Coalescence of gravitationally settling bubbles

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Introduction

Spherical particles in a viscous flow

Particles move vertically at Stokes velocity

Merging: realistic models are complex $\Rightarrow$ simplified models:

1. any two bubbles can merge
2. merging is restricted to bubbles of similar sizes

bubbles merge upon touching ...

Model simple to state, rich in features:

- different stationary regimes,
- self-similar solutions,
- role of local and nonlocal mergings.

Analytical prediction confronted with numerical simulations.

Deriving a kinetic equation

Our model

σ volume, r radius: $\sigma = \frac{4\pi r^3}{3}$

Stokes velocity: $u(r) = \frac{2g\rho}{9\eta} r^2$

g: free fall acceleration,

ρ: density of liquid,

η: dynamic viscosity.

Experimentally, valid for bubbles up to ≈ 1mm.

$$u(\sigma) \propto \sigma^{2/3}$$

The collision integral

Distribution of bubbles characterized by density $n(\sigma, z, t)$

$$\partial_t n + u\partial_z n = \int_{0<\sigma_1<\sigma_2<+\infty} \mathrm{d}\sigma_1 \mathrm{d}\sigma_2 \Big[(u_2 - u_1)\pi(r_1 + r_2)^2 n_1 n_2 \delta(\sigma - \sigma_1 - \sigma_2) - |u - u_1|\pi(r + r_1)^2 n n_1 \delta(\sigma_2 - \sigma - \sigma_1)\Big] \ .$$

Define the interaction kernel:

$$R_{\sigma 12} = |u_2 - u_1|\pi(r_1 + r_2)^2 n_1 n_2 \delta(\sigma - \sigma_1 - \sigma_2)$$

then

$$\frac{\mathrm{d}n}{\mathrm{d}t} = \int_0^{+\infty} \mathrm{d}\sigma_1 \int_0^{+\infty} \mathrm{d}\sigma_2 \left(R_{\sigma 12} - R_{1\sigma 2} - R_{2\sigma 1}\right) \ .$$

This allows merging of bubbles of any sizes.

Collision efficiency

Introduce collision efficiency between bubbles: $0 \leq \mathcal{E}_{12} \leq 1$.

$\longrightarrow \qquad R_{\sigma 12}\mathcal{E}_{12} - R_{1\sigma 2}\mathcal{E}_{\sigma 2} - R_{2\sigma 1}\mathcal{E}_{\sigma 1}$

$$\mathcal{E}_{12} = \begin{cases} 1 & \text{if } q^{-1} < \sigma_1/\sigma_2 < q, \\ 0 & \text{otherwise.} \end{cases}$$

$q > 1$: maximum allowed volume ratio for bubble merging

Kolmogorov-Zakharov solution

Scaling ($n \sim \sigma^\nu$) stationary solution with non-zero flux, called a Kolmogorov-Zakharov (KZ) solution

Cascade of a conserved quantity (void volume)

Zakharov transform

Rigorous derivation through Zakharov transform

Adimensionalize by $\sigma_1 = \sigma_1' \sigma$, $\sigma_2 = \sigma_2' \sigma$:

$$R_{\sigma 12} = C \sigma^{2\nu+1/3} |{\sigma_2'}^{2/3} - {\sigma_1'}^{2/3}| ({\sigma_1'}^{1/3} + {\sigma_2'}^{1/3})^2 {\sigma_1'}^{\nu} {\sigma_2'}^{\nu} \delta(1 - \sigma_1' - \sigma_2')$$

Zakharov transformation: pass in $R_{1\sigma 2}$ to new variables $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$

$$\sigma_1' = \frac{\tilde{\sigma}_2}{\tilde{\sigma}_1}\ , \qquad \sigma_2' = \frac{1}{\tilde{\sigma}_1}\ ,$$

$R_{1\sigma 2}$ transforms to $\sigma_1^{-10/3-2\nu} R_{\sigma 12}$. Similar transform for $R_{2\sigma 1}$.

Combine transformed terms:

$$0 = \int_0^{+\infty} \mathrm{d}\sigma_1 \int_0^{+\infty} \mathrm{d}\sigma_2 \; (1 - \sigma_1^{-10/3-2\nu} - \sigma_2^{-10/3-2\nu}) R_{\sigma 12} \; .$$

$$\longrightarrow \quad -10/3 - 2\nu = 1 \quad \longrightarrow \quad \boxed{\nu = -13/6}$$

KZ solution is only a *true* solution of the kinetic equation if the collision integral on the RHS of the latter (prior to the Zakharov transformation) converges.

This property is called locality, and it physically means that the bubble kinetics is dominated by mergings of bubbles with comparable (rather than very different) sizes.

Dimensional analysis

For our simple setup one could derive the KZ distribution without recourse to the Zakharov transform (cf. Falkovich' lectures).

