Charged test particles in turbulent magnetoplasmas

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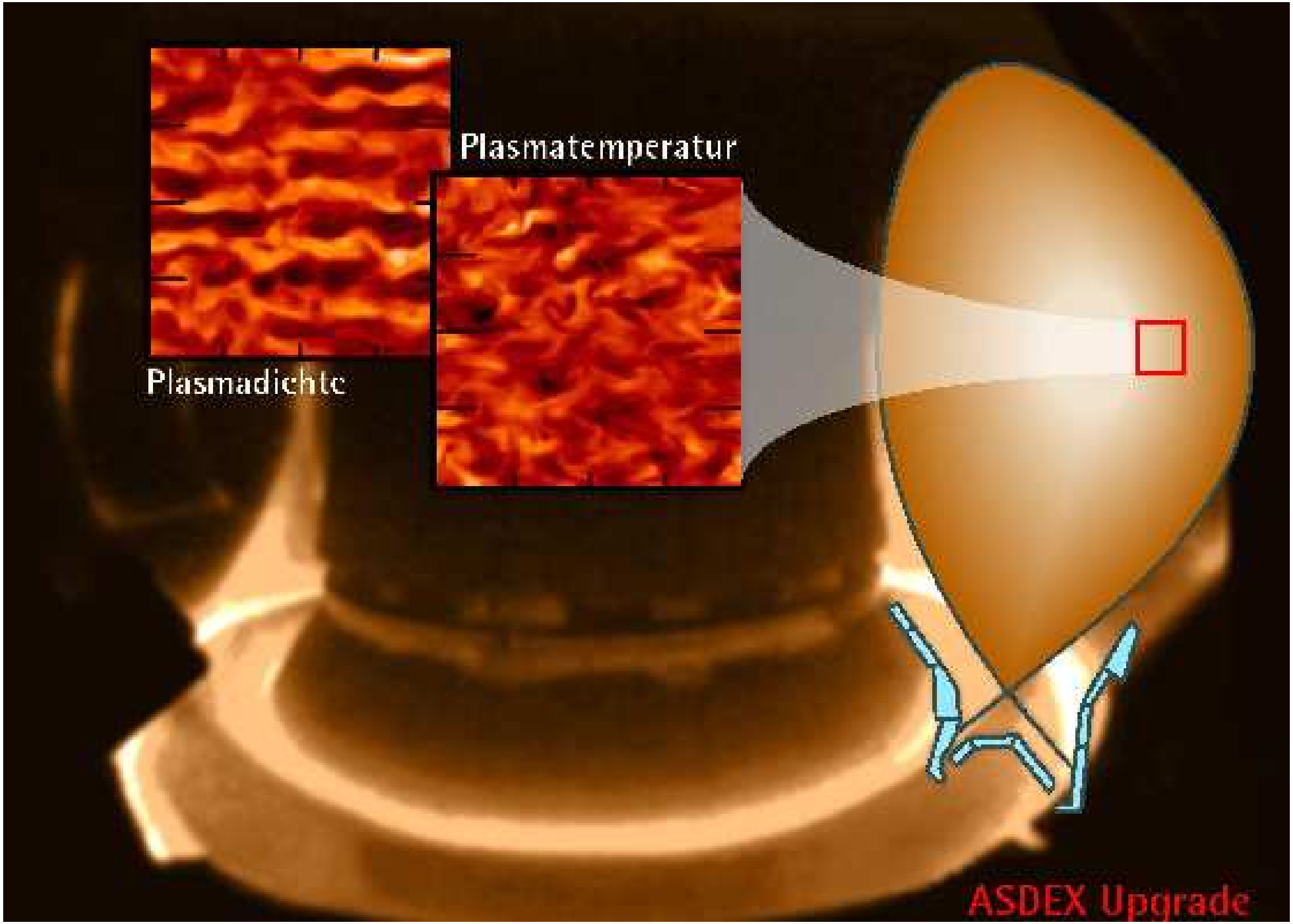
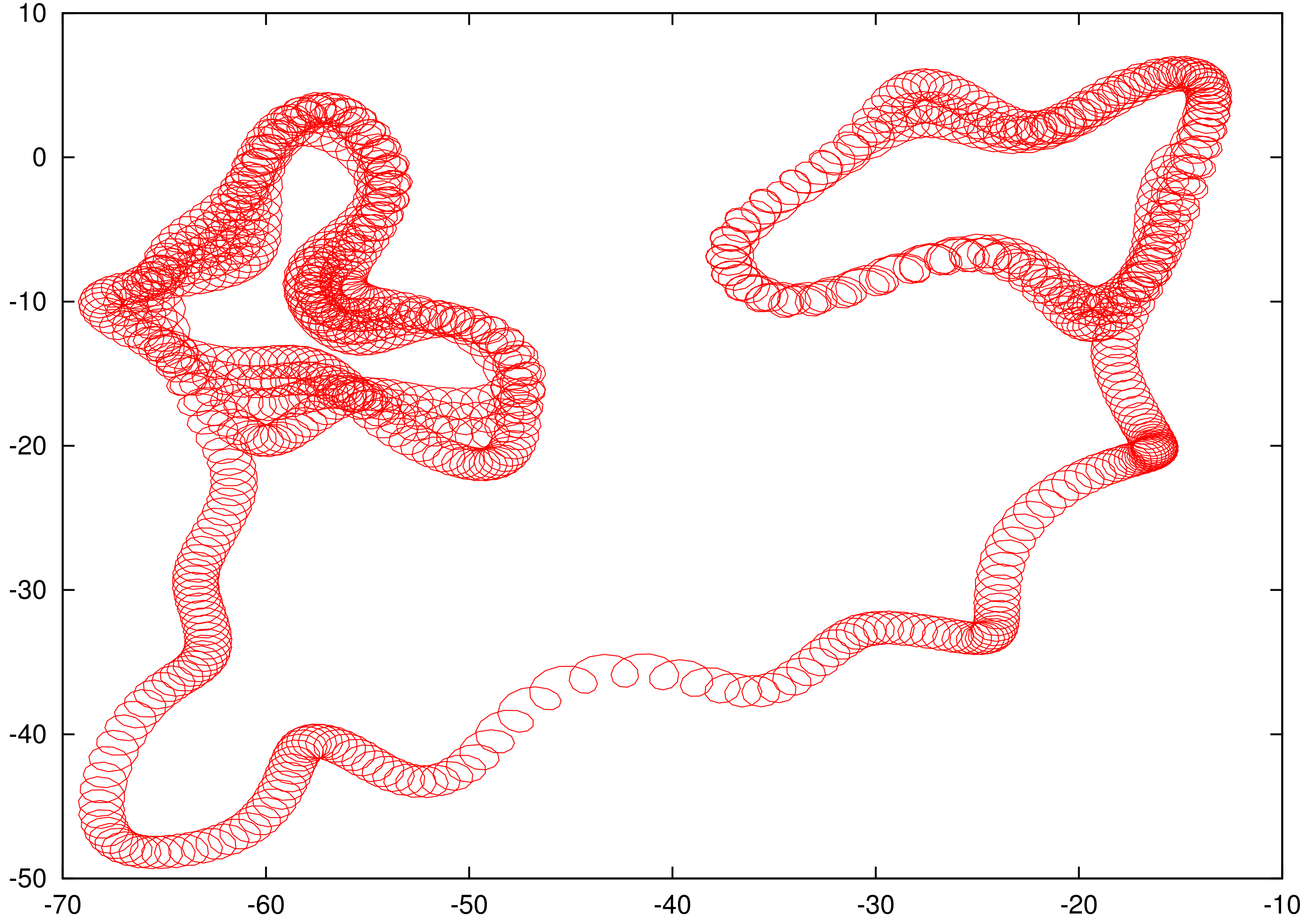
Turbulence in magnetized fusion plasmas



Test particles with finite gyroradii

We consider the diffusion of test particles with finite gyroradii in a 2D electrostatic potential (magnetic field perpendicular to the plane).



