Turbulent spectra generated by singularities

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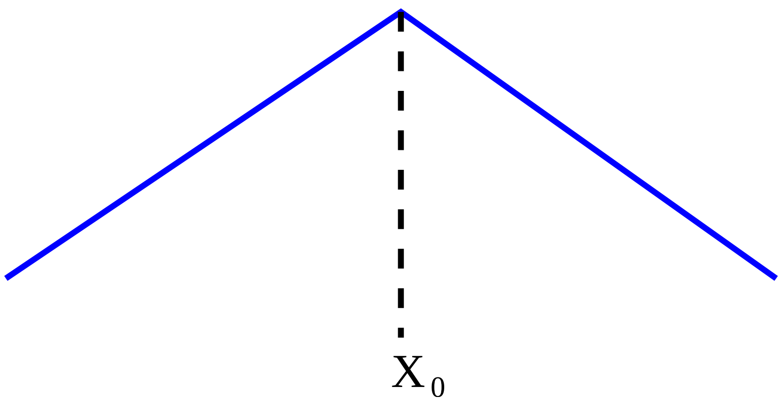
OUTLINE

- Motivation: collapses and turbulent spectra
- Turbulence of water (gravity) waves
- 2D hydrodynamic turbulence at $Re \gg 1$
- Numerical experiment
- Acoustic turbulence

Motivation: collapses and turbulent spectra

It is well known that singularities give the power type behavior of the Fourier amplitudes that provides appearance of power tails for turbulent spectra.

- (1958) Phillips spectrum for gravity waves on the fluid surface. Surface singularities are wedges:



$\Leftarrow \; z = \eta(x,t) \Rightarrow$

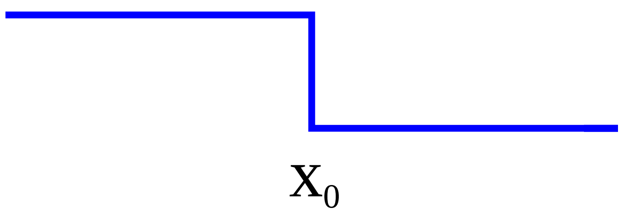
$\eta_{xx} \sim \delta(x - x_0)$

or $\eta_k \sim k^{-2}$.

Hence, according to Phillips, $E_k = 2\pi k \cdot g\langle|\eta_k|^2\rangle \sim k^{-3}$ or $E_\omega \sim \omega^{-5}$ where $\omega = \sqrt{gk}$ is assumed.

Motivation: collapses and turbulent spectra

- (1967) WT Zakharov-Filonenko spectrum: $E_\omega \sim P^{2/3}\omega^{-4}$, P the energy flux.
- (1967) Kraichnan spectrum for 2D hydrodynamic turbulence at $Re \gg 1$: $E_k \sim \eta^{2/3}k^{-3}$ where η is the enstrophy flux.
- (1971) Saffman spectrum: $E_k \sim k^{-4}$. It appears due to vorticity discontinuities which are observed in many numerics.
- (1973) Kadomtsev-Petviashvili spectrum. According to KP acoustic turbulence is a random set of shocks:



$$\rho_x \sim \delta(x - x_0),$$
$$\rho_k \sim k^{-1} \;\Rightarrow E_\omega \sim \omega^{-2}.$$

- (1951, unpublished) Burgers found this spectrum in 1D.

Turbulence of water waves

Phillips *implicitly* assumed the singularities are point like although they are distributed on the whole lines.

0D: The temporal autocorrelation function (at some spatial point) $K(\tau) = \langle \eta(t+\tau)\eta(t) \rangle$ gives the turbulent spectrum as its Fourier transform: $E_\omega = g \int_{-\infty}^{\infty} K(\tau) e^{i\omega\tau} d\tau$.

Assume that

$$\frac{\partial^2 \eta}{\partial t^2} = \sum_i \Gamma_i \delta(t - t_i) + \text{regular terms},$$

with random both Γ_i and t_i .

For the singular part the Fourer transform is

$$\eta_\omega = -\frac{1}{2\pi\omega^2} \sum_i \Gamma_i e^{-i\omega t_i}.$$

Turbulence of water waves

Hence after averaging we have

$$E_\omega = \frac{g}{2\pi T}\langle|\eta_\omega|^2\rangle = \frac{g\nu}{2\pi\omega^4}\overline{\Gamma^2}$$

where $\nu = N/T$ is the cusp appearance frequency, N the number of discontinuities during the averaging time T. This spectrum has the same power dependence as Zakharov-Filonenko WT spectrum $E_\omega \sim P^{2/3}\omega^{-4}$. Notice that in WT ω- and k spectra are connected with each other. This follows from

$$E_{k\omega} = \varepsilon(k)\,\delta(\omega - \omega_k),$$

so that

$$E_\omega = 2\pi k\,\frac{dk}{d\omega}\,\varepsilon(k(\omega)).$$

Turbulence of water waves

In the strong nonlinear regime it is not so. Consider singularity of the wedge type parallel to y -axis with length $l = x_1 - x_2$ centered at (x_0, y_0) :

$$\frac{\partial^2 \eta}{\partial y^2} = \Gamma(x)\delta(y - y_0) + \text{regular terms}$$

Here $\Gamma(x) = 0$ outside the interval $[x_1, x_2]$ including endpoints $\Gamma(x_{1,2}) = 0$. Hence

$$\eta_k = -\frac{1}{k_y^2} e^{-ik_y y_0} \int_{x_1}^{x_2} \Gamma(x) e^{-ik_x x} dx,$$

with $\mathbf{k} = (k_x, k_y)$.