Locality of interaction with small and large bubbles, as dependent on the scaling exponent of $n(\sigma)$:

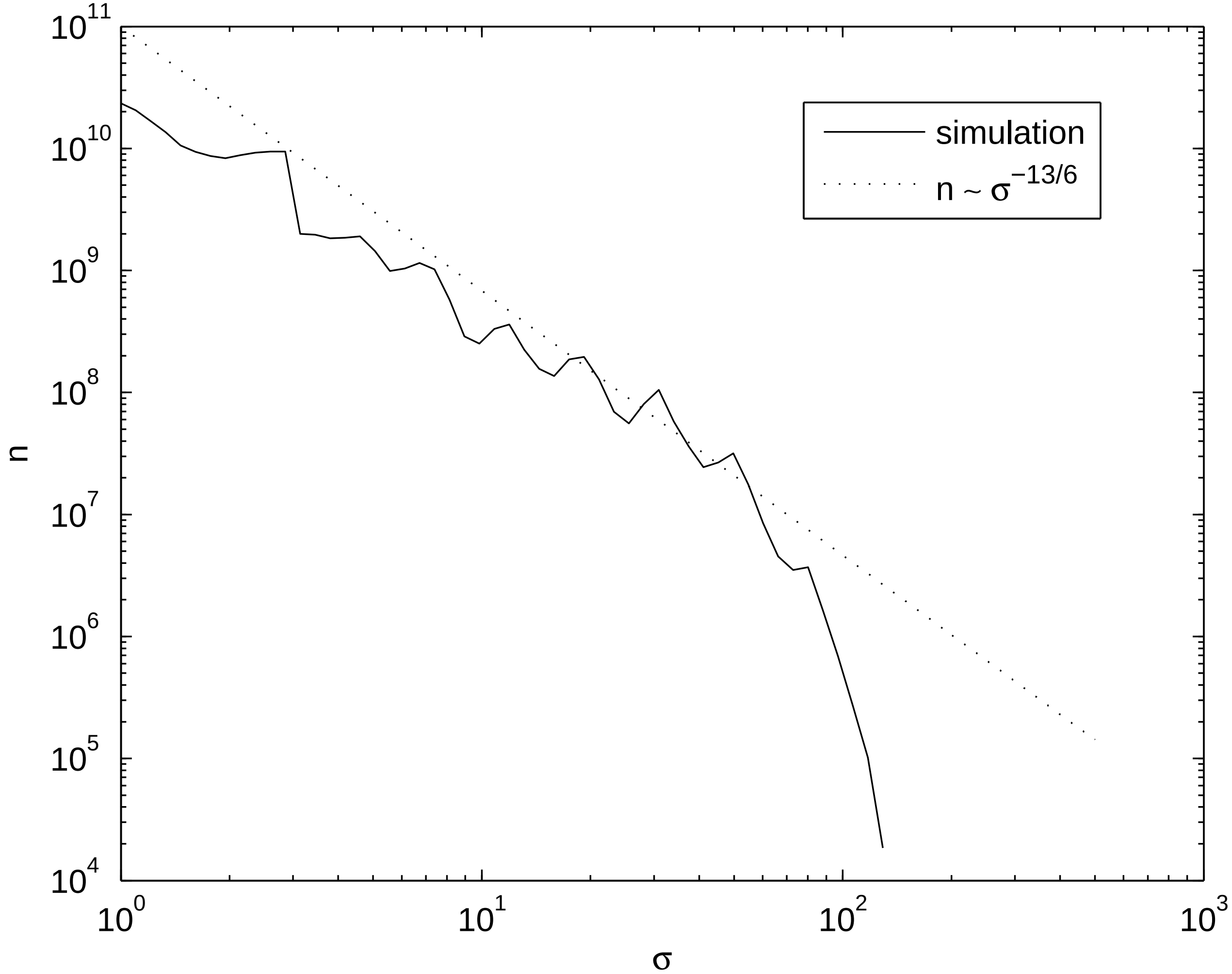
	$\nu < -\frac{7}{3}$	$-\frac{7}{3} \leq \nu \leq -2$	$-2 < \nu$
upper	local	non-local	
lower	non-local		local

KZ spectrum in the system with forced locality

Locality of interactions, and therefore validity of the KZ solution, are immediately restored if one modifies the model by introducing the local collision efficiency kernel.

This kernel is a homogeneous function of degree zero in the σ, therefore KZ exponent obtained via the Zakharov transformation remains the same.

DNS of KZ spectrum with forced locality



Kinetics dominated by nonlocal interactions

Non-locality $\longrightarrow$ reduce kinetic equation to a differential equation.

Contrib. from non-local interactions with smallest bubbles ($\sigma_1 \ll \sigma$):

$$-c_1 \partial_\sigma (\sigma^{4/3} n) \qquad \text{where} \qquad c_1 = \int_{\sigma_{\text{min}}} n_1 \sigma_1 \text{d}\sigma_1 \ .$$

Contrib. from non-local interactions with largest bubbles ($\sigma_1 \gg \sigma$):

$$-c_2 n \qquad \text{where} \qquad c_2 = \int^{\sigma_{\text{max}}} n_1 \sigma_1^{4/3} \text{d}\sigma_1 \ .$$