$E \times B$ drift motion

The $E \times B$ drift velocity

Consider a homogeneous electrostatic field

- Lorentz transport (exact solution):

$$\frac{d}{dt}\vec{v} = \pm\vec{E}(t) \pm \vec{v} \times \vec{e}_z.$$

- Differentation w. r. t. time:

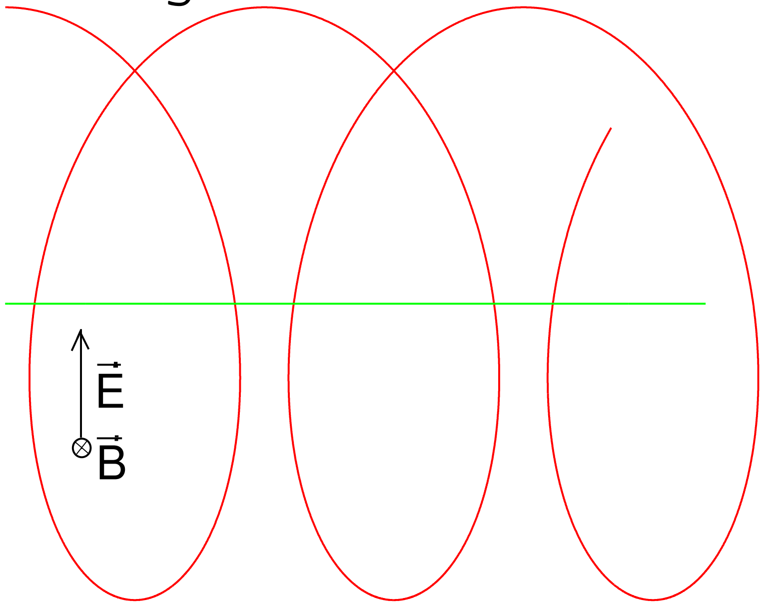
$$\left(1 + \frac{d^2}{dt^2}\right)\vec{v} = \vec{E}(t) \times \vec{e}_z \pm \dot{\vec{E}}(t) \equiv \vec{v}^{\mathrm{dr}} + \vec{v}^{\mathrm{P}}$$

- For $\bar{\omega} \ll 1$, the $E \times B$ drift dominates:

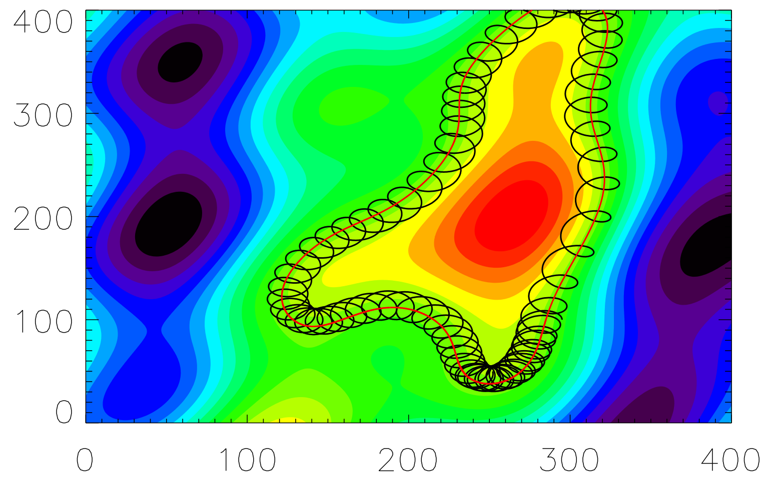
$$\vec{v}^{\mathrm{dr}} = \vec{E} \times \vec{e}_z = \begin{pmatrix} E_y \\ -E_x \end{pmatrix} = -\varepsilon_{ij}\frac{\partial \phi(\vec{x}, t)}{\partial x_j}$$

Examples for drift motion

Homogeneous static E field:



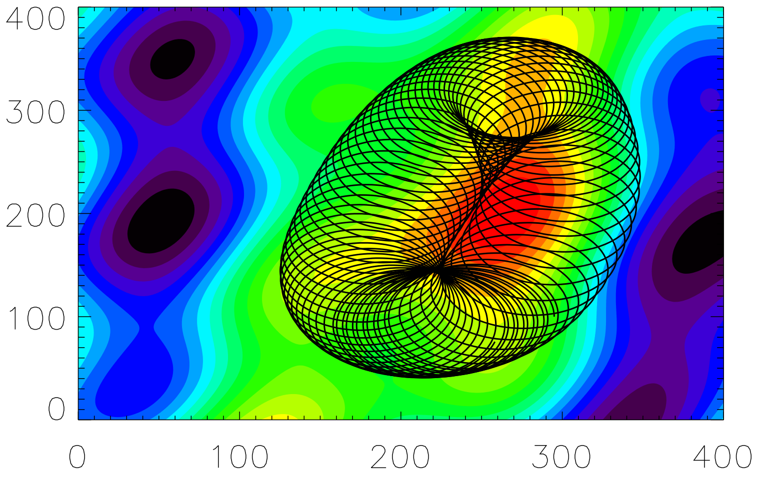
Random static E field:



- Particles are trapped on equipotential lines
  $\longrightarrow$ No transport in a static potential
- Trapping effects are reduced in a time dependent electric field

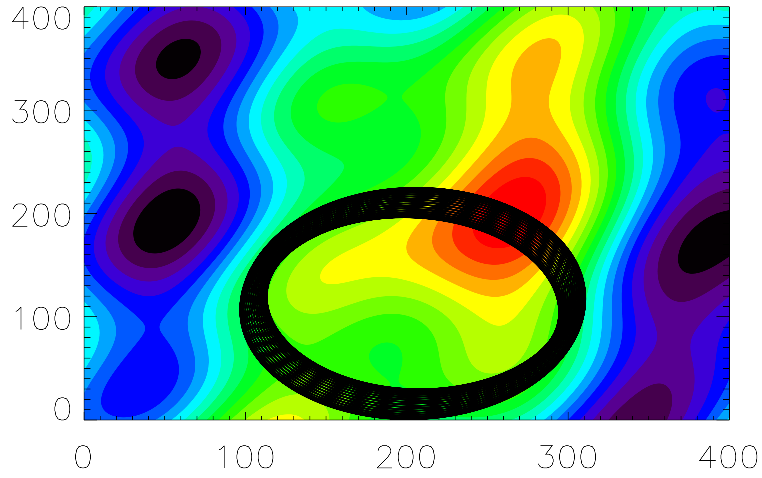
Finite gyroradius effects

5-fold increase of gyroradius:



Drift approximation roughly valid

10-fold increase of gyroradius:



Gyromotion averages the field

$\Longrightarrow$ Drift strongly reduced

The gyrokinetic approximation

The $E \times B$ drift approximation is only exact for homogeneous fields or vanishing gyroradius.
Extend to finite gyroradii by averaging the potential over one gyration period of the test particle.

- Ansatz:

$$\langle \Phi \rangle (\vec{x}) = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \Phi(\vec{x} + \vec{\rho}); \quad \vec{\rho} \text{ Gyroradius}$$

- Applying a Fourier transformation we receive:

$$\langle \Phi \rangle (\vec{x}) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} d\vec{k}\, e^{i\vec{k}\vec{x}} \phi(\vec{k}) \cdot J_0(|\vec{k}|\rho);$$

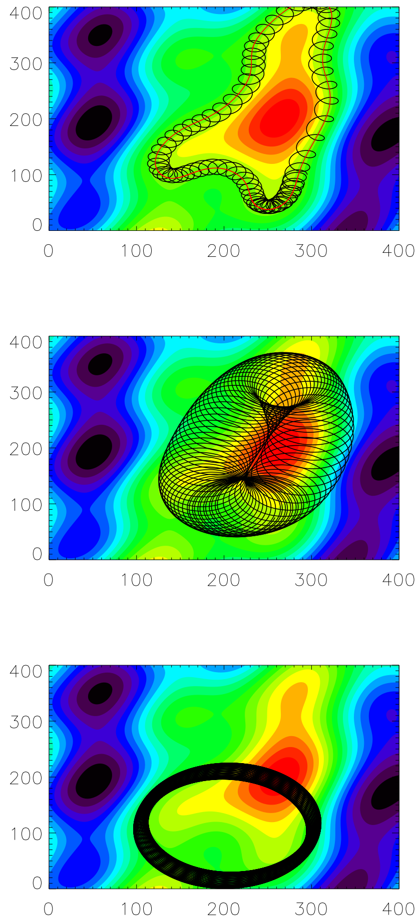
J_0 Bessel function of order 0

- It can be shown that the gyrokinetic approximation is valid for $k^2 \langle \Phi \rangle < 1$

The motivation of this work

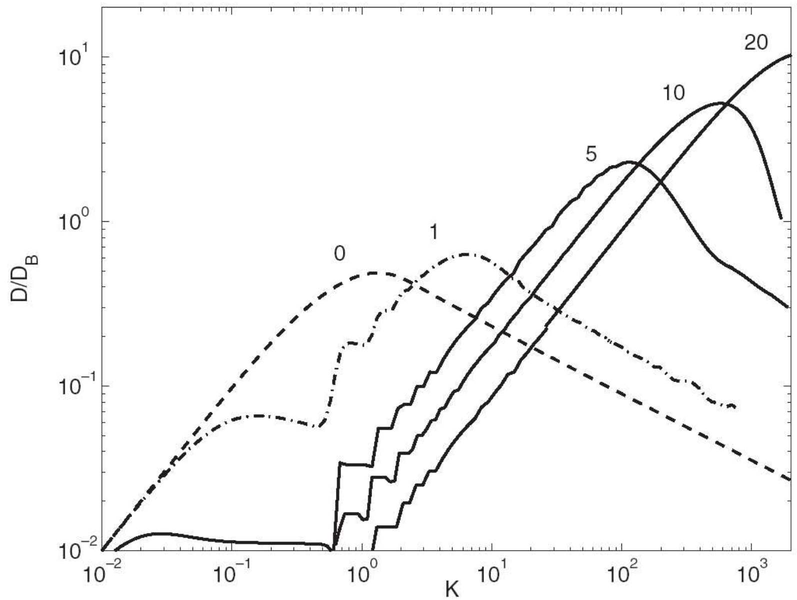
Naive expectations vs. recent results

Single trajectory simulation



Transport expected to decrease

Vlad, Spineanu et al. *Plasma Phys. Contr. Fusion* **47** (2005)



Kubo Number $K = \frac{V \tau_C}{\lambda}$

According to that publication, the diffusion coefficient may increase with increasing gyroradius for large K

The decorrelation trajectory method

(zero gyroradius)

Connection between Diffusion and Correlations

Idea: "A stationary homogeneous Gaussian stochastic field is completely determined by its Eulerian correlation function."