Turbulence of water waves

Summation with respect to all crests gives

$$\eta_k = -\sum_\alpha \frac{e^{-i(\mathbf{k}\mathbf{n}_\alpha)y_\alpha}}{(\mathbf{k}\mathbf{n}_\alpha)^2} \int_{x_{1\alpha}}^{x_{2\alpha}} \Gamma_\alpha(x) e^{-i(\mathbf{k}\tau_\alpha)x} dx.$$

Here normal and tangent vectors $\mathbf{n}_\alpha$ and τ_α define orientation of the crest α.

Spectrum is determined after averaging $|\eta_k|^2$ against all random variables.

Average with respect (x_α, y_α) distributed uniformly gives

$$\overline{|\eta_k|^2} = N \langle \left| \frac{1}{(\mathbf{k}\mathbf{n})^4} \int_{x_1}^{x_2} \Gamma(x) e^{-i(\mathbf{k}\tau)x} dx \right|^2 \rangle.$$

Here N is the mean number of discontinuities inside area S.

Turbulence of water waves

We are interested in short wave asymptotics when $kL \gg 1$ with L being the characteristic length of breaks. Then for all angles, except $\theta_k \leq \theta_0 = (kL)^{-1}$, the spectrum $\tilde{\epsilon}(\mathbf{k})$ can be estimated as

$$\tilde{\epsilon}_2(\mathbf{k}) \approx \frac{gn}{2\pi^2} \frac{\langle (\Gamma')^2 \rangle}{(\mathbf{kn})^4 (\mathbf{k}\tau)^4},$$

where $\Gamma' \equiv \Gamma'(x_{1.2})$. For narrow cone of angles $\tilde{\epsilon}(\mathbf{k})$

$$\tilde{\epsilon}_1(\mathbf{k}) \approx \frac{gn}{4\pi^2 k^4} \langle (\bar{\Gamma} l)^2 \rangle, \;\; \theta_k \leq (kL)^{-1}.$$

where n is the density of breaks,

$$\bar{\Gamma} l = \int_{x_1}^{x_2} \Gamma(x) dx, \;\; l = x_1 - x_2, \;\; L = \langle l \rangle.$$

Turbulence of water waves

Hence the spectrum is obtained after average with respect to angles: $E(\mathbf{k}) = k\overline{\tilde{\epsilon}(\mathbf{k})}$.

The isotropic case:

$$E(k) = \frac{gn}{\pi^2 k^4 L}\left[\langle(\bar{\Gamma} l)^2\rangle + \frac{2}{3}\langle(\Gamma')^2\rangle(L^3 + a^3)\right]$$

that by one power differs from the Phillips spectrum. Here a is the mean bending size of discontinuities.

This spectrum is in correspondence with the spectrum $E_\omega \sim \omega^{-4}$ because in the isotropic case the Fourier transform of the correlation function $K(\mathbf{r}) = \langle \eta(\mathbf{r}+\mathbf{x},t)\eta(\mathbf{x},t)\rangle$ will have the same power, i.e. $\sim k^{-4}$.

Note that if $\omega = \sqrt{gk}$ then $E_\omega \sim \omega^{-7}$ instead of ω^{-4}!

Turbulence of water waves

Strong anisotropy:

If the angular width $\Delta\theta$ of the distribution is narrow enough, $\Delta\theta < \theta_0 = 1/(kL)$, then in this cone of angles the spectrum will fall $\sim k^{-3}$, like for the Phillips spectrum.

With $k > k^* = (L\Delta\theta)^{-1}$ the spectrum gets another power: k^{-4} and, respectively, with increasing k the angular width of the spectrum becomes more narrow decreasing like $(kL)^{-1}$ that in the k-space results in JETS.

2D hydrodynamic turbulence

The situation with 2D turbulence is analogous to the water waves case, if, following Saffman, one assumes, that vorticity Ω undergoes jumps with widths $\delta \ll L$, the characteristic scale of turbulence.

For 2D turbulence sharp vorticity gradients were observed in many numerical experiments (Lilly, 1971; McWilliams, 1984; Kida, 1985; Brachet, Meneguzzi, & Sulem, 1986; Okhitani, 1991).