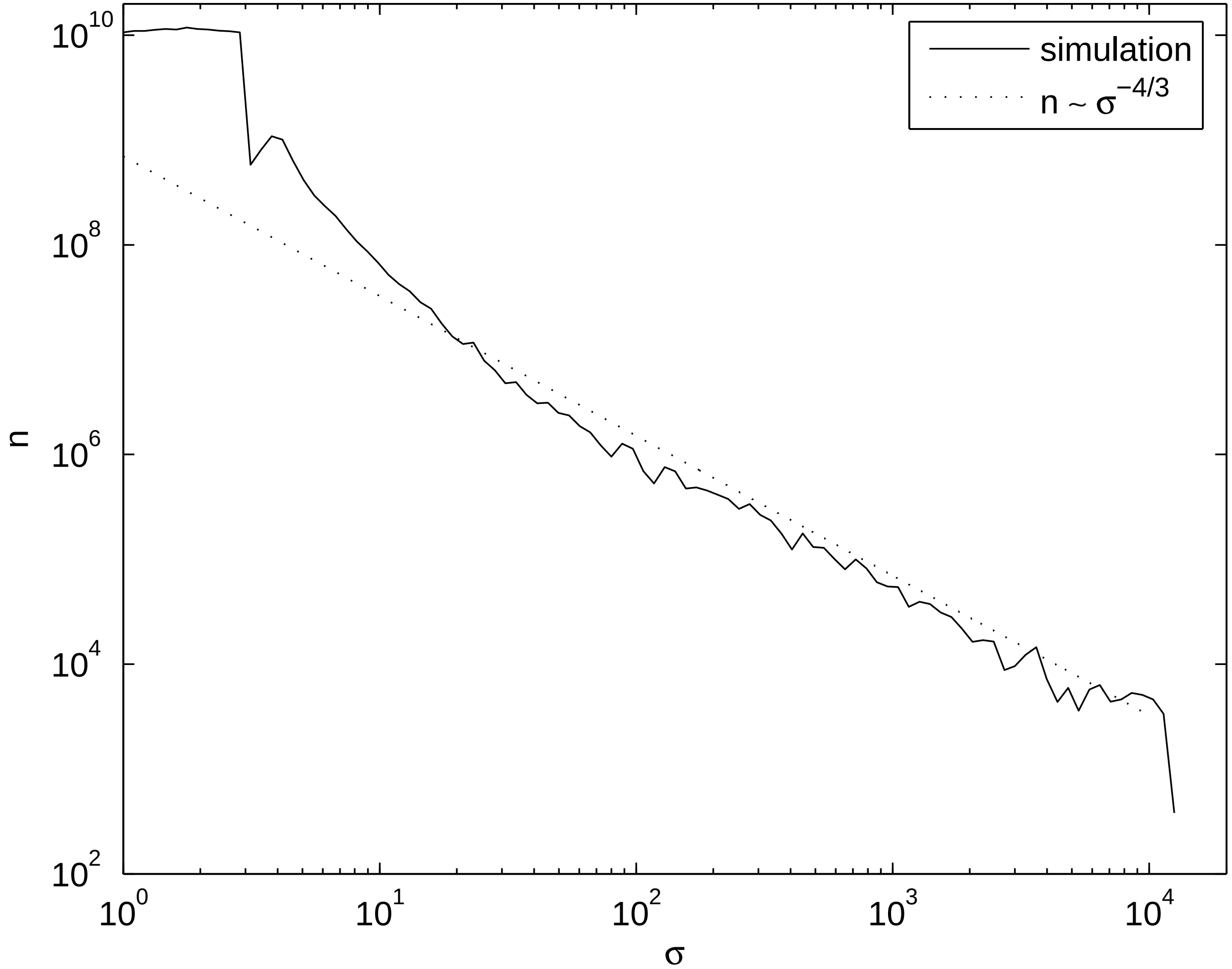
$\rightarrow$ Effective kinetic equation (nonlocal interactions dominant):

$$\frac{\text{d}n}{\text{d}t} = -c_1 \partial_\sigma (\sigma^{4/3} n) - c_2 n$$

Steady state: $\text{d}n/\text{d}t = 0$, leads to $\boxed{n = A\sigma^{-4/3} e^{\frac{3c_2}{c_1}\sigma^{-1/3}}}$, A const. > 0.

Not a pure power law!

DNS of steady state without forced locality



Self-similar solutions

Stationary homogeneous solutions: physical meaning ?
Assumes homogeneity in space + sink at **large volumes**
$\longrightarrow$ not realistic

More realistic to consider:

1. time dependent solutions without sink,
2. height dependent steady-state, sink = surface

Asymptotics: <u>self-similar solutions</u> of the kinetic equation.

Locality:

1. no forced locality ... still local ...
2. local collisional efficiency
3. super-local model $\rightarrow$ Burgers equation

Steady state z–dependent solution

Time-independent state.

Self-similar: verifies scaling relation $\quad n(\sigma, z) = z^{\alpha} f(z^{\beta}\sigma)$

To determine α and β we need two relationships.