Definitions:

- Eulerian autocorrelation function:

$$E(\vec{x}, t) = \langle \phi(0,0)\phi(\vec{x}, t) \rangle$$

- Further Eulerian correlations:

$$E_{ij}(\vec{x}, t) \equiv \langle v_i^{dr}(0,0) v_j^{dr}(\vec{x}, t) \rangle = -\varepsilon_{in}\varepsilon_{jm} \frac{\partial^2 E(\vec{x}, t)}{\partial x_n \partial x_m}$$

$$E_{\phi j}(\vec{x}, t) \equiv \langle \phi(0,0) v_j^{dr}(\vec{x}, t) \rangle = -\varepsilon_{in} \frac{\partial E(\vec{x}, t)}{\partial x_n}$$

Connection between Diffusion and Correlations

The Taylor formula

$$D_x(t) \equiv \frac{1}{2}\frac{d}{dt}\langle x(t)^2\rangle = \int_0^t d\tau\, L_{xx}(\tau),$$

Lagrangian autocorrelation:

$$L_{xx}(\tau) \equiv \langle v_x^{\mathrm{dr}}(\vec{x}(0),0)\, v_x^{\mathrm{dr}}(\vec{x}(\tau),\tau)\rangle$$

The diffusion coefficient can be calculated from the Lagrangian correlation of the velocity

- Problem: $E(\vec{x},t) \rightarrow E_{ii}(\vec{x},t) \rightarrow L_{ii}(t) \rightarrow D(t)$

„Classical“ method: Corrsin approximation

$$L_{ij}(t) = \int d\vec{x}\, E_{ij}(\vec{x},t) P(\vec{x},t), \qquad P(\vec{x},t) \equiv \frac{1}{2\pi\langle x(t)^2\rangle}\exp\left(-\frac{x^2}{2\langle x(t)^2\rangle}\right)$$

$\Longrightarrow$ **But:** No trapping effects

The decorrelation trajectory method ($E \times B$ drift)
M Vlad, F Spineanu (1998 - 2005)

We consider subensembles $S : \phi(0,0) = \phi^0; \quad v^{dr}(0,0) = v^0$

$$
\begin{aligned}
L_{ij}(t) &= \langle v_i[\vec{x}(0), 0] v_j[\vec{x}(t), t] \rangle \\
&= \iint d\phi^0 \, d\vec{v}^0 \, P_1(\phi^0, \vec{v}^0) \langle v_i[\vec{x}(0), 0] v_j[\vec{x}(t), t] \rangle_S \\
&= \iint d\phi^0 d\vec{v}^0 P_1(\phi^0, \vec{v}^0) v_i^0 \langle v_j[\vec{x}(t), t] \rangle_S \\
&\equiv \iint d\phi^0 d\vec{v}^0 P_1(\phi^0, \vec{v}^0) v_i^0 V_j^L(t; S)
\end{aligned}
$$

where

$$
V_j^L(t; S) \equiv \langle \, v_j^{\mathrm{dr}}(\vec{x}(t), t) \rangle_S
$$

denotes the mean *Lagrangian* drift velocity in the subensemble

The decorrelation trajectory method ($E \times B$ drift)

The average *Eulerian* drift velocity in the subensemble $S(\Phi^0, \vec{v}^0)$ can be calculated as

$$V_j^E(\mathbf{x}, t; S) \equiv \langle v_j^{\mathrm{dr}}(\mathbf{x}, t)\rangle_S \;=\; \phi^0 \frac{E_{\phi j}(\mathbf{x}, t)}{E(0,0)} + v_1^0 \frac{E_{1j}(\mathbf{x}, t)}{E_{11}(0,0)} + v_2^0 \frac{E_{2j}(\mathbf{x}, t)}{E_{22}(0,0)}$$

Analogous:

$$\Phi(\mathbf{x}, t; S) \equiv \langle \phi(\mathbf{x}, t)\rangle_S = \phi^0 \frac{E(\mathbf{x}, t)}{E(0,0)} + v_1^0 \frac{E_{1\phi}(\mathbf{x}, t)}{E_{11}(0,0)} + v_2^0 \frac{E_{2\phi}(\mathbf{x}, t)}{E_{22}(0,0)}$$

Therefore:

$$V_i^E(\mathbf{x}, t; S) = -\varepsilon_{ij} \frac{\partial \Phi(\mathbf{x}, t; S)}{\partial x_j}$$

The decorrelation trajectory method ($E \times B$ drift)

- Key concept: 'Decorrelation trajectory'

$$\frac{dX_i}{dt} = V_i^E(\vec{X}, t; S) = -\varepsilon_{ij} \frac{\partial \Phi(\vec{X}, t; S)}{\partial X_j}$$

- Express $V_j^L(t; S)$ in terms of $V_j^E(\mathbf{x}, t; S)$ via the Ansatz

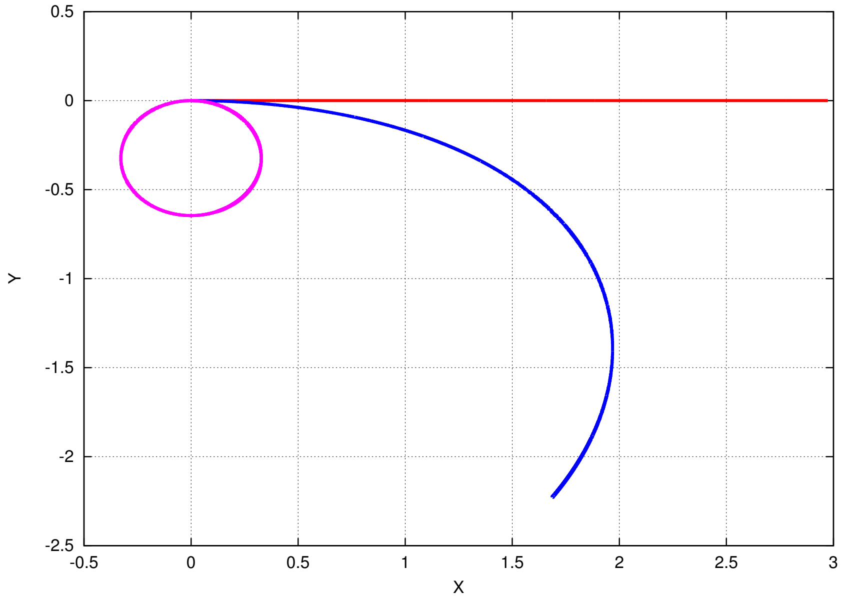
$$V_j^L(t; S) \cong V_j^E(\vec{X}(t; S), t; S) \,.$$

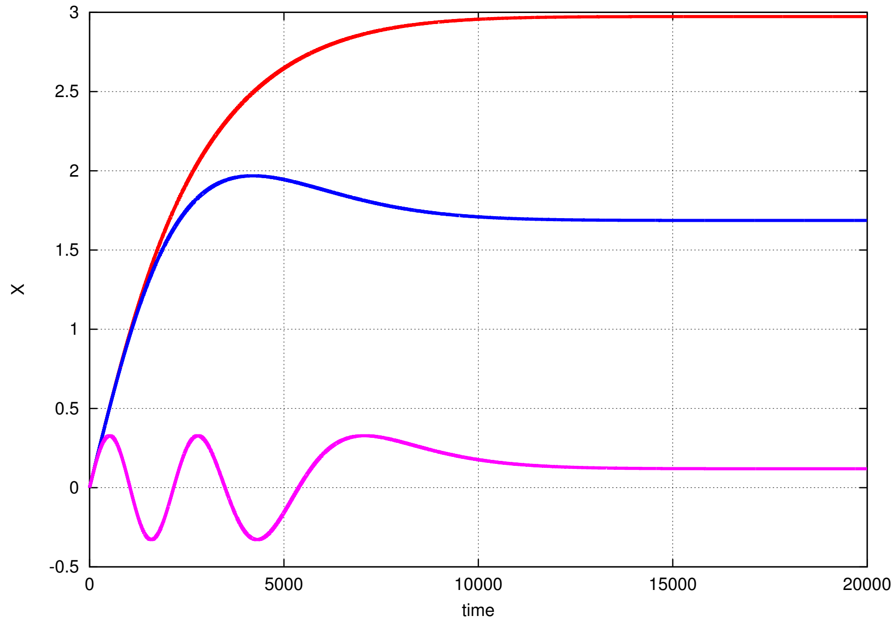
- With this assumption the diffusion coefficient can be rewritten as

$$D_x(t) = \iint d\phi^0 \, d\mathbf{v}^0 \, P_1(\phi^0, \mathbf{v}^0) \, v_x^0 \, X(t; S) \,.$$

The decorrelation trajectory method ($E \times B$ drift)

