On the elimination of the sweeping interactions from theories of hydrodynamic turbulence

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Some Publications.

- This presentation is based on
  1. E. Gkioulekas (2006): Ph.D. thesis, University of Washington (Advisor: Ka-Kit Tung)
  2. E. Gkioulekas: *Physica D*, under review. [nlin.CD/0506064]
- Other relevant papers include:
  1. U. Frisch, *Proc. R. Soc. Lond. A* **434** (1991), 89–99.
  2. U. Frisch, *Turbulence: The legacy of A.N. Kolmogorov*, Cambridge University Press, Cambridge, 1995.
  3. V.S. L'vov and I. Procaccia, *Phys. Rev. E* **52** (1995), 3840–3857.
  4. V.S. L'vov and I. Procaccia, *Phys. Rev. E* **54** (1996), 6268–6284.
  5. R.J. Hill, J. Fluid. Mech. **353** (1997), 67–81.
  6. U. Frisch, J. Bec, and E. Aurell, "Locally homogeneous turbulence: Is it a consistent framework?", [nlin.CD/0502046], 2005.

K41 prediction

- In three-dimensional turbulence there is an energy cascade from large scales to small scales is driven by the nonlinear term of the Navier-Stokes equations
- Using dimensional analysis we get K41 prediction

$$S_n(\mathbf{x}, r\mathbf{e}) = \langle \{[\mathbf{u}(\mathbf{x} + r\mathbf{e}, t) - \mathbf{u}(\mathbf{x}, t)] \cdot \mathbf{e}\}^n \rangle \tag{1}$$

$$= C_n (\varepsilon r)^{n/3}, \text{ for } \eta \ll r \ll \ell_0 \tag{2}$$

$$E(k) = C\varepsilon^{2/3} k^{-5/3}, \text{ for } \ell_0^{-1} \ll k \ll \eta^{-1} \tag{3}$$

- Including intermittency corrections, the real behaviour in the inertial range is:

$$S_n(\mathbf{x}, r\mathbf{e}) = C_n (\varepsilon r)^{n/3} (r/\ell_0)^{\zeta_n - n/3} \tag{4}$$

$$E(k) \sim C\varepsilon^{2/3} k^{-5/3} (k\ell_0)^{5/3 - \zeta_2} \tag{5}$$

- How do we understand: dimensional analysis, intermittency, and universality?

Outline of presentation

- Review of the following ideas:
  - Similarity analysis
  - Frisch reformulation of K41
  - Analytical theories: MSR theory and L'vov-Procaccia theory
- Hierarchical definition of local homogeneity
- Sufficient condition to eliminate sweeping
- Same condition needed to prove 4/5 law
- Stronger condition needed to use the Belinicher-L'vov quasi-Lagrangian transformation
- Open question: More rigorous elimination of sweeping

Similarity analysis I

- Similarity analysis is a generalization of dimensional analysis.

1. E. Hopf, *Statistical hydromechanics and functionals calculus*, J. Ratl. Mech. Anal. **1** (1952) 87–123.
2. S. Moiseev, A. Tur, V. Yanovskii, *Spectra and expectation methods of turbulence in a compressible fluid*, Sov. Phys. JETP **44** (1976) 556–561.
3. A.G. Sazontov, *The similarity relation and turbulence spectra in a stratified medium*, Izv. Atmos. Ocean. Phys. **15** (1979), 566–570.
4. S.S. Moiseev and O.G. Chkhetiani, *Helical scaling in turbulence*, JETP **83** (1996), 192–198.
5. H. Branover, A. Eidelman, E. Golbraikh, and S. Moiseev, *Turbulence and structures: chaos, fluctuations, and helical self organization in nature and the laboratory*, Academic Press, San Diego, 1999.

Similarity analysis II

- Assume gaussian delta-correlated forcing with forcing spectrum $F(k)$ parameterized as

$$F(k) = \varepsilon F_0(k\ell_0) \tag{6}$$

- Using the Hopf formalism, it can be shown rigorously that the energy spectrum satisfies

$$E(k, t|\nu, \varepsilon, \ell_0) = \lambda^{-(2\beta+1)} E(k, \lambda^{1-\beta} t|\lambda^{1+\beta}\nu, \lambda^{3\beta-1}\varepsilon, \lambda\ell_0) \tag{7}$$

- From the conditions $\partial E/\partial t = 0$ and $\partial E/\partial \beta = 0$ we find that

$$E(k, t|\nu, \varepsilon, \ell_0) = \varepsilon^{2/3} k^{-5/3} E_0(k\ell_0, k\eta) \tag{8}$$

  with $\eta \equiv (\nu^3/\varepsilon)^{1/4}$. If F_0 fixed, then E_0 is fixed.
- Assume the limits $\ell_0 \to +\infty$ and $\eta \to 0$ converge (similarity assumption). Likewise for structure functions.