Introduce $\tau = z^{\beta}\sigma$ to replace all occurrences of σ:

$$z^{\alpha-\frac{2}{3}\beta-1}[\alpha f(\tau) + \beta\tau f'(\tau)] = z^{2\alpha-\frac{7}{3}\beta} \int\limits_{0<\tau_1<\tau_2<+\infty} \mathrm{d}\tau_1 \mathrm{d}\tau_2 \left(T_{\tau 12} - T_{21\tau}\right)$$

where $T_{\tau 12} = C|\tau_2^{2/3} - \tau_1^{2/3}|(\tau_1^{1/3} + \tau_2^{1/3}) f(\tau_1) f(\tau_2) \delta(\tau - \tau_1 - \tau_2)$

Equal powers of z on both sides $\rightarrow$

$$\alpha - \frac{2}{3}\beta - 1 = 2\alpha - \frac{7}{3}\beta \ .$$

Constant flux of mass through a given height z:

$$\int n(z,\sigma) u \sigma \mathrm{d}\sigma = \int z^{\alpha} f(\tau) z^{-2\beta/3} \tau^{2/3} z^{-\beta} \tau z^{-\beta} \mathrm{d}\tau$$

The total power of z should be 0 for z to vanish $\rightarrow$

$$\alpha - \frac{8}{3}\beta = 0 \ .$$

we find

$$\alpha = -\frac{8}{3} \ , \qquad \beta = -1 \ ,$$

implying $\boxed{n(\sigma, z) = z^{-8/3} f(\sigma/z)}$.

Unsteady height independent solution

We can treat similarly the case of distribution independent of z but dependent on time.

solution of the form $n(\sigma, t) = t^{\alpha} f(t^{\beta}\sigma)$.

Introduce $\tau = t^{\beta}\sigma$

$$t^{\alpha-1}[\alpha f(\tau) + \beta\tau f'(\tau)] = t^{2\alpha-\frac{7}{3}\beta} \int_{0<\tau_1<\tau_2<+\infty} \mathrm{d}\tau_1 \mathrm{d}\tau_2 \, (T_{\tau 12} - T_{21\tau})$$

first relationship

$$\alpha - 1 = 2\alpha - \frac{7}{3}\beta \ .$$

Conservation of mass: $\int n(t,\sigma)\sigma \mathrm{d}\sigma = \int t^{\alpha} f(\tau) t^{-\beta} \tau t^{-\beta} \mathrm{d}\tau$: gives

$$\alpha - 2\beta = 0 \ .$$

We arrive at

$$\alpha = 6 \ , \qquad\qquad \beta = 3 \ ,$$

implying $n(\sigma, t) = t^6 f(\sigma t^3)$.

These exponents seem impossible:
$\alpha, \beta > 0 \ \Rightarrow$ distribution of sizes gets peaked around 0 as $t \to \infty$:
$\to$ contradiction for a coagulation process ....

Locality of the self-similar solutions

Locality was assumed in derivations above.
Asymptotic behavior of self-similarity function $f(\tau)$?

Hypothesis (self-consistent):

- at very large τ the collision integral is dominated by contributions of the range of much smaller τ
- at very small τ the collision integral is dominated by contributions of the range of much larger τ.

Consider large τ: in the z-dependent steady state:

$$u\partial_z n = -c_1 \partial_\sigma (\sigma^{4/3} n)$$

which for $f(\tau)$ reduces to

$$\tau^{2/3}[\alpha f + \beta \tau f'] = -c_1 \tau^{1/3}[\frac{4}{3} f + \tau f']$$

Asymptotics $f(\tau) \sim \tau^{-\alpha/\beta}$
Substituting α and β we get $\boxed{f(\tau) \sim \tau^{-8/3}}$.

Small τ range: dominant contribution to collision integral: non-local interaction with large bubbles:

