mathbf{U}}^2 + \bar{\mathbf{B}}^2)rdrdz + \frac{1}{2}\int (\overline{\mathbf{u}'^2} + \overline{\mathbf{b}'^2})\rho(\mathbf{u}',\mathbf{b}')rdrdz = E^0 + E'$$
$$\bar{H}_c = \int \bar{\mathbf{U}}\cdot\bar{\mathbf{B}}\ rdrdz + \int \overline{\mathbf{u}'\cdot\mathbf{b}'}\ \rho(\mathbf{u}',\mathbf{b}')rdrdz = H_c^0 + H_c'$$

$$\frac{E'_M}{E^0_M} = \frac{H'_c}{H^0_c} < \frac{E'_k}{E^0_k}$$

Solutions = minima of (Kinetic) energy at fixed I, H_m, H_c and L
(Energy is the fastest decaying invariant in presence of a small viscosity)

Axisymmetric HD

(Phys. Rev. E **73**, 046308, 2006)

Equations

$$\sigma = rU \qquad \xi = \frac{\omega}{r} = -\Delta_* \psi$$

$$\begin{cases} \partial_t \sigma + \{\psi, \sigma\} = 0 \\ \partial_t \xi + \{\psi, \xi\} = \partial_z \left(\frac{\sigma^2}{4y^2} \right) \end{cases}$$

Conserved quantity :

$$\partial_t (rU) + \mathbf{U_p} \cdot \nabla (rU) = 0$$

$\Longrightarrow$ An infinite set of constraints

Stationary solutions :

$$\partial_t \sigma = \partial_t \xi = 0 \qquad \Rightarrow \qquad \begin{aligned} \sigma &= f(\psi) \\ \xi &= -\Delta_* \psi = \frac{f(\psi)}{2y} f'(\psi) + g(\psi) \end{aligned}$$

Constraints :

$$\begin{aligned} E &= \frac{1}{2} \int \mathbf{U}^2 d\mathbf{x} \\ I &= \int G(\sigma) d\mathbf{x} \\ H_K &= \int \xi F(\sigma) d\mathbf{x} \end{aligned}$$

Nonlinear Dynamical stability : $\delta(E + I + H_K) = 0$

$$\Rightarrow \begin{cases} \psi + F(\sigma) = 0 \\ \frac{\sigma}{2y} + G'(\sigma) + \xi F'(\sigma) = 0 \end{cases}$$

Straightforward approach

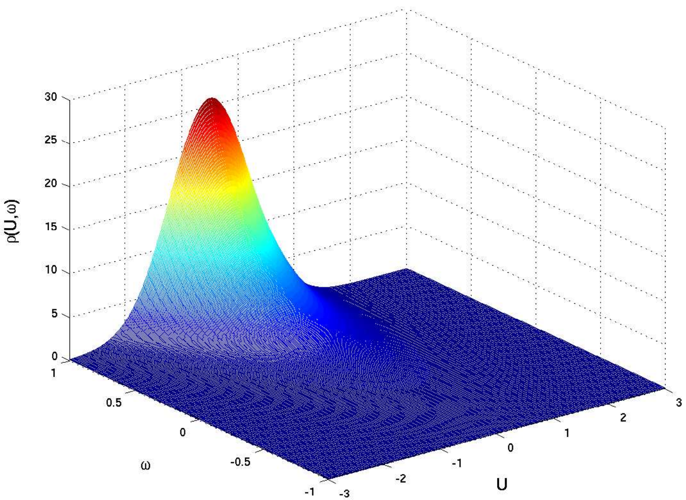
Max. of $\rho(U,\omega)$ under the constraints :

$$\bar{E} = \frac{1}{2}\int \mathbf{U}^2 \rho(U,\omega) dU d\omega d\mathbf{x}$$
$$\bar{I} = \int G(\sigma) \rho(U,\omega) dU d\omega d\mathbf{x}$$
$$\bar{H}_K = \int F(rU)\frac{\omega}{r} \rho(U,\omega) dU d\omega d\mathbf{x}$$

Casimirs are Fragile constraints

$\Rightarrow$ Non-universal distributions $\rho(U,\omega)$

$$\rho(U,\omega) \propto \exp\left[-\frac{\beta U^2}{2} - G(rU) - \{\beta\psi + \mu F(rU)\}\frac{\omega}{r}\right]$$