Examples for decorrelation trajectories





$$E(\vec{x}, t) = a \cdot e^{-b|\vec{x}|^2} e^{-ct^2}$$

- $\phi^0 = 0$
- $\phi^0 = 0.001$
- $\phi^0 = 0.01$

Advantages of this method

- Only the autocorrelation function of a potential needs to be known
- Smooth autocorrelation functions lead to smooth subensemble potentials
- Decorrelation trajectories are not very chaotic
- Reduced number of trajectories to be calculated

Crucial point:

- Validity of $V^L(t;S) \cong V^E(\vec{X}(t;S),t;S)$ cannot be proven

The decorrelation trajectory method

(finite gyroradii)

Extension to particles with finite gyroradii

Claim by the authors: The DCT method can be extended to Lorentz transport in a straightforward way.
Definition of new subensemble values:

$$\mathbf{\Xi}(t,S) \equiv \langle \vec{\xi}(t) \rangle_S, \qquad \mathbf{\Pi}(t,S) \equiv \langle \vec{\rho}(t) \rangle_S,$$

$$\begin{aligned} \frac{d\Xi_i}{dt} &= -\varepsilon_{ij} \frac{\partial \Phi(\mathbf{\Xi} + \mathbf{\Pi}, t; S)}{\partial \Xi_j} \\ \frac{d\Pi_i}{dt} &= \varepsilon_{ij} \left[\frac{\partial \Phi(\mathbf{\Xi} + \mathbf{\Pi}, t; S)}{\partial \Xi_j} + \Pi_j \right] \end{aligned}$$

Using this method, high diffusion coefficients for large gyroradii are obtained.
Question: Is this correct?

Extension to particles with finite gyroradii

Pseudo-gyrokinetic approximation instead of full Lorentz transport *("Method A")*:

$$\Psi(\Xi, \rho, t; S) = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \, \Phi(\Xi + \vec{\rho}(\varphi), t; S)$$

This means:

$$\begin{aligned} E^{\mathrm{eff,A}}(\vec{x}, t, \rho) &= \frac{1}{2\pi} \int_0^{2\pi} d\varphi \, E(\vec{x} + \vec{\rho}(\varphi), t) \\ &= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} d\vec{k} \, e^{i\vec{k}\vec{x}} E(\vec{k}, t) J_0(|\vec{k}|\rho) \end{aligned}$$

$\longrightarrow$ This approach is not in line with standard gyrokinetics!

Extension to particles with finite gyroradii

Alternative approach *("Method B")*:

First gyroaverage the potential:

$$\langle\phi\rangle(\vec{x}) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} d\vec{k}\, e^{i\vec{k}\vec{x}}\phi(\vec{k}) \cdot J_0(|\vec{k}|\rho)$$

Then calculate the Eulerian correlation function:

$$E^{\mathrm{eff,B}}(\vec{x},\rho) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} d\vec{k}\, e^{i\vec{k}\vec{x}} E(\vec{k},t) J_0^2(|\vec{k}|\rho)$$

$\Longrightarrow$ The two methods lead to different results!

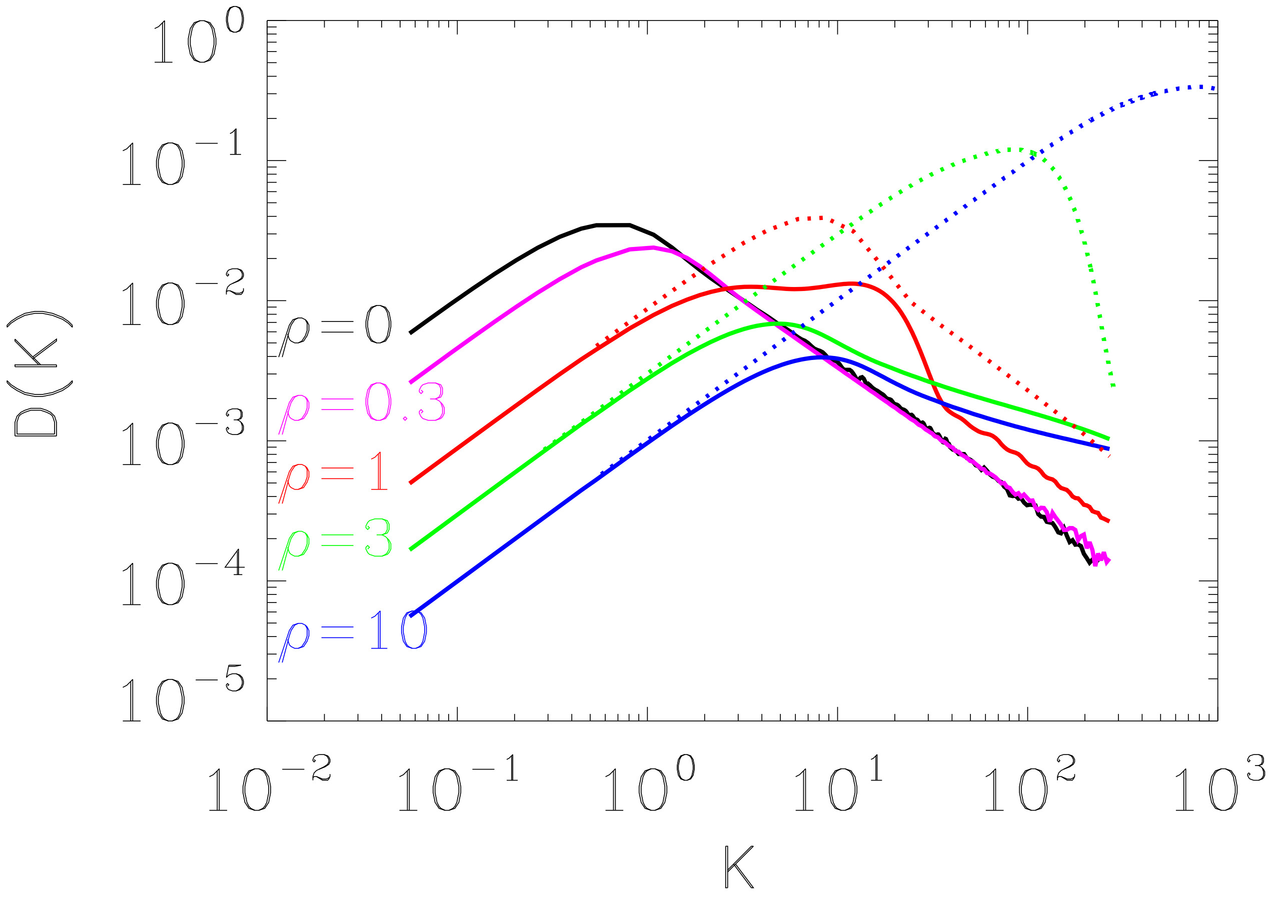
Numerical results

- Ansatz:

$$E(\vec{x},t) = A{\cdot}e^{-|\vec{x}|^2}e^{-t^2/\tau_C^2}$$

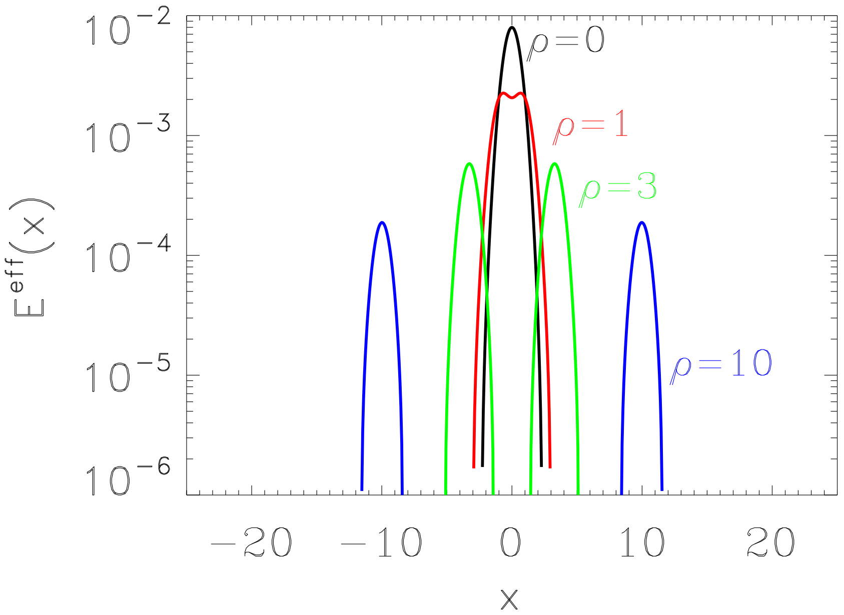
- Parameter quantifying turbulence: **Kubo number**