$$u\partial_z n = -c_2 n$$

which for $f(\tau)$ reduces to

$$\tau^{2/3}[\alpha f + \beta\tau f'] = -c_2 f$$

This can be solved explicitly and yields

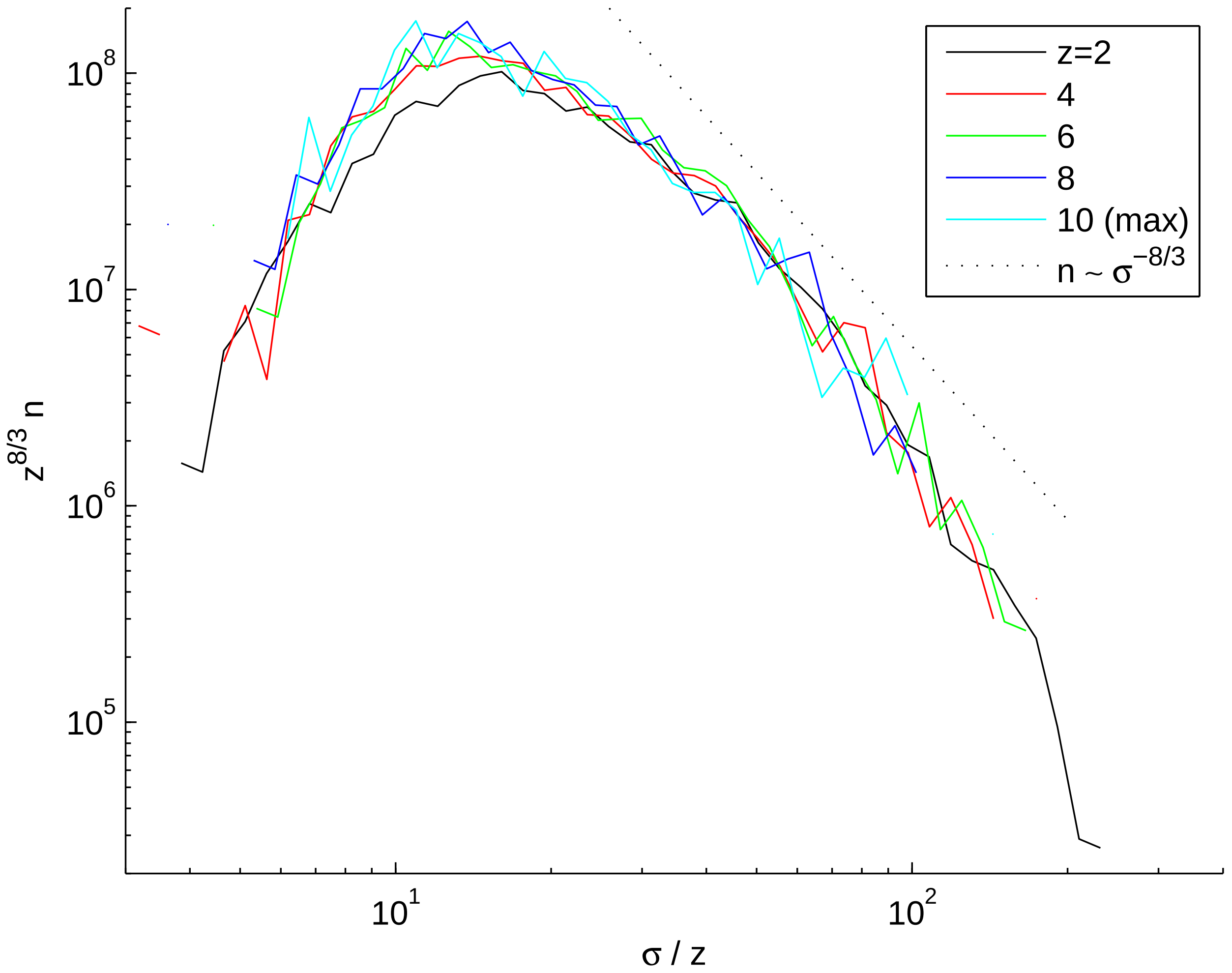
$$\boxed{f(\tau) = Ce^{-\frac{3c_2}{2}\tau^{-2/3}}\tau^{-8/3}}$$

where $C > 0$

Very strong stretched exponential decay of f at small τ
$\rightarrow$ self-consistency of our hypotheses

Self-similar solutions local, even without local collisional efficiency!

Numerical verification of self-similarity without forced locality



Burgers equation for local interaction case

"Super-local" model:

preserves the essential scalings of the original kinetic equation

$$\partial_t n + u\partial_z n = -\sigma^{-1}\partial_\sigma(\sigma^{13/3}n^2)$$

reminiscent of Burgers' equation.

Same self-similarity exponents as above, in either case of stationary or homogeneous self-similar solutions.

Time independent solution

Steady state in t only $\rightarrow$

$$u\partial_z n = -\sigma^{-1}\partial_\sigma(\sigma^{13/3}n^2)\ .$$

Turn this into Burgers' equation by introducing new variable s such that $\sigma = s^\alpha$ and the new function $g(s) = Cs^\beta n(\sigma(s))$.

Then $\partial_z g = -(C\alpha)^{-1}s^{\beta-8\alpha/3+1}\partial_s(s^{13\alpha/3-2\beta}g^2)$.

Setting $\beta - 8\alpha/3 + 1 = 0$ and $13\alpha/3 - 2\beta = 0$ and $(C\alpha) = 2$:

$$\partial_z g = -g \partial_s^2 g$$

$\rightarrow \quad \alpha = 2$, $\beta = 13/3$ and $C = 1$.