Such tendency can be understood if within the Euler equation one introduces the divergence-free vector $\mathbf{B}$ (di-vorticity),

$$B_x = \frac{\partial \Omega}{\partial y}, \qquad B_y = -\frac{\partial \Omega}{\partial x}.$$

where $\mathbf{B}$ obeys the equation

2D hydrodynamic turbulence

$$\frac{\partial \mathbf{B}}{\partial t} = \text{rot}\,[\mathbf{v} \times \mathbf{B}].$$

This vector field is frozen-in, changes due to the velocity component $\mathbf{v_n}$, normal to $\mathbf{B}$. By introducing new trajectories,

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}_n(\mathbf{r}, t); \quad \mathbf{r}|_{t=0} = \mathbf{a},$$

$\mathbf{B}$ is expressed through the mapping $\mathbf{r} = \mathbf{r}(\mathbf{a}, t)$ and its Jacobian J (analog of VLR):

$$\mathbf{B}(\mathbf{r}, t) = \frac{(\mathbf{B}_0(\mathbf{a}) \cdot \nabla_a)\mathbf{r}(\mathbf{a}, t)}{J}$$

J is not fixed, i.e., the mapping is compressible, that is a reason of appearance of sharp gradients in 2D Euler.

2D hydrodynamic turbulence

The spectrum in this case is found by the same scheme.

We will assume that $L^{-1} \ll k \ll \delta^{-1}$.

By considering one vorticity jump,

$$\frac{\partial \Omega}{\partial y} = G(x)\ \delta(y - y_0) + \text{regular terms}$$

with $G(x)$ vanishing outside the interval $[x_1 x_2]$ and at $x = x_{1,2}$, we find first Ω_k for one jump, then after summation we get the Fourier amplitude for the whole ensemble of discontinuities:

$$\Omega_k = -i \sum_\alpha \frac{e^{-i(\mathbf{kn})y_\alpha}}{(\mathbf{kn}_\alpha)} \int_{x_{1\alpha}}^{x_{2\alpha}} G_\alpha(x) e^{-i(\mathbf{k}\boldsymbol{\tau}_\alpha)x} dx.$$

2D hydrodynamic turbulence

The spectrum $\epsilon(k)$ is given as follows

$$\epsilon_1(k) = \frac{n}{8\pi^2 k^4} \langle (\bar{G} l)^2 \rangle, \quad \theta_k \leq \theta_0;$$

$$\epsilon_2(k) = \frac{n}{4\pi^2 k^2} \, \frac{\langle (G')^2 \rangle}{(\mathbf{kn})^2 (\mathbf{k}\tau)^4}, \quad \theta_k > \theta_0,$$

where n is the density of discontinuities.

Hence, after averaging over angles we have

in the isotropic case - the Saffman spectrum

$$E(k) = \frac{n}{2\pi^2 k^4 L} \left[\langle (\bar{G} l)^2 \rangle + \frac{2L^4}{3} \langle (G')^2 \rangle \right],$$

2D hydrodynamic turbulence

in the strong anisotropic case - the combination of spectra of the Saffman and Kraichnan types:

$$\max_\theta E(k) \sim k^{-3} \text{ if } \Delta\theta < \theta_0 = (kL)^{-1} \text{ (Kraichnan);}$$

$$\max_\theta E(k) \sim k^{-4} \text{ if } \Delta\theta > \theta_0 = (kL)^{-1} \text{ (Saffman)}$$

The latter assumes also that the angle distribution becomes more narrow with increasing k with $\theta_0 = (kL)^{-1}$, i.e., the formation of jets at large k.

Numerical experiment

To support the above arguments and reveal the direct connection between the formation of the sharp vorticity gradients and the tail of the energy spectrum we have performed a numerical study of the evolution of decaying 2D turbulence.

Numerically we solved the Euler equation with hyperviscosity:

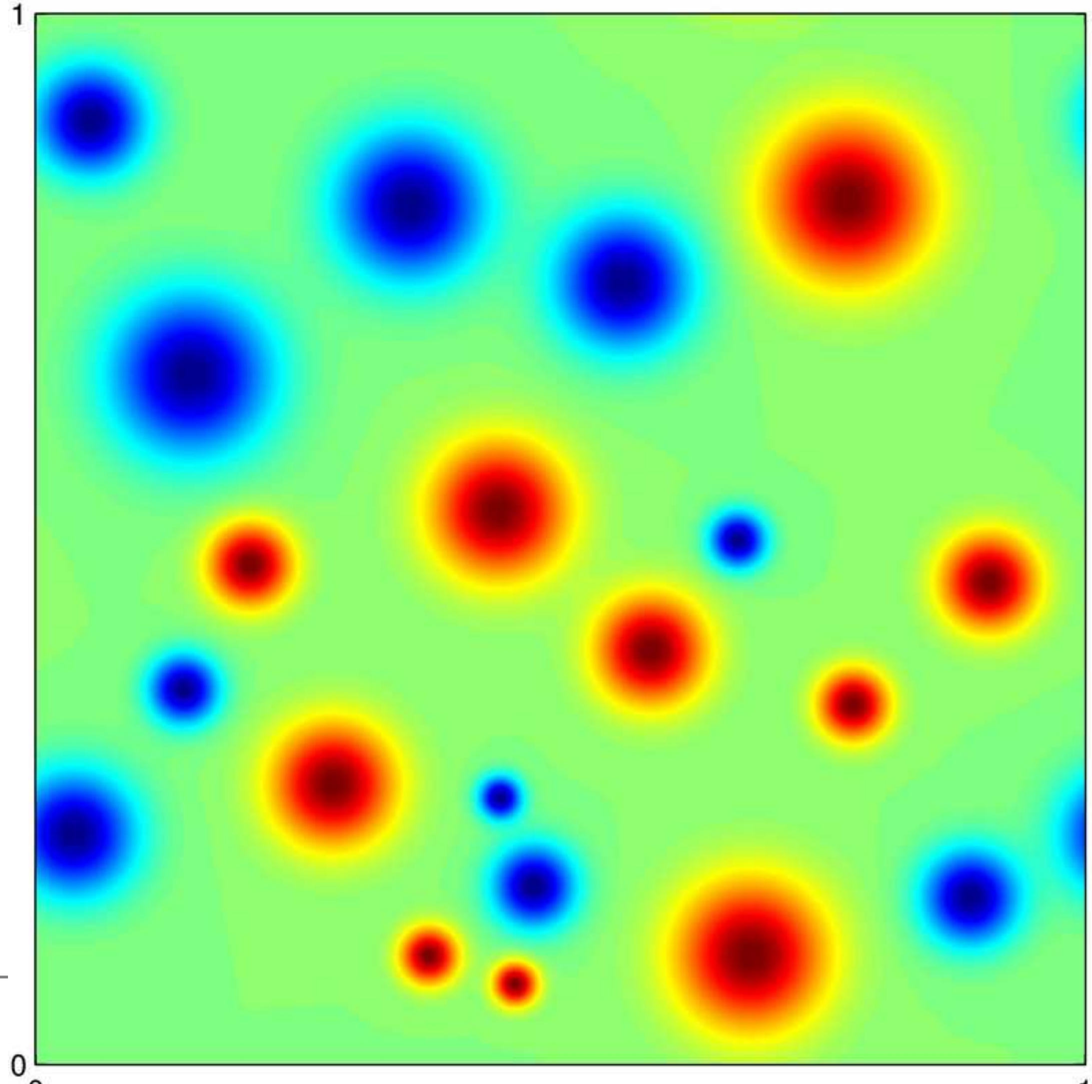
$$\frac{\partial \omega}{\partial t} + \{\omega, \psi\} = \mu_{2n} \nabla^{2n} \omega,$$

where ψ is the streamfunction and $n = 3$ and $\mu_6 = 10^{-20}$. In our case the energy decreased by less than 0.002% . We used a double periodic domain