$\rightarrow$ There must be a bound on vorticity : $\omega < \omega^*$

True for axisymmetric equation (Cowling, 1934)

But not in the general case

Alternative formulation

Assumptions :

- The fluctuations of $\xi = \omega/r$ are neglected

$$\begin{cases} \xi = \bar{\xi} \\ \rho(U,\omega) = \rho(\sigma) \end{cases}$$

- Vorticity tends to zero at long time
  $\Rightarrow$ Energy is approximately conserved on the coarse-grained scale
  $(\dot{E} = -\nu \int \omega^2 d\mathbf{x})$

Maximisation procedure :

$$\delta S - \beta\delta\bar{E} - \mu\delta\bar{H}_K - \gamma\delta\bar{\Gamma} - \alpha\delta\bar{I}_1 - \sum_{n>1} \delta\bar{I}_n = 0$$

with :

$$\begin{aligned} \bar{E} &= \frac{1}{2}\int \xi\psi \, d\mathbf{x} + \int \frac{\bar{\sigma}^2}{4y} \, d\mathbf{x} \\ \bar{H}_K &= \int \xi\bar{\sigma} \, d\mathbf{x} \qquad \Gamma = \int \xi \, d\mathbf{x} \\ \bar{I}_n &= \int \sigma^n \rho(\sigma) \, d\rho d\mathbf{x} \end{aligned}$$

$$\begin{cases} \beta\psi = -\mu\bar{\sigma} - \gamma \\ \rho(\sigma) \propto \chi(\sigma) e^{-\left[\frac{\beta\bar{\sigma}}{2y} + \mu\xi + \alpha\right]\sigma} \end{cases}$$

where : $\chi(\sigma) = \exp\left[-\sum_{n>1} \alpha_n \sigma^n\right]$

$\Rightarrow$ Non universal distribution of the fluctuations of σ due to the Casimirs (fragile constraints)

Equilibrium States

$$\begin{cases} \beta\psi = -\mu\bar{\sigma} - \gamma \\ \rho(\sigma) = \frac{1}{Z}\chi(\sigma)e^{-\left[\frac{\beta\bar{\sigma}}{2y}+\mu\xi+\alpha\right]\sigma} = \frac{1}{Z}\chi(\sigma)e^{-\Psi\sigma} \end{cases}$$

$$Z(\Psi) = \int_{-\infty}^{+\infty} \chi(\sigma)e^{-\Psi\sigma}d\sigma$$

where $\Psi = \beta\frac{\bar{\sigma}}{2y} + \mu\xi + \alpha$

Mean Field

$$\bar{\sigma} = -\frac{\partial \ln Z}{\partial \Psi} \equiv F(\Psi)$$

$$\Rightarrow \begin{cases} \Psi = \beta\frac{\bar{\sigma}}{2y} + \mu\xi + \alpha = F^{-1}(\sigma) \\ \beta\psi = -\mu\bar{\sigma} - \gamma \end{cases}$$

Mean fields are stationary solutions of the axisymmetric equations

Most probable value $\langle\sigma\rangle$

Obtained for maximum of

$$\mathcal{F}(\sigma) = -\Psi\sigma + \ln\chi(\sigma)$$

$$\Rightarrow \begin{cases} \langle\sigma\rangle = [(\ln\chi)'^{-1} = G(\Psi) \\ \beta\psi = -\mu\bar{\sigma} - \gamma \end{cases}$$

The most probable field is not a stationary solution

(Rk : $\langle\sigma\rangle = \bar{\sigma}$ only for Gaussian fluctuations)

Conclusion

MHD axisymmetric flows

- Density probability $\rho(\mathbf{u}, \mathbf{b})$
- Mean fields are stationary solutions of MHD equations
- Gaussian Universal fluctuations
- Principle of minimum energy state

HD axisymmetric flows

- Density probability $\rho(\sigma)$ ($\sigma = rU$)
  - Non universal distribution
  - Mean fields are stationary solutions of HD equations
- Density probability $\rho(\mathbf{u})$
  - Mean fields are not stationary solutions (Reynolds stress)
  - Non universal distribution
  - Vorticity has to be bounded

Route to statistical mechanics of 3D turbulence

2D		Axisymmetry (2D and 1/2)		3D
Coherent structure (controls dynamics)	$\longrightarrow$	Stretching term in the vorticity equation But bounded !	$\longrightarrow$	Singularity for vanishing viscosity ? Weak Solutions ?