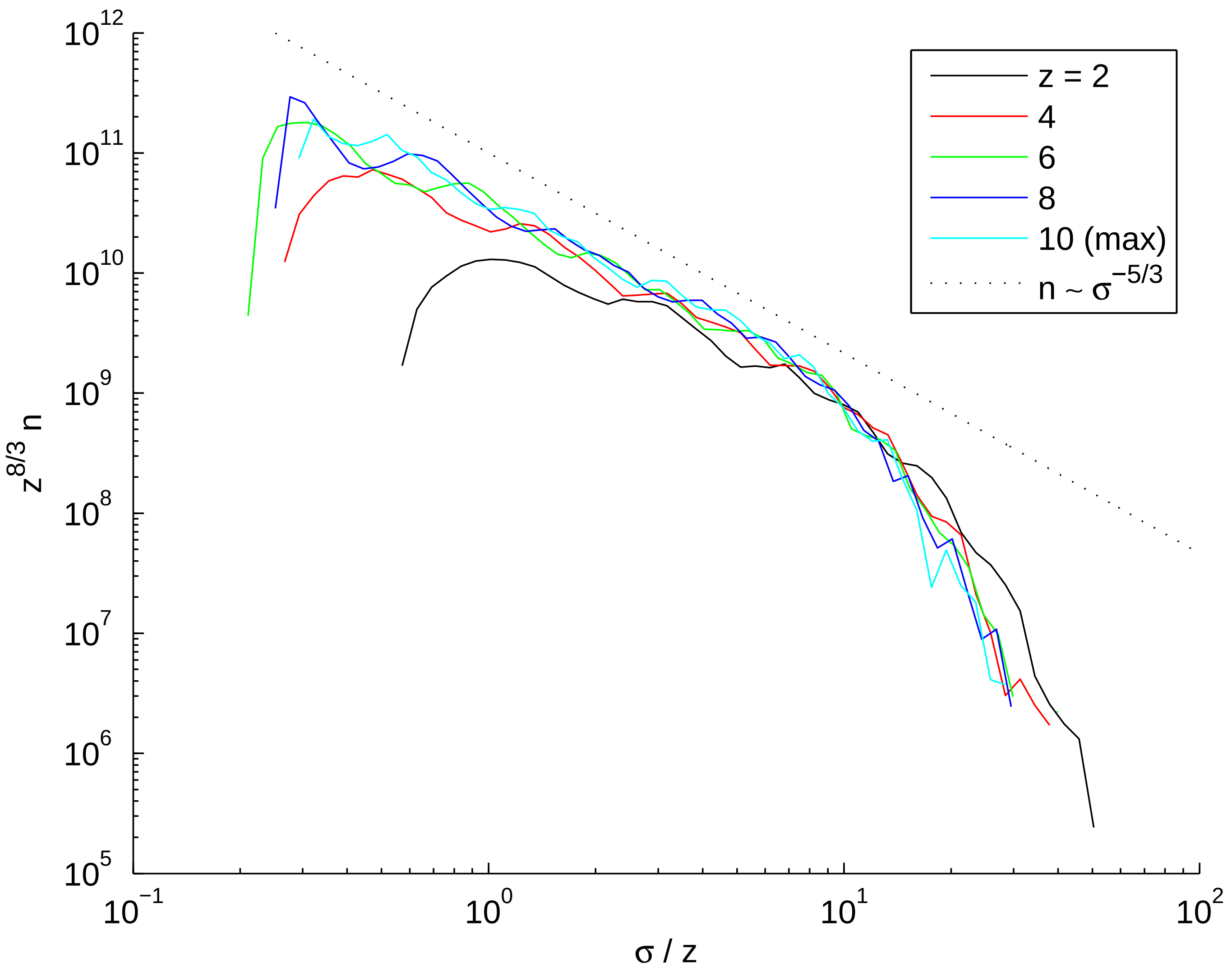
Conservation of total bubble volume $\rightarrow$ conservation of $\int g(s)\mathrm{d}s$ $\rightarrow$ usual Burgers dynamics even for the weak solutions.

At "time" z: shock is at $s_* \sim z^{1/2}$ and its height is $g_* \sim z^{-1/2}$.

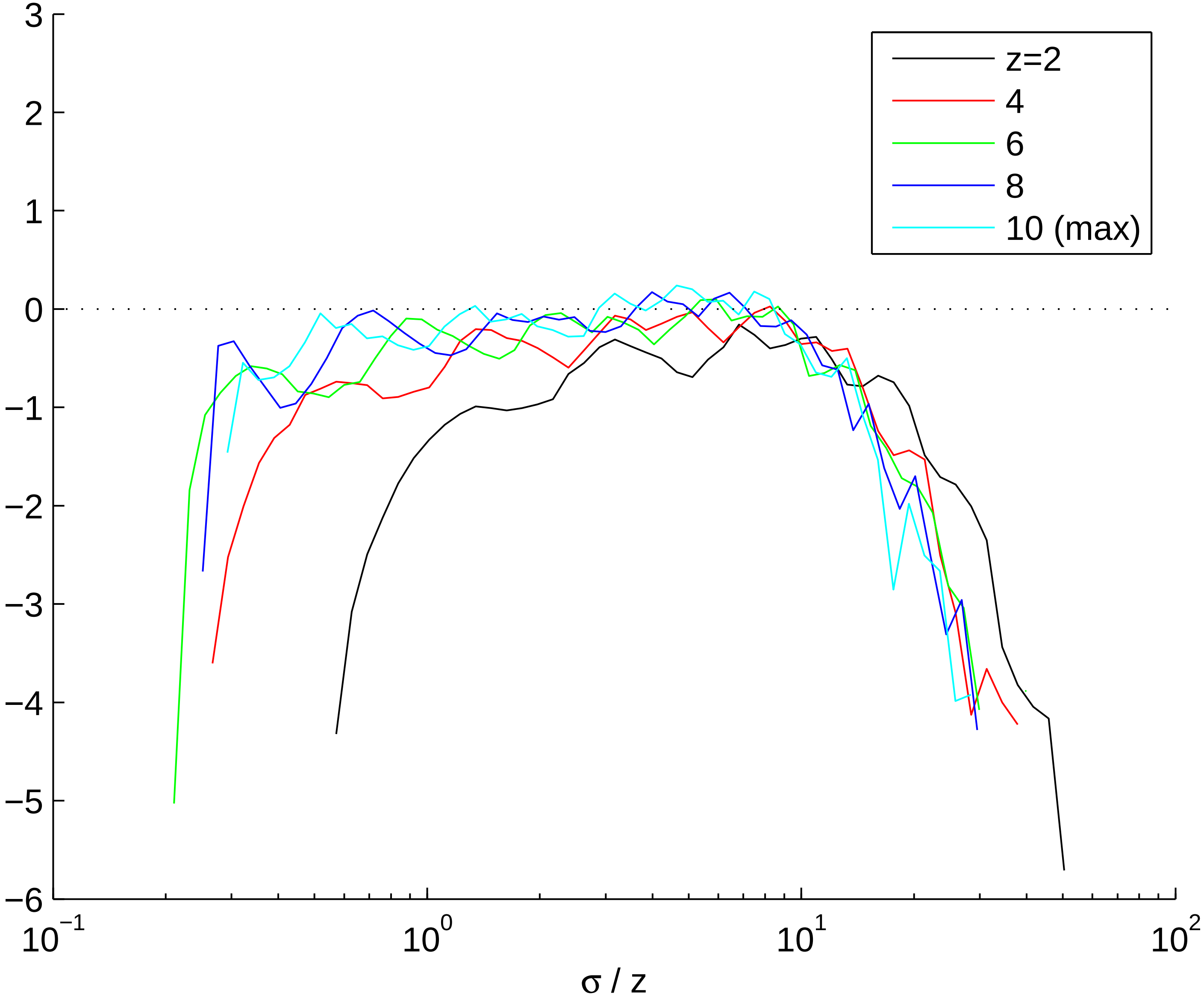
$\rightarrow$ In original variables: $\sigma_* \sim z^{\alpha/2} = z$ and $n_0 \sim z^{-\beta/2} z^{-1/2} = z^{-8/3}$.