by employing a high resolution fully de-aliased spectral scheme. The domain size was taken to be unity and the resolution was 2048×2048.

Numerical experiment

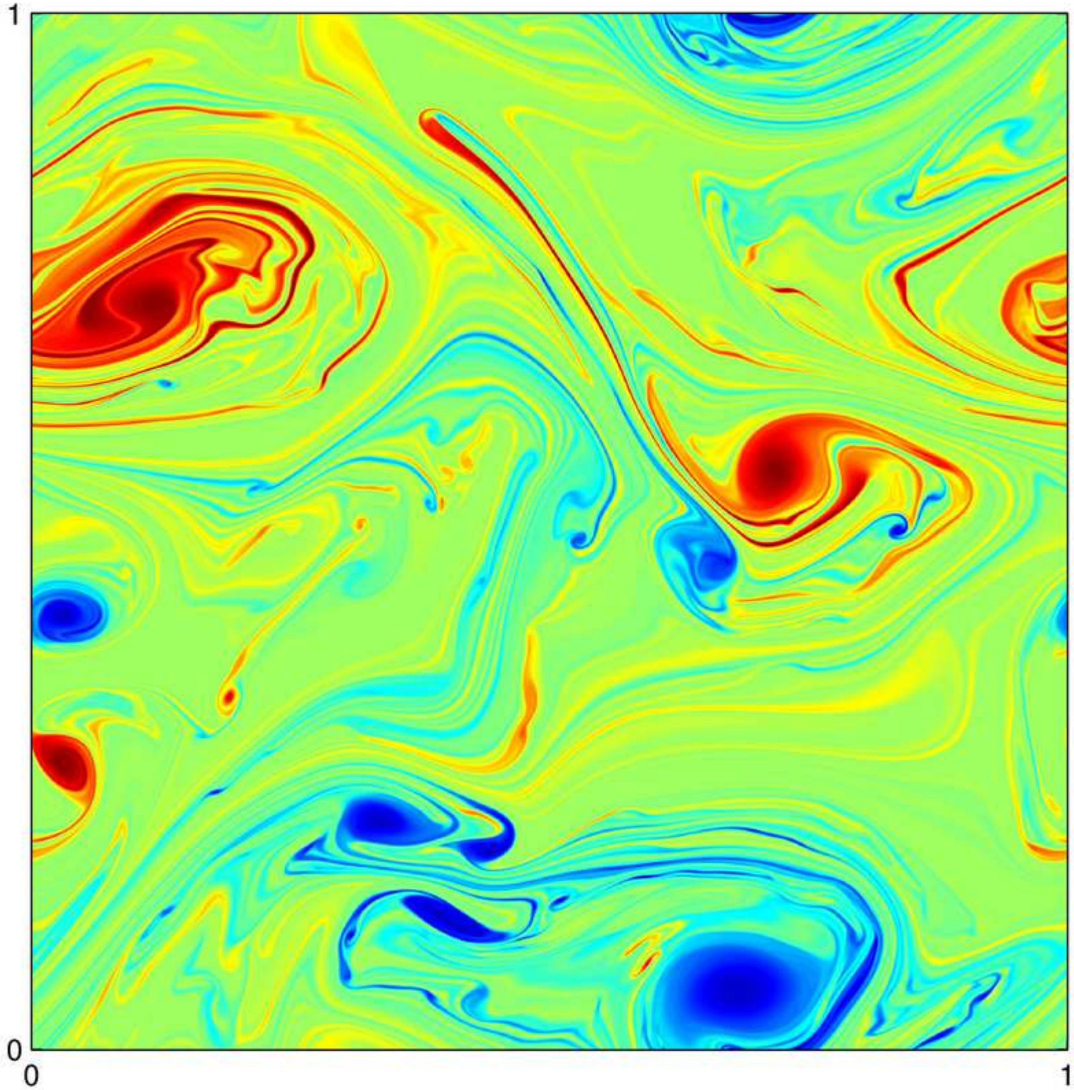
The time scale corresponded to inverse maximal value of vorticity, ω_0^{-1}.