Frisch reformulation of K41. I

- Define the Eulerian velocity differences w_α:

$$w_\alpha(\mathbf{x}, \mathbf{x}', t) = u_\alpha(\mathbf{x}, t) - u_\alpha(\mathbf{x}', t). \tag{9}$$

- H1: Local homogeneity/isotropy/stationarity

$$w_\alpha(\mathbf{x}, \mathbf{x}', t) \overset{\mathbf{x},\mathbf{x}'}{\sim} w_\alpha(\mathbf{x} + \mathbf{y}, \mathbf{x}' + \mathbf{y}, t) \ , \forall \mathbf{y} \in \mathbb{R}^d. \tag{10}$$

$$w_\alpha(\mathbf{x}, \mathbf{x}', t) \overset{\mathbf{x},\mathbf{x}'}{\sim} w_\alpha(\mathbf{x}_0 + A(\mathbf{x} - \mathbf{x}_0), \mathbf{x}_0 + A(\mathbf{x}' - \mathbf{x}_0), t) \ , \forall A \in SO(d). \tag{11}$$

$$w_\alpha(\mathbf{x}, \mathbf{x}', t) \overset{\mathbf{x},\mathbf{x}'}{\sim} w_\alpha(\mathbf{x}, \mathbf{x}', t + \Delta t) \ , \forall \Delta t \in \mathbb{R}. \tag{12}$$

- H2: Self-similarity

$$w_\alpha(\lambda\mathbf{x}, \lambda\mathbf{x}', t) \overset{\mathbf{x},\mathbf{x}'}{\sim} \lambda^h w_\alpha(\mathbf{x}, \mathbf{x}', t) \tag{13}$$

- H3: Anomalous energy sink

Frisch reformulation of K41. II

- The argument
  - H1 and H3 $\Longrightarrow$ 4/5 law $\Longrightarrow \zeta_3 = 1$
  - H2 $\Longrightarrow \zeta_n = nh$
  - Therefore: $\zeta_n = n/3 \Longrightarrow k^{-5/3}$ scaling
- 2005: Frisch questions self-consistency of local homogeneity
- Proof of 4/5 law
  - 1959: Proof by Monin using local homogeneity (in Russian)
  - 1975: Reprinted by Monin and Yaglom book
  - 1995: Frisch proof uses global homogeneity
  - 1996: Lindborg notes that the pressure gradient/velocity field correlations cannot be eliminated by local isotropy
  - 1997: Problem corrected by Hill
  - 1999: Rasmussen proof uses global homogeneity
  - 2006: Gkioulekas notes that local homogeneity not sufficient.

Analytical theories.

- In the beginning: Quasinormal closure models.
- 1957: Kraichnan showed that they give negative $E(k)$.
- 1958: Kraichnan DIA theory $\Longrightarrow k^{-3/2}$ scaling.
- 1961: Wyld shows that DIA is 1-loop line-renormalized diagrammatic theory
- 1962: Experiments confirm $k^{-5/3}$ scaling.
- 1964: Kraichnan notes the need to eliminate the sweeping interactions via a Lagrangian teansformation.
- 1965: LHDIA theory $\Longrightarrow$ Locality $\Longrightarrow k^{-5/3}$ scaling.
- 1973: Martin-Siggia-Rose theory (MSR theory)
- 1977: Phythian reformulates MSR theory in terms of path integrals.

MSR theory. I. Formulation

- In MSR formalism we have a quadratic problem of the form

$$\frac{\partial u_\alpha}{\partial t} = P_{\alpha\delta} V_{\beta\gamma\delta} u_\gamma u_\delta + \mathcal{D}_{\alpha\beta} u_\alpha + P_{\alpha\beta} f_\beta \tag{14}$$

with gaussian forcing: $Q_{\alpha\beta} = \langle f_\alpha f_\beta \rangle$.

- Define the correlators

$$F_{\alpha\beta} = \langle u_\alpha u_\beta \rangle\,, \quad G_{\alpha\beta} = \langle \delta u_\alpha / \delta f_\beta \rangle \tag{15}$$

- The Dyson-Wyld equations are

$$\frac{\partial G_{\alpha\beta}(t)}{\partial t} = \mathcal{D}_{\alpha\gamma} G_{\gamma\beta}(t) + P_{\alpha\beta}\delta(t) + \int_0^t dt_1\ P_{\alpha\gamma}\Sigma_{\gamma\delta}(t_1) G_{\delta\beta}(t - t_1) \tag{16}$$

$$F_{\alpha\beta}(t) = \int dt_1 \int dt_2\ G_{\alpha\gamma}[Q_{\gamma\delta}(t - t_1 + t_2) + \Phi_{\gamma\delta}(t - t_1 + t_2)] G_{\delta\beta}(t_2) \tag{17}$$

MSR theory. II. Diagram expansion

- The operators $\Sigma_{\alpha\beta}$ and $\Phi_{\alpha\beta}$ can be represented with a Feynman diagram expansion

$$\Sigma_{\alpha\beta} = \Sigma^1_{\alpha\beta} + \Sigma^2_{\alpha\beta} + \cdots \quad (18)$$

$$\Phi_{\alpha\beta} = \Phi^1_{\alpha\beta} + \Phi^2_{\alpha\beta} + \cdots \quad (19)$$