$$n(\sigma, z) = \begin{cases} z^{-8/3}(\sigma/z)^{-5/3} & \text{if } \sigma \leq z \\ 0 & \text{if } \sigma > z \end{cases}$$

Remarkably, the $-5/3$ scaling of the self-similar function $f(\tau)$ is indeed observed in the numerical simulation of the bubbles with the forced locality collisional efficiency.



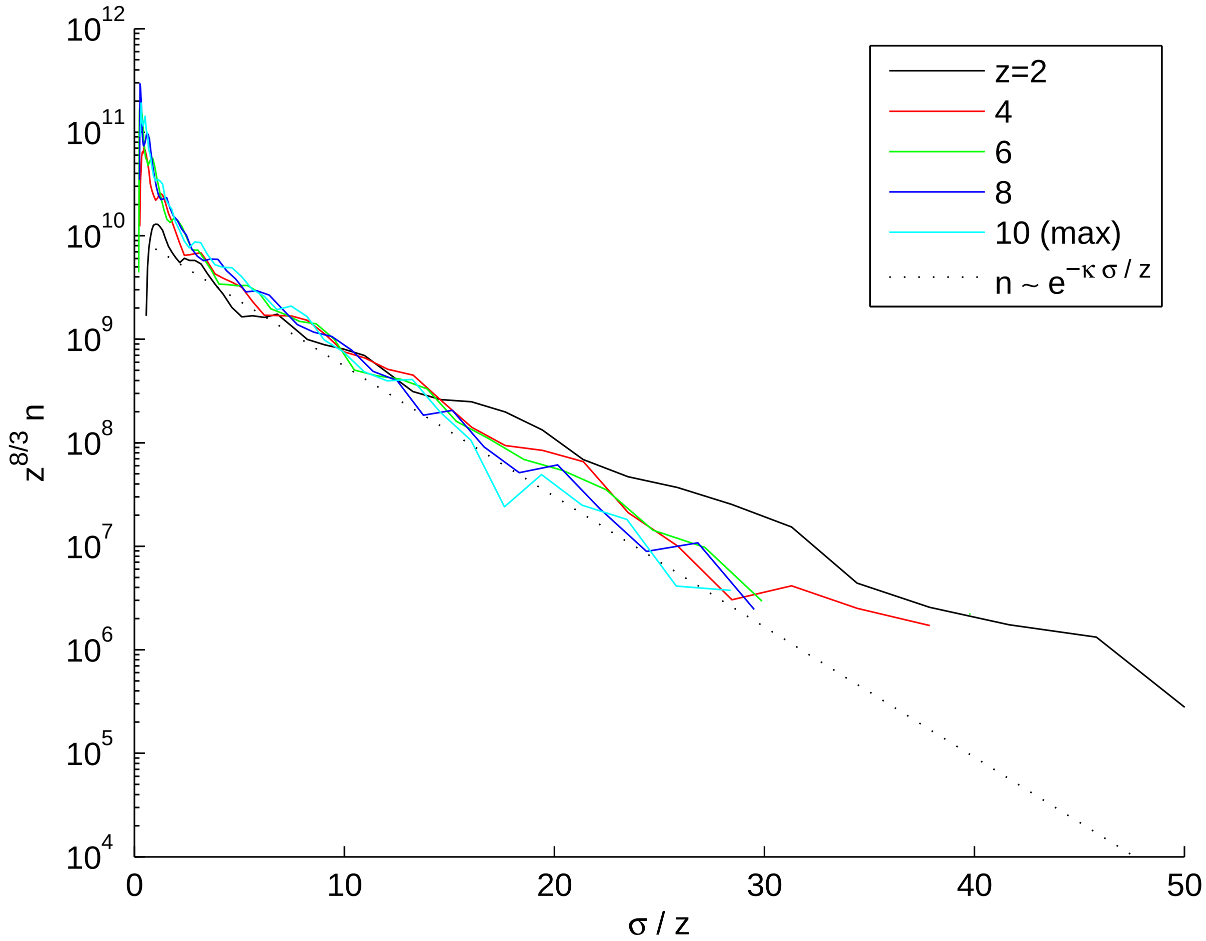
Despite its simplicity, the super-local model is quite efficient!