Fig.1. Initial distribution



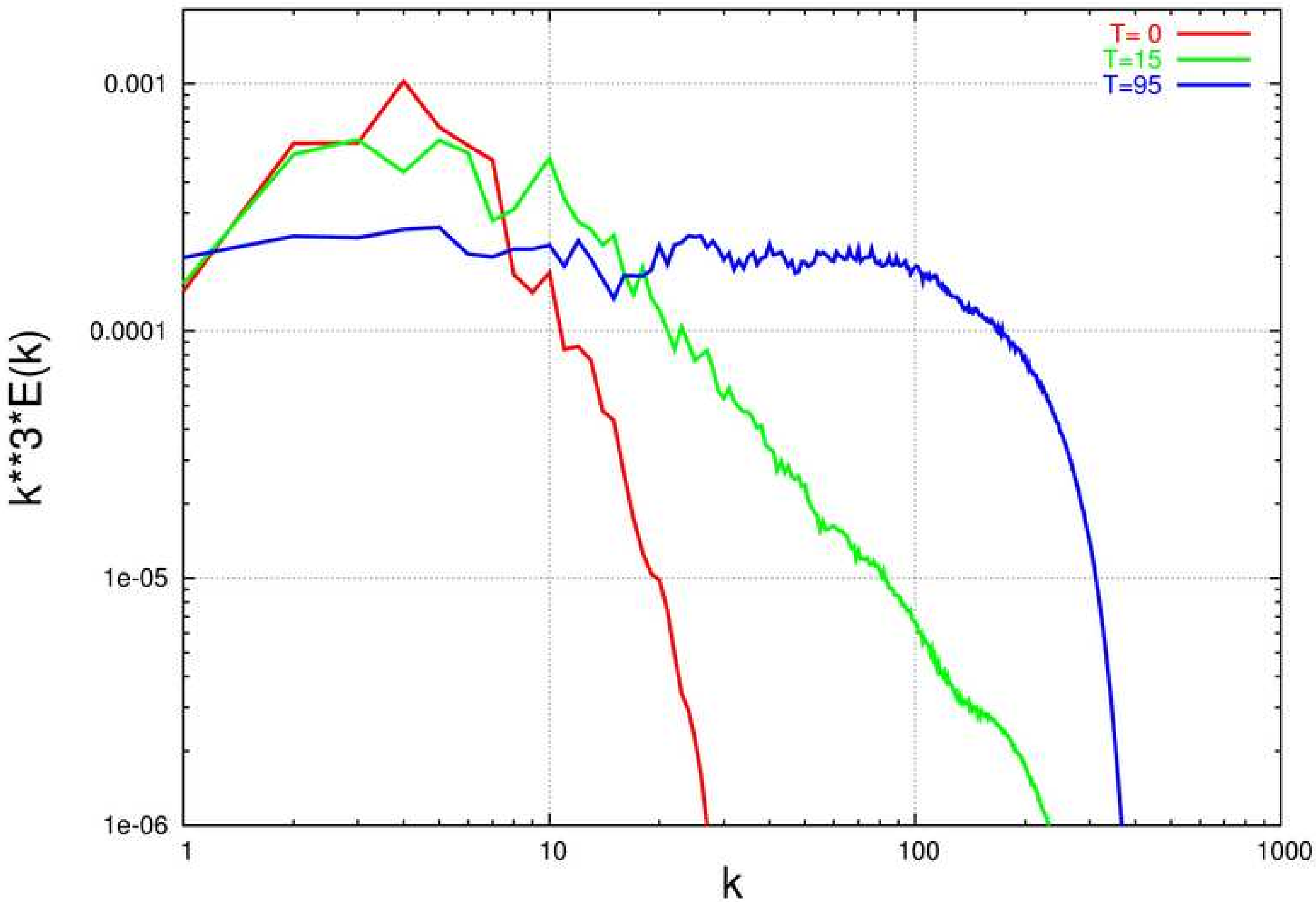
Numerical experiment

Fig.2. Vorticity field at time 95 corresponding to 10 vortex turnover times. Maximum $\omega_0 = 1$ and minimum is -1.



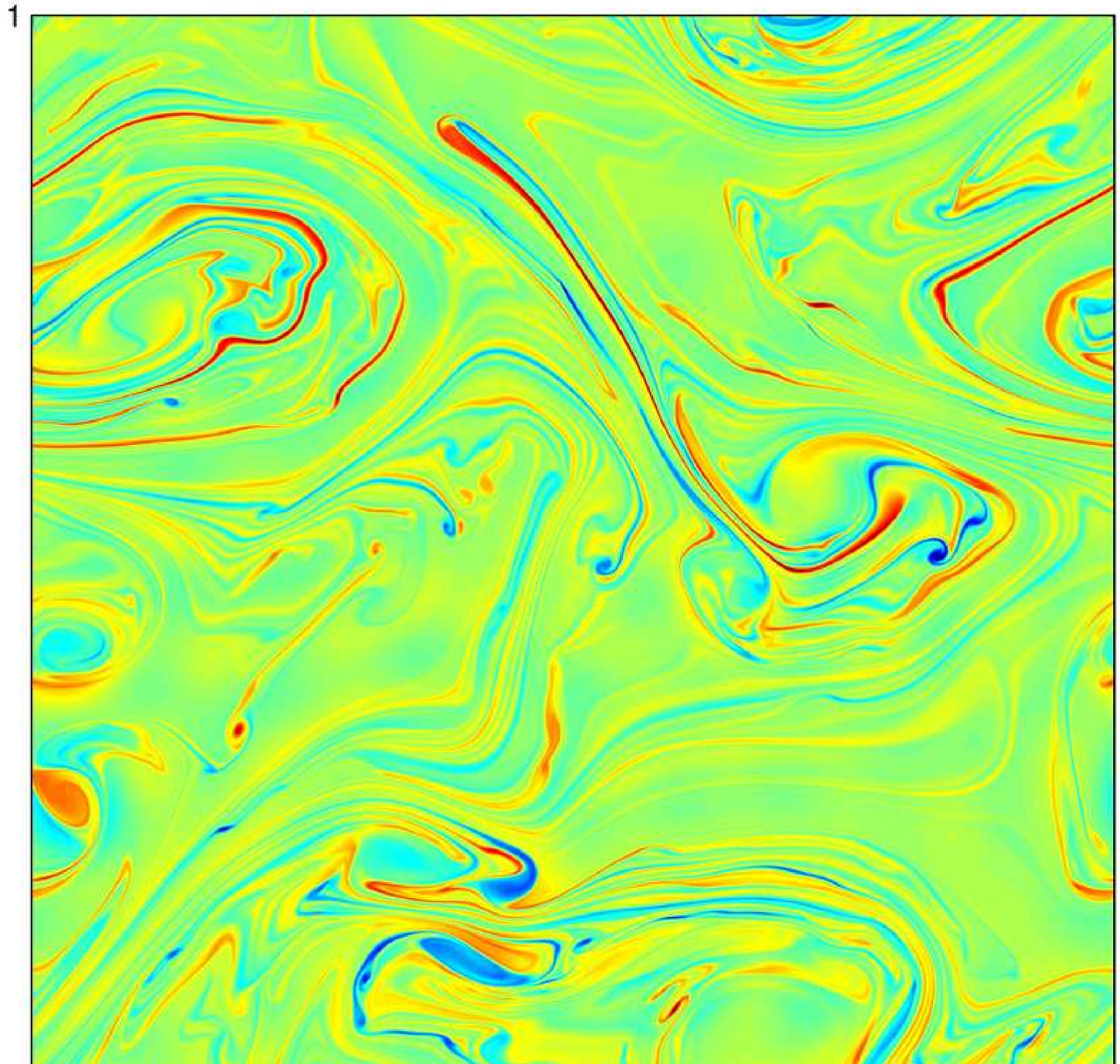
Numerical experiment

Fig.3. Compensated energy spectrum at different times $k^3E(k)$ corresponding to the vorticity field in Fig. 2.