$$K \equiv \frac{V\tau_C}{\lambda_C} = \frac{\tau_C}{\tau_{\mathrm{fl}}}$$

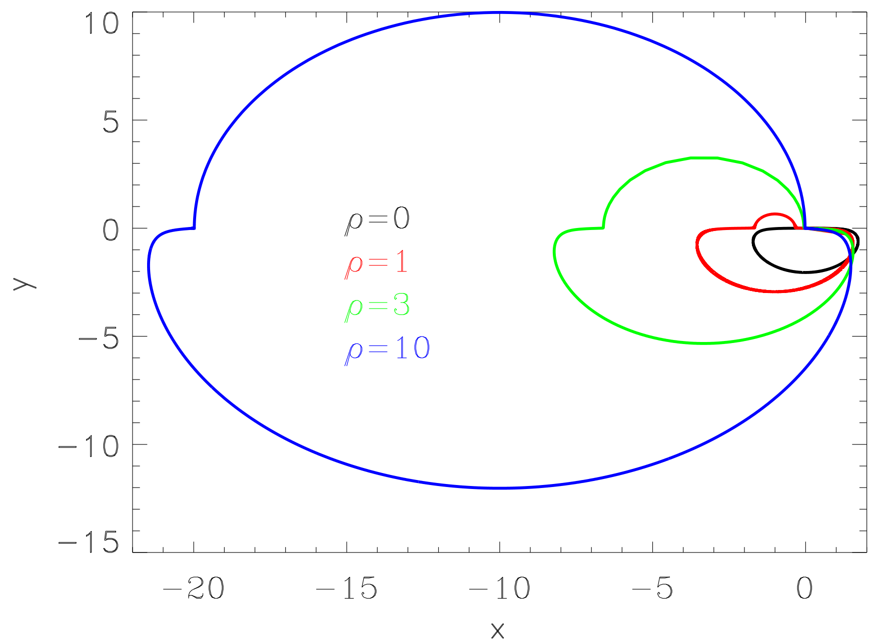


- Modification of the result of Vlad et. al.
- Moderate increase of transport still possible

Explanation of the increase of D ('method A')



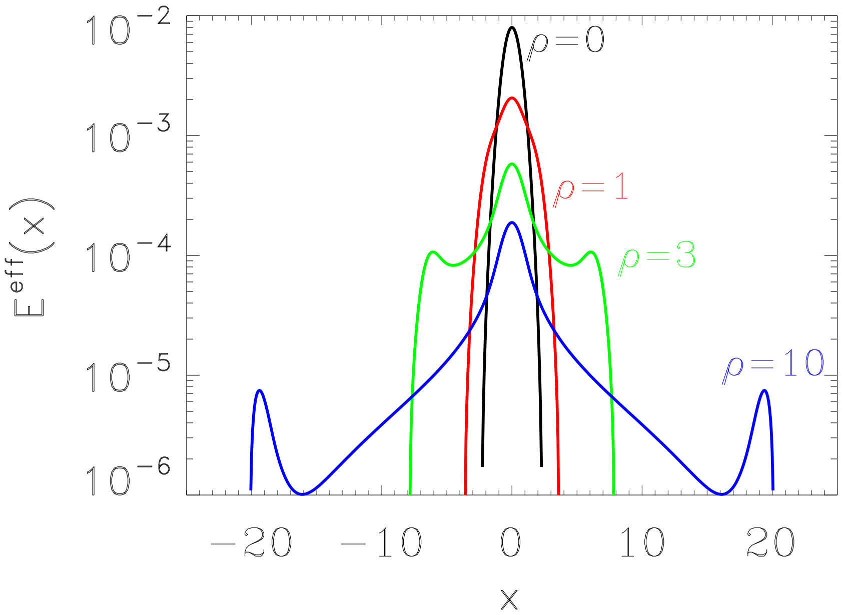
$E^{\text{eff},A}$ for different gyroradii



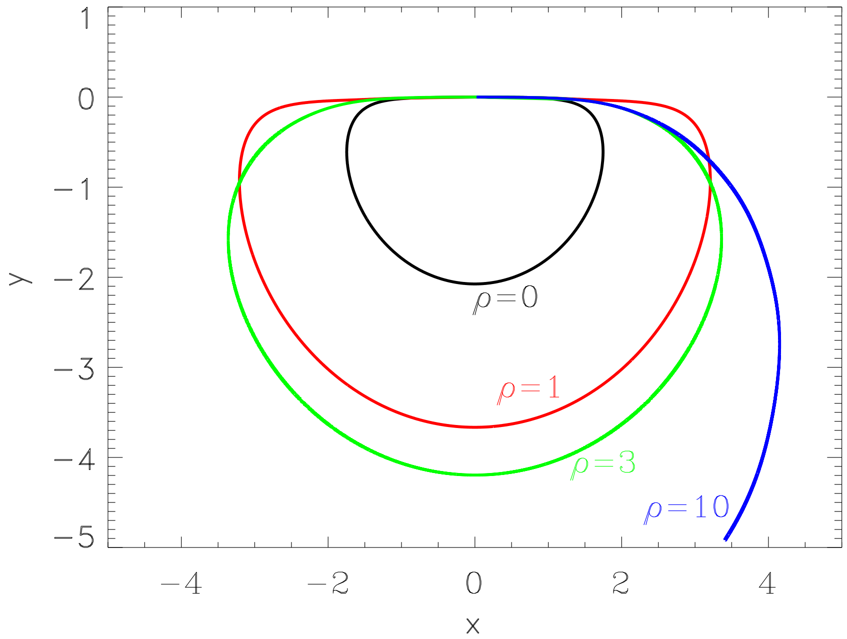
'Decorrelation trajectories' in a certain subensemble for different gyroradii

- Special ring structure of $E^{\text{eff},A}$ leads to wider particle trajectories

Explanation of the increase of D ('method B')



$E^{\text{eff,B}}$ for different gyroradii



'Decorrelation trajectories' in a certain subensemble for different gyroradii

- Widening of $E^{\text{eff,B}}$ leads to wider particle trajectories

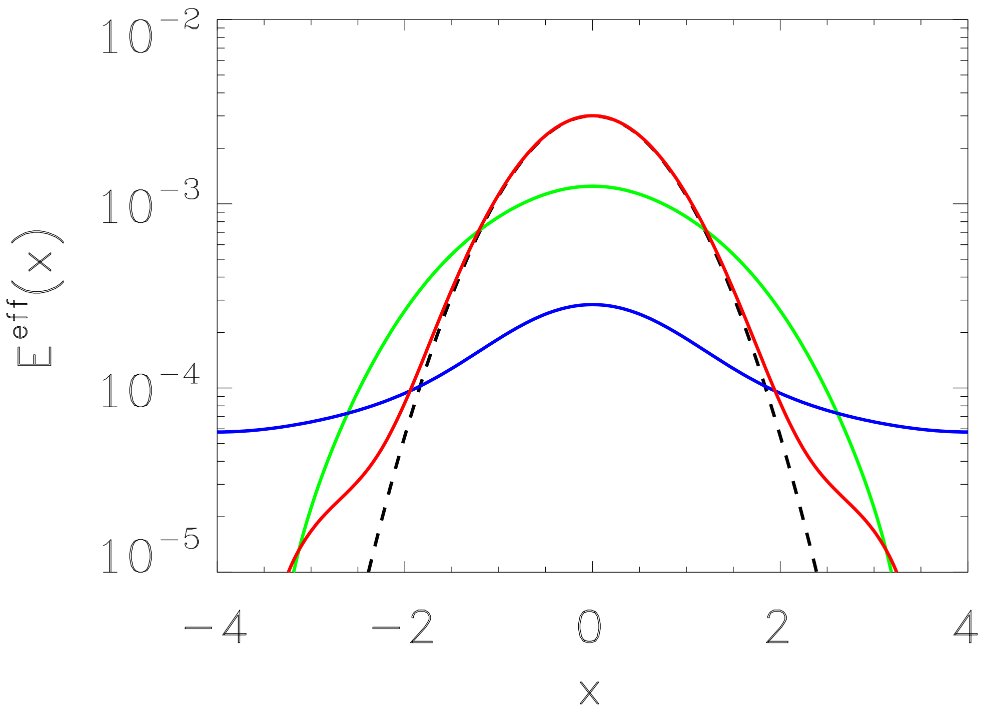
Comparison with direct numerical simulations

DCT method vs. direct numerical simulations

Creation of an isotropic stochastic potential:

$$\phi(\vec{x},t) = \sum_{i=1}^{N} A_i \sin(\vec{k}_i\vec{x} + \omega_i t + \varphi_i)$$