3
2
1
0
−1
−2
−3
−4
−5
−6
10^{-1}
10^{0}
10^{1}
10^{2}
σ / z
z=2
4
6
8
10 (max)

No shock ... add diffusion regularisation to Burgers' equation

... Correct prediction of large σ tail:



Depth independent solution

Independence of z only $\rightarrow$

$$\partial_t n = -\sigma^{-1}\partial_\sigma(\sigma^{13/3}n^2) \ .$$

We turn this into Burgers' equation as above ...
Then $\partial_t g = -(C\alpha)^{-1}s^{\beta-2\alpha+1}\partial_s(s^{13\alpha/3-2\beta}g^2)$. Set $\beta - 2\alpha + 1 = 0$ and $13\alpha/3 - 2\beta = 0$ and $C\alpha = 2$: werecover Burgers' equation. Happens for $\alpha = -6$, $\beta = -13$ and $C = -1/3$.

In order to know what happens at shocks we need to know what is the quantity conserved by evolution, even at shocks. We know that the original system conserves the volume $\int n\sigma \mathrm{d}\sigma$, which translates for g to conservation of $(\alpha/C)\int g(s)s^{2\alpha-\beta-1}\mathrm{d}s$, and since $2\alpha - \beta - 1 = 0$ this simply means conservation of $\int g(s)\mathrm{d}s$. Thus once again we really deal with the usual Burgers dynamics.

If the initial distribution of n is peaked around σ_0 with height n_0 then the initial distribution of g is peaked around $s_0 = \sigma_0^{1/\alpha}$ with height $g_0 = C s_0^{\beta} n_0$. It is convenient to suppose that the peak is of compact support. The peak evolves to give a shock and to good approximation we get a single sawtooth shock, the area under the triangle being conserved, and also the first point where $g = 0$ behind the shock. From this we get speed and height of the shock.

Since $C < 0$ the shock moves to the left and hits 0 in finite time, which for n means, since $\alpha < 0$, that there is a finite-time singularity at infinite volume.

Conclusions

Analytical formulae consolidated by direct numerical simulations for

- stationary homogeneous solution
  - with locality (Kolmogorov-Zakharov)
  - without locality
- time-stationary self-similar solution
  - with forced locality
  - without forced locality

Remarkably the crude mean-field hypothesis on which the kinetic equation was based seems to give satisfactory results, at least in the numerically studied cases.

Future directions:

- Can we show validity of mean field theory ?
- Time-dependent spatially homogenous case

Appendix: Locality of power-law distributions

We start with the small σ end. Introduce $f(\sigma_1,\sigma_2) = |u_2 - u_1|(r_1 + r_2)^2 n_1 n_2$. Then the contribution to the collision integral from the bubbles having small size, $\sigma_1 << \sigma$ is well approximated by

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{\ll} n = \int_{\sigma_{\mathrm{min}}}^{\sigma/2} \mathrm{d}\sigma_1 [f(\sigma_1, \sigma - \sigma_1) - f(\sigma_1,\sigma)] \approx \int_{\sigma_{\mathrm{min}}}^{\sigma/2} \mathrm{d}\sigma_1 \sigma_1 \frac{\partial f(\sigma_1,\sigma)}{\partial \sigma} \ .$$

For small σ_1 we also have $f(\sigma_1,\sigma) \sim \sigma^{4/3} n_1 n$ so we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{\ll} n \sim - \left[\int_{\sigma_{\mathrm{min}}} n_1 \sigma_1 \mathrm{d}\sigma_1 \right] \partial_\sigma (\sigma^{4/3} n) \ .$$

The interaction is local at small scales iff the integral above remains finite when $\sigma_{\mathrm{min}} \to 0$. This is equivalent to $\nu > -2$.

Let us now examine locality at the high σ end. Contribution to $\dot{n}$

from bubbles having large size is

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{\gg} n = -\int_{\sigma}^{\sigma_{\max}} \mathrm{d}\sigma_1 f(\sigma_1, \sigma).$$

For $\sigma_1 \gg \sigma$ we have $f(\sigma_1, \sigma) \sim \sigma_1^{4/3} n_1 n$ and therefore

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{\gg} n \sim -n \int^{\sigma_{\max}} n_1 \sigma_1^{4/3} \mathrm{d}\sigma_1 \ .$$

The interaction is local at large scales iff the integral above remains finite when $\sigma_{\max} \to \infty$. This is equivalent to $\nu < -7/3$.

We thus get the picture that for $\nu < -7/3$ the interaction is local at large scales but non-local at small scales. For $-7/3 \leq \nu \leq -2$ both ends are non-local. And for $\nu > -2$ interaction is non-local at large scales but local at small scales. In particular we never have locality at both ends.