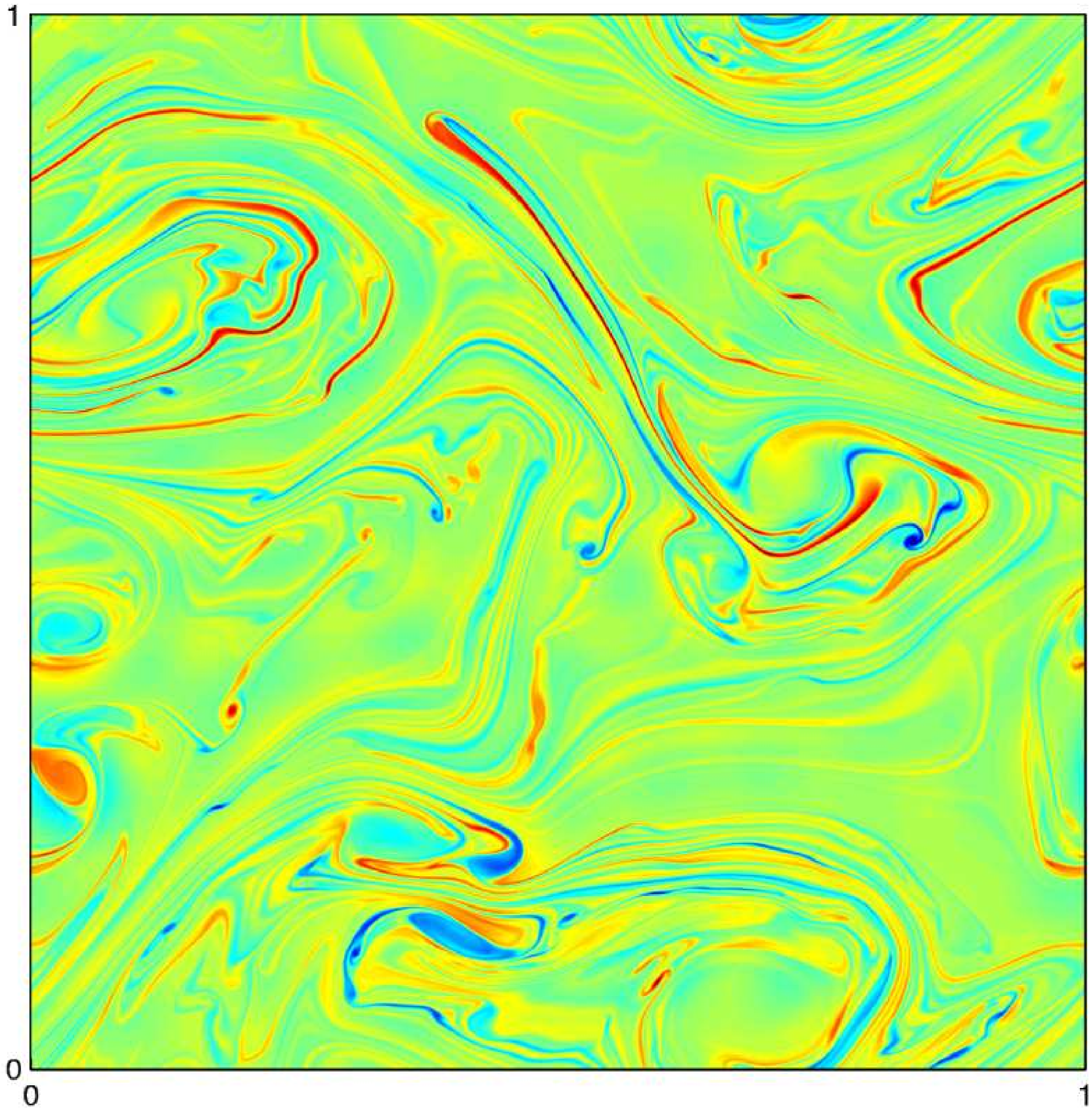
Numerical experiment

Fig.4. The di-vorticity field $\mathbf{B}$ at $T = 95$. The modulus of di-vorticity, the maximum (red) value is 673.