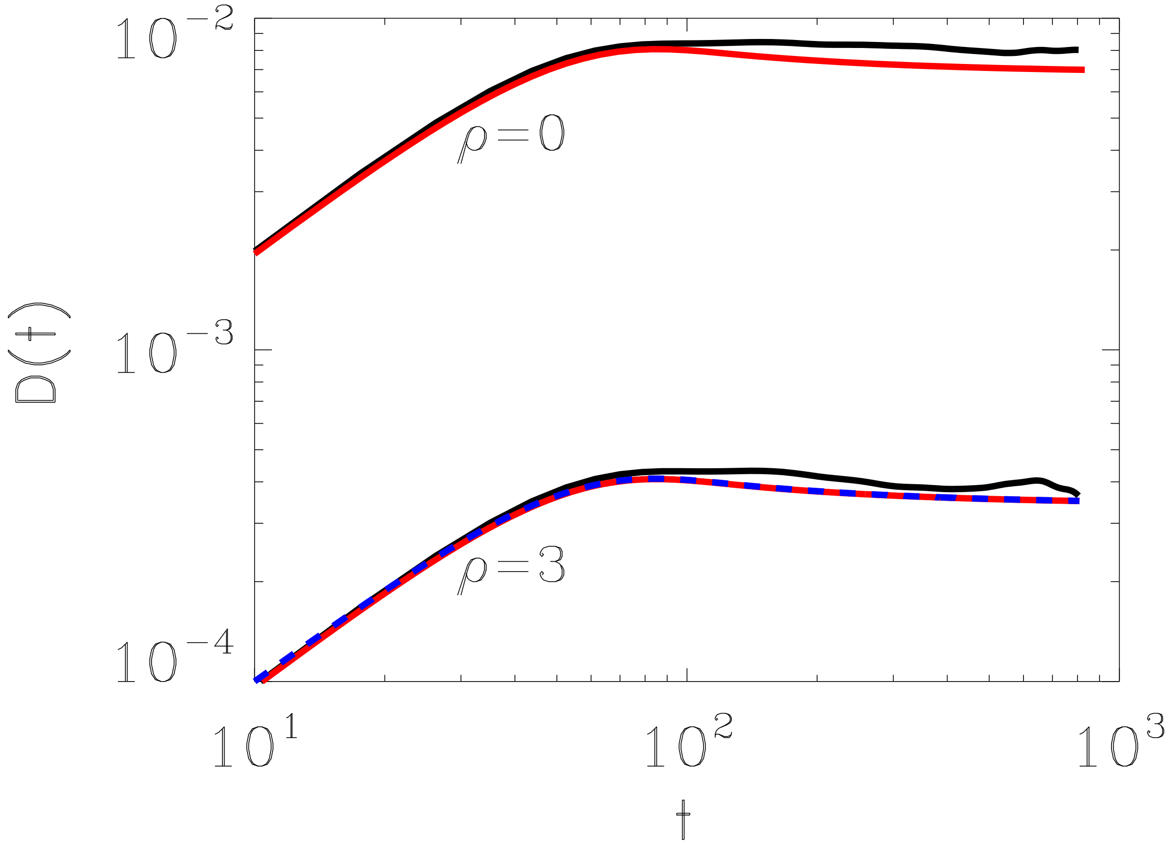
$$E(\vec{x},t) \equiv \langle \phi(0,0)\phi(\vec{x},t)\rangle = \sum_{i=1}^{N} \frac{A_i^2}{2}\cos(\vec{k}_i\vec{x} + \omega_i t)$$



- $\rho = 0$
- $\rho = 0.75$
- $\rho = 3$
- $E(x) = Ae^{-x^2}$

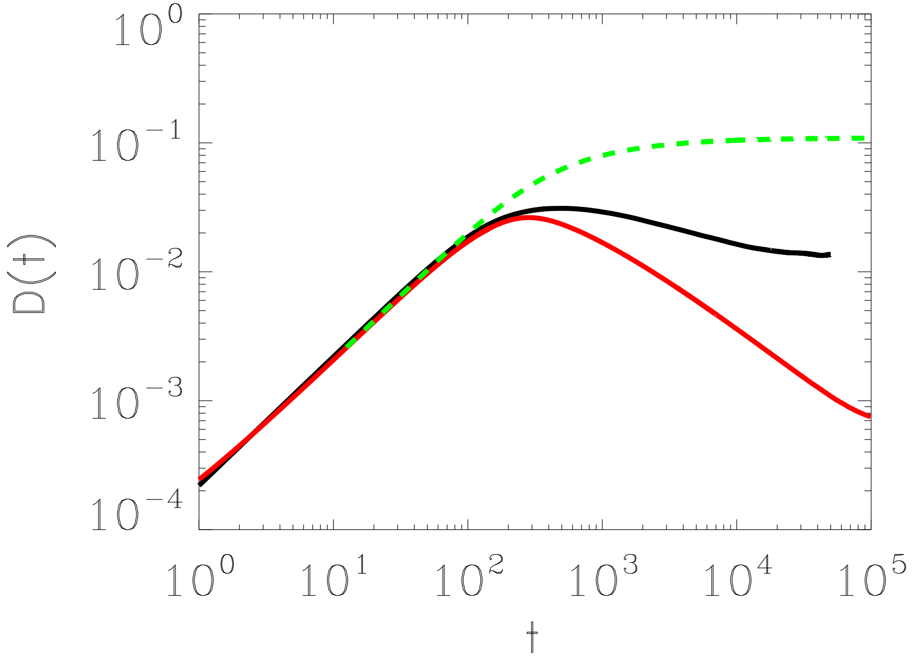
DCT method vs. direct numerical simulations

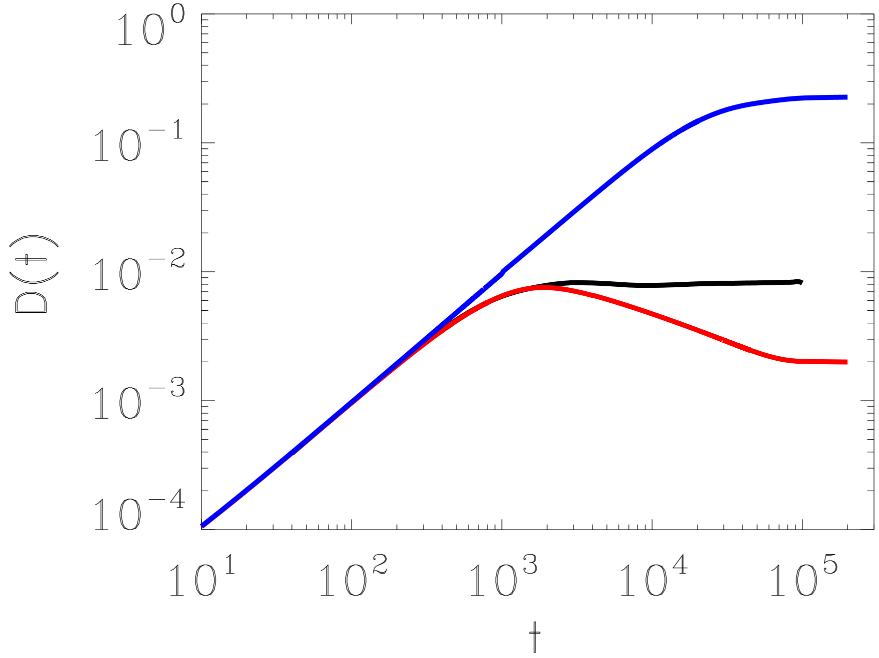
$K = 0.18$



- Direct numerical simulation
- Decorrelation trajectory method (B)
- Decorrelation trajectory method (A)

DCT method vs. direct numerical simulations





$K = 180,\ \rho = 0$

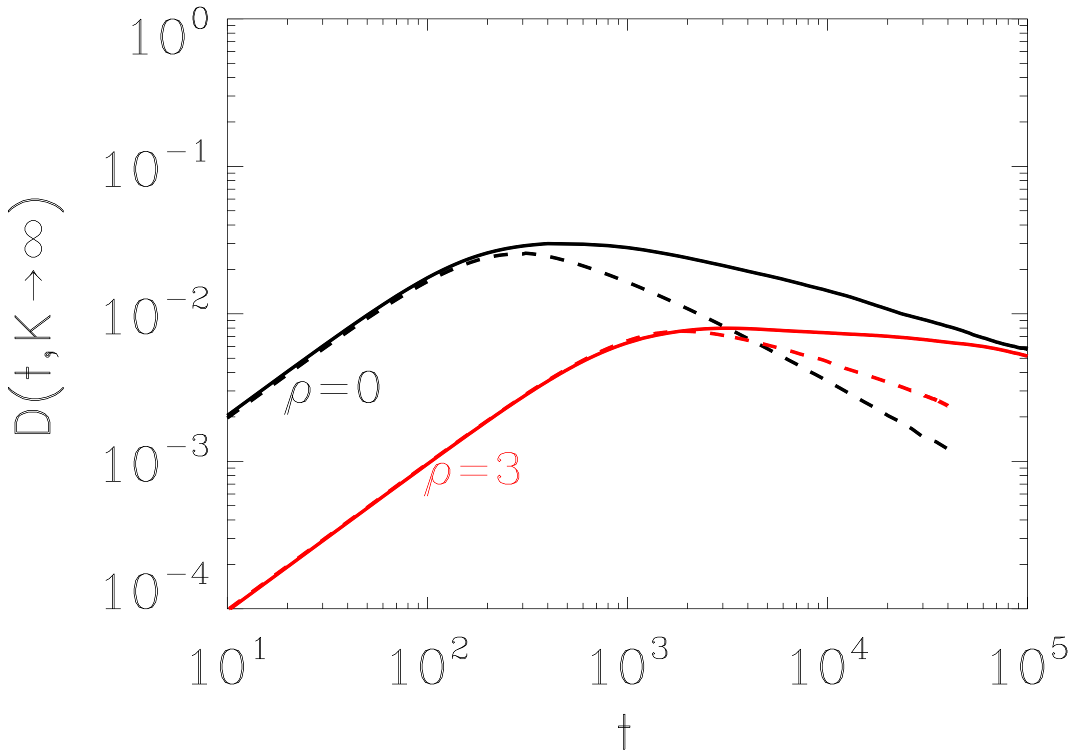
- Direct Simulation
- Decorrelation trajectory method
- Corrsin approximation