- In Eulerian formulation, the 1-loop approximation gives DIA:

$$\Sigma_{\alpha\beta} \approx \Sigma^1_{\alpha\beta} = (V_{\alpha A\Gamma} + V_{\alpha\Gamma A})(V_{\beta B\Delta} + V_{\beta\Delta B})G_{AB}F_{\Gamma\Delta} \quad (20)$$

$$\Phi_{\alpha\beta} \approx \Phi^1_{\alpha\beta} = V_{\alpha A\Gamma}(V_{\beta B\Delta} + V_{\beta\Delta B})F_{AB}F_{\Gamma\Delta} \quad (21)$$

- Problem: In Eulerian formulation, IR divergences arise from sweeping interactions
  - 1987: Belinicher-L'vov quasi-Lagrangian transformation
  - 1995-2001: L'vov and Procaccia go beyond the LHDIA theory

Quasi-Langrangian transformation. I. Definition

- Let $u_\alpha(\mathbf{x}, t)$ be the Eulerian velocity field, and let $\rho_\alpha(\mathbf{x}_0, t_0|t)$ be the position of the unique fluid particle initiated at $(\mathbf{x}_0, t_0)$ at time t relative to its initial position at time t_0.
- First, we introduce $v_\alpha(\mathbf{x}_0, t_0|\mathbf{x}, t)$ as

$$\begin{aligned} \rho_\alpha(\mathbf{x}_0, t_0|t) &= \int_{t_0}^{t} d\tau\, u_\alpha(\mathbf{x}_0 + \rho(\mathbf{x}_0, t_0|\tau), \tau) \\ v_\alpha(\mathbf{x}_0, t_0|\mathbf{x}, t) &= u_\alpha(\mathbf{x} + \rho(\mathbf{x}_0, t_0|t), t). \end{aligned} \tag{22}$$

- Then, we subtract the velocity of the fluid particle uniformly:

$$\begin{aligned} w_\alpha(\mathbf{x}_0, t_0|\mathbf{x}, t) &= v_\alpha(\mathbf{x}_0, t_0|\mathbf{x}, t) - \frac{\partial}{\partial t}\rho_\alpha(\mathbf{x}_0, t_0|t) \\ &= v_\alpha(\mathbf{x}_0, t_0|\mathbf{x}, t) - v_\alpha(\mathbf{x}_0, t_0|\mathbf{x}_0, t) \\ &= u_\alpha(\mathbf{x} + \rho(\mathbf{x}_0, t_0|t), t) - u_\alpha(\mathbf{x}_0 + \rho(\mathbf{x}_0, t_0|t), t). \end{aligned} \tag{23}$$

Quasi-Langrangian transformation. II. Navier-Stokes equations

- Let $w_\alpha(\mathbf{x}_0, t_0|\mathbf{x}, t)$ be defined as

$$W_\alpha(\mathbf{x}_0, t_0|\mathbf{x}, \mathbf{x}', t) \equiv w_\alpha(\mathbf{x}_0, t_0|\mathbf{x}, t) - w_\alpha(\mathbf{x}_0, t_0|\mathbf{x}', t) \tag{24}$$

$$= v_\alpha(\mathbf{x}_0, t_0|\mathbf{x}, t) - v_\alpha(\mathbf{x}_0, t_0|\mathbf{x}', t). \tag{25}$$

- Differentiating with respect to time gives an equation of the form

$$\frac{\partial W_\alpha}{\partial t} + \mathcal{V}_{\alpha\beta\gamma} W_\beta W_\gamma = \nu(\nabla^2_{\mathbf{x}} + \nabla^2_{\mathbf{x}'}) W_\alpha + F_\alpha, \tag{26}$$

where $\mathcal{V}_{\alpha\beta\gamma}$ is a bilinear integrodifferential operator of the form

$$\mathcal{V}_{\alpha\beta\gamma} W_\beta W_\gamma \equiv \iint d\mathbf{X}_\beta d\mathbf{X}_\gamma \, V_{\alpha\beta\gamma}(\mathbf{x}_0|\mathbf{X}_\alpha, \mathbf{X}_\beta, \mathbf{X}_\gamma) W_\beta(\mathbf{X}_\beta) W_\gamma(\mathbf{X}_\gamma). \tag{27}$$

- All the terms, and especially the nonlinear term, are written in terms of velocity differences!

Quasi-Langrangian transformation. III. The theory

- Consider the Dyson-Wyld equations with

$$F_{\alpha\beta}(\mathbf{x}_0, t_0|\mathbf{x}_1, \mathbf{x}_2, t) = \langle w_\alpha(\mathbf{x}_0, t_0|\mathbf{x}_1, t) w_\beta(\mathbf{x}_0, t_0|\mathbf{x}_2, t)\rangle \quad (28)$