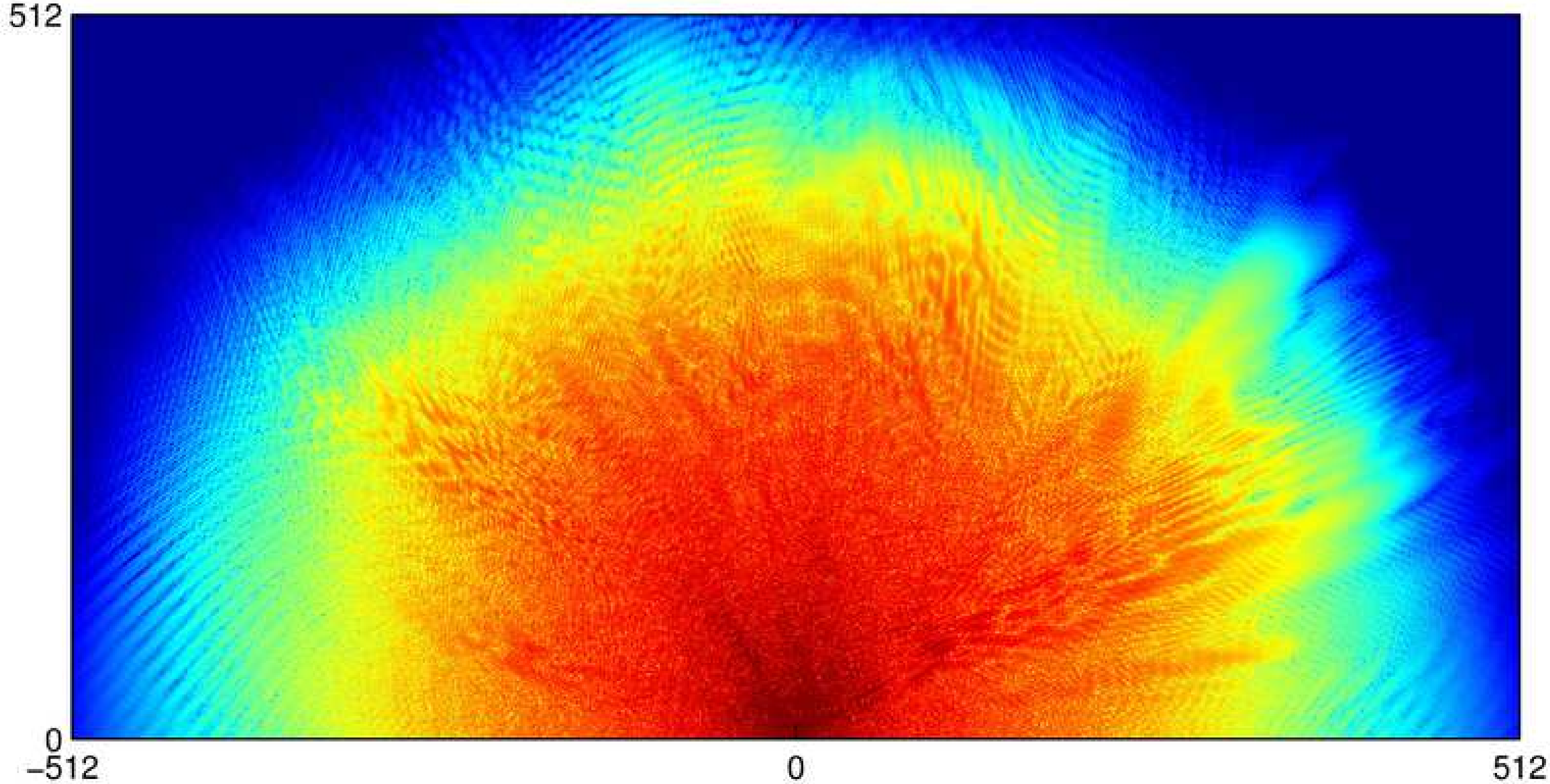
Numerical experiment

Fig.5. High pass filtered vorticity field from Fig. 2, $k > 10$.



Numerical experiment

Fig.6. 2D energy spectrum $\epsilon(k_x, k_y)$, logarithmic scale.



Acoustic turbulence

For acoustic turbulence, following Kadomtsev and Petviashvili, singularities are shocks. Therefore for 1D case we have

$$E_1(k) = \frac{n_1 c_s^2}{2\pi\rho_0 k^2}\,\overline{(\Delta\rho)^2}.$$

where n_1 is the density of shocks (per unit length). This result was obtained first by Burgers (1951).

In 3D isotropic case the spectrum can be easily found from 1D spectrum.

Acoustic turbulence

The pair density correlation function

$$\phi(y_1) = \langle \rho(x_1 + y_1, x_2, x_3)\rho(x_1, x_2, x_3) \rangle$$

has the Fourier spectrum

$$\phi_k = \frac{n_1}{2\pi k^2}\,\overline{(\Delta\rho)^2}.$$

Hence the energy spectrum is given by

$$E_3(k) = \frac{2n_1 c_s^2}{\pi \rho_0 k^2}\,\overline{\Delta\rho^2}.$$

This is the KP spectrum.

Conclusion

- For water waves the frequency spectrum due to surface cusps has the same dependence $(\sim \omega^{-4})$ as the WT Zakharov-Filonenko spectrum and can be considered as its continuation.
- For the 2D water wave spectrum the situation is very different:
  i) in the isotropic case $E(k) \sim k^{-4}$, strongly differs from the Phillips spectrum,
  ii) the Phillips spectrum can be obtained for very anisotropic distribution.

Conclusion

- For 2D hydrodynamic turbulence we have reproduced in the isotropic case the Saffman spectrum. The Kraichnan-type spectrum has been found for very anisotropic case as intermediate asymptotics at $k < k^* = (L\Delta\theta)$; for $k > k^*$ we have found jets with decreasing angular width.
- The performed numerics for 2D decay turbulence have demonstrated that the spectrum tails appear due to sharp vorticity gradients.
- For acoustic turbulence our approach leads to the results of K & P.