$K = 180,\ \rho = 3$

- Direct Simulation
- Decorrelation trajectory method (B)
- Decorrelation trajectory method (A)

DCT method vs. direct numerical simulations

$K = \infty$ (static potential)



Solid line:
direct simulation

Dashed line:
DCT method

Result:

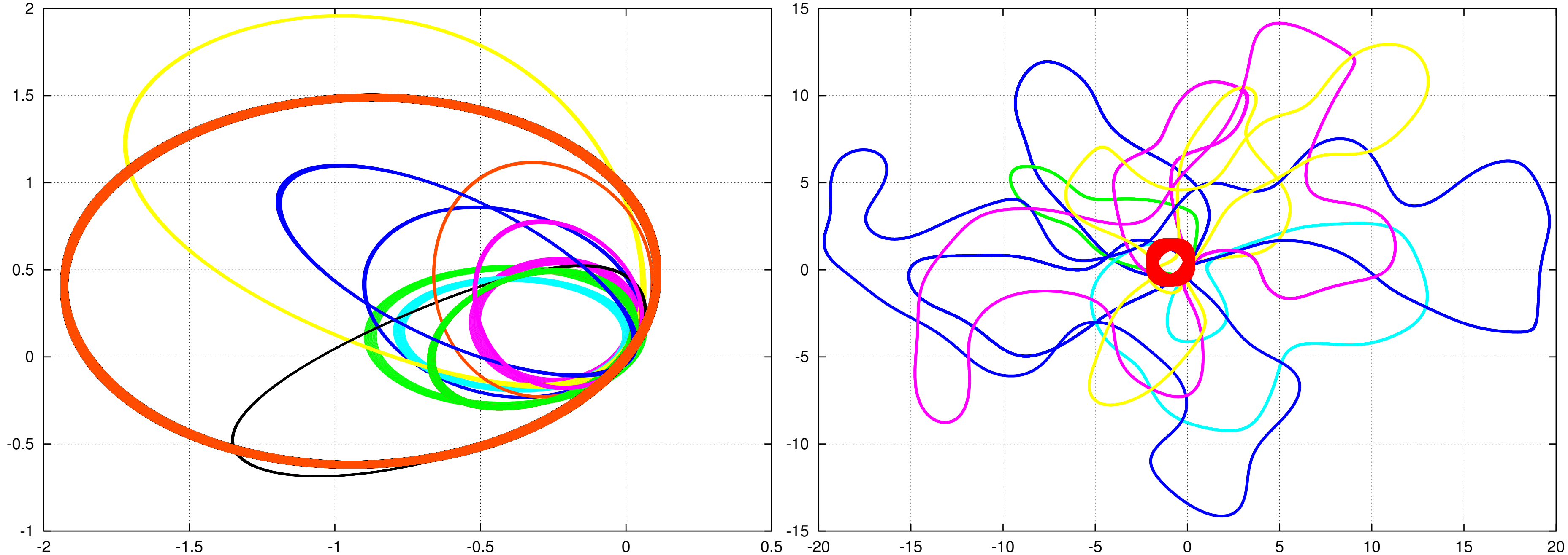
- DCT method: Wrong quantitative results for $K > 1$ and $t > \lambda_C/V$
- However, basic qualitative behavior is recovered

Why does the DCT method fail for large K?

Remember the assumption of the DCT method:

Average trajectory $\equiv$ "decorrelation trajectory"

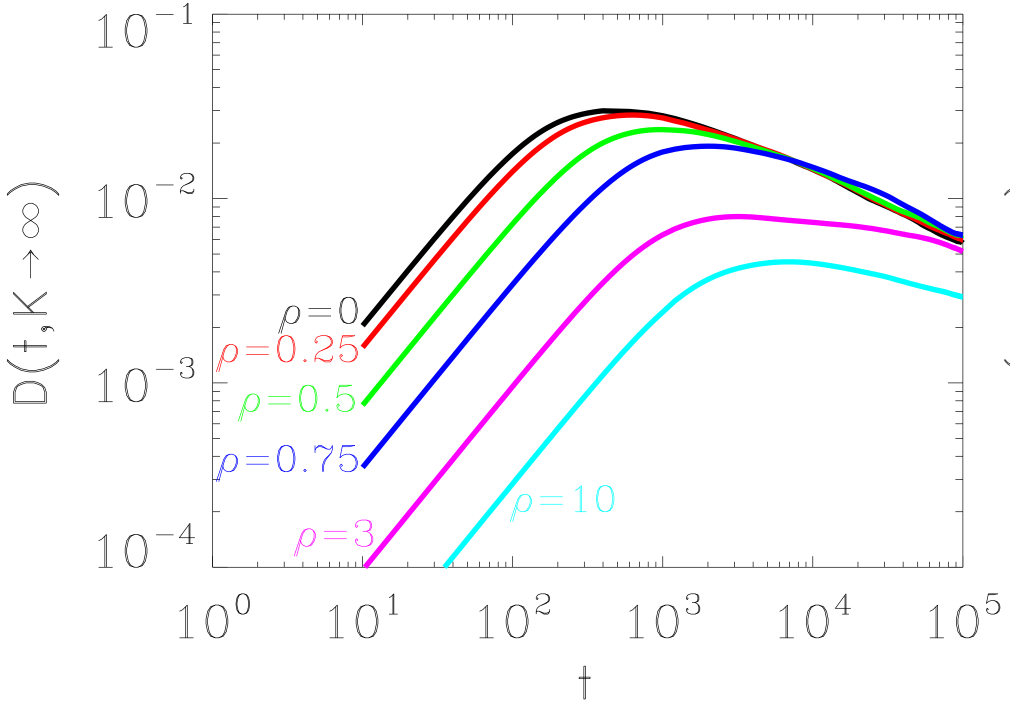
$\rightarrow \langle v^{dr}(\vec{x}(t),t)\rangle_S \approx \langle v^{dr}(\langle\vec{x}(t)\rangle_S,t)\rangle_S$

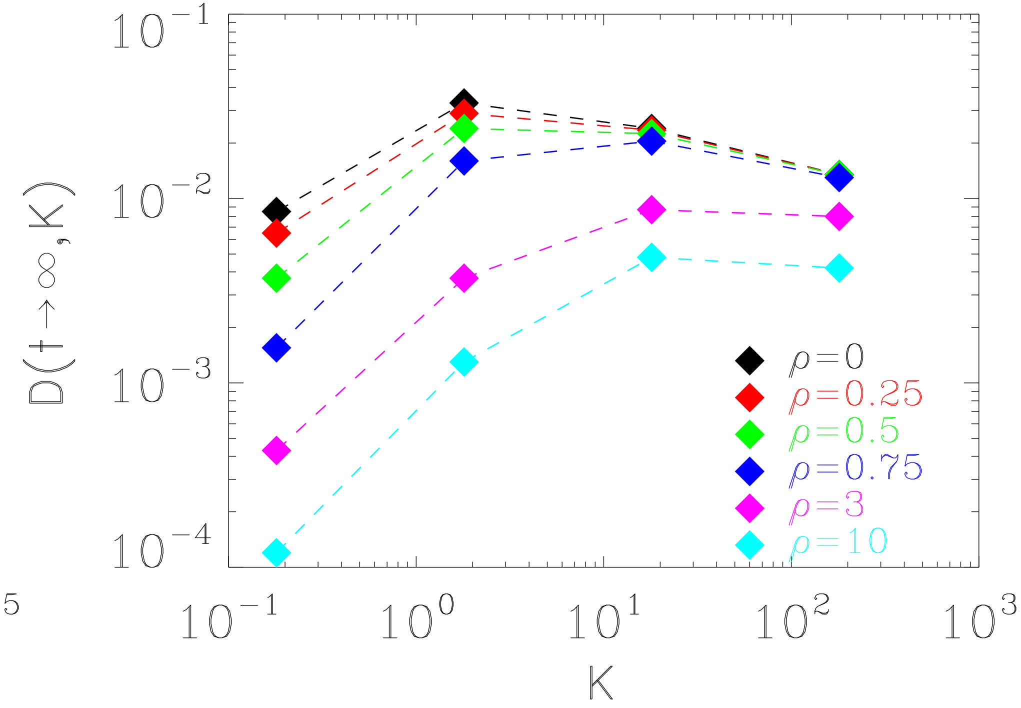


Qualitative comparison of a "decorrelation trajectory" with real trajectories of the same subensemble

- red: decorrelation trajectory
- *other colors:* real trajectories

Further direct numerical simulations





Observations:

- $K \lesssim 1$: Monotonic reduction of D with increasing ρ
- $K \gtrsim 1$: D stays constant for $\rho \lesssim 1$
  Reduced reduction for $\rho > 1$

An analytical approach

An analytical approach

Well known behavior of $D(K)$:

- $K \ll 1$: $D(K) \propto \lambda_C V K = \tau_C V^2$
- $K \gg 1$: $D(K) \propto \lambda_C V K^{\gamma-1} = \frac{\lambda^{2-\gamma} V^{\gamma}}{\tau_C^{1-\gamma}}$; $\gamma \approx 0.7$ [*Isichenko 1991*]

Goal: Find expressions for V^{eff} and λ_C^{eff} from $E^{\text{eff}}(\vec{x})$

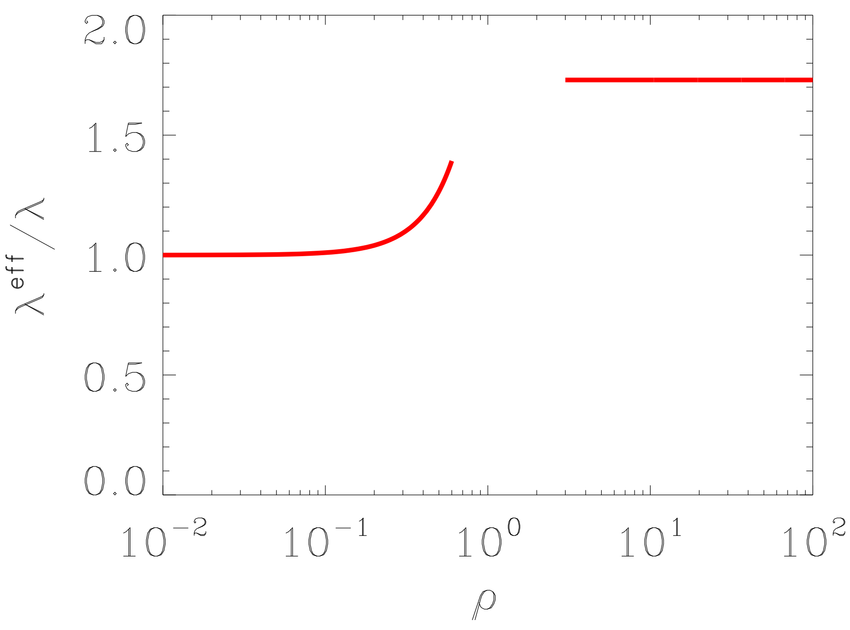
$$E^{\text{eff}}(\vec{x}, \rho) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} d\vec{k}\, e^{i\vec{k}\vec{x}} E(\vec{k}) J_0^2(|\vec{k}|\rho)$$

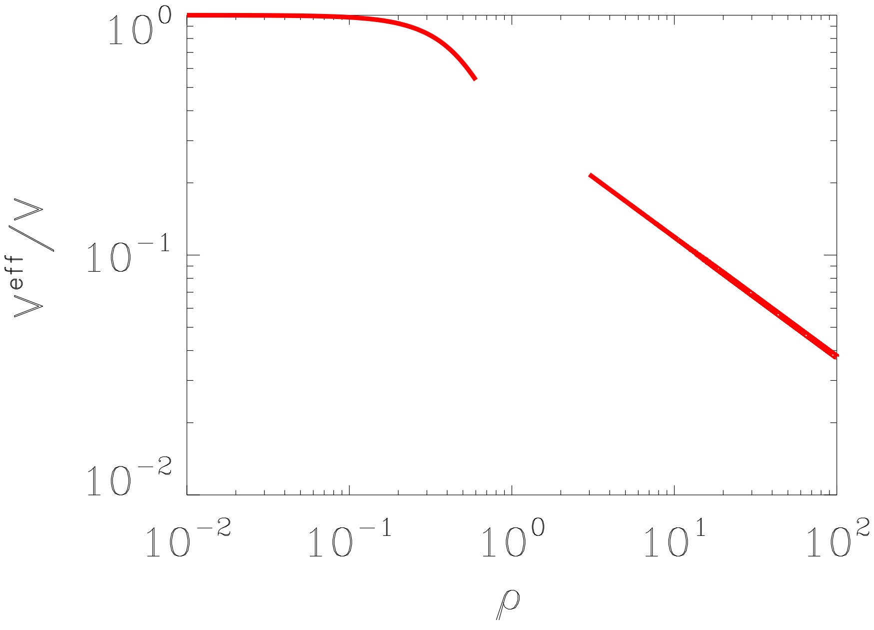
Approximate $J_0^2(|\vec{k}|\rho)$ in the limit of $|\vec{k}|\rho < 1$ and $|\vec{k}|\rho \gg 1$

An analytical approach

Results:

- $\lambda^{\text{eff}}_{\text{small}\,\rho} = \lambda\left(1 + \rho^2 + \mathcal{O}(\rho^4)\right)$
- $V^{\text{eff}}_{\text{small}\,\rho} = V\left(1 - 2\rho^2 + \mathcal{O}(\rho^4)\right)$
- $\lambda^{\text{eff}}_{\text{large}\,\rho} \approx 1.73$
- $V^{\text{eff}}_{\text{large}\,\rho} \equiv \sqrt{-\frac{\partial^2}{\partial x^2}E^{\text{eff}}} = V\sqrt{\frac{1}{4\sqrt{\pi}\rho}}$

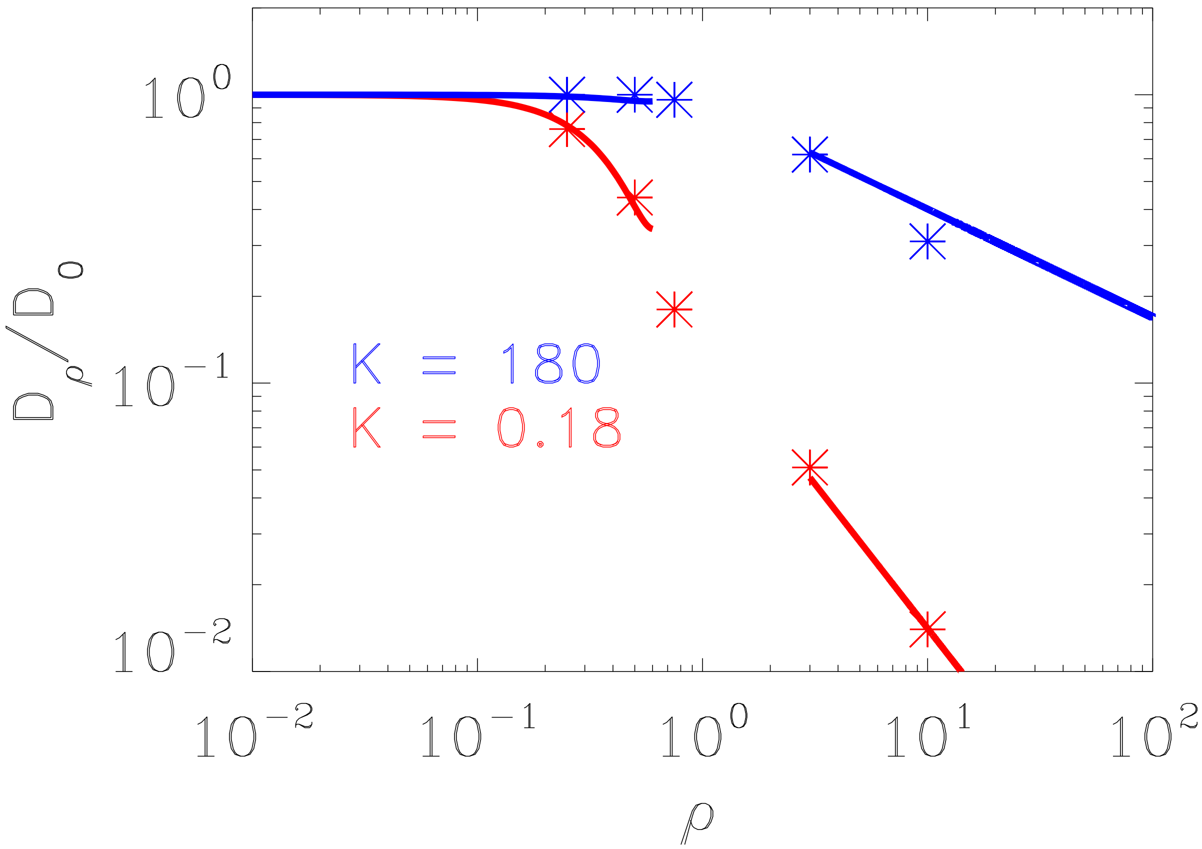




An analytical approach

Final results:

- $\rho \lesssim 1 : D_\rho/D_{\rho=0} \approx 1 + [2 - 3\gamma]\,\rho^2 + \left[\frac{3}{2} - \frac{21}{4}\gamma + \frac{9}{2}\gamma^2\right]\rho^4$
- $\rho \gtrsim 1 : D_\rho/D_{\rho=0} \approx 1.73^{2-\gamma}\,(4\sqrt{\pi}\rho)^{-\gamma/2}$
- $\gamma = 2$ for $K \lesssim 1$; $\gamma \approx 0.7$ for $K \gtrsim 1$



Numerical results

vs.

analytical approach

Summary

- Finite gyroradius effects on the turbulent $E{\times}B$ advection of test particles have been investigated systematically
- They strongly depend on the Kubo number of the potential
- **Small** $\mathbf{K}$ (linear regime): Transport is reduced monotonically with increasing gyroradius ρ
- **Large** $\mathbf{K}$ (nonlinear regime): Transport keeps constant up to $\rho \approx \lambda_C$; then it drops slowly
- Various applications in fusion research and plasma astrophysics

www.ipp.mpg.de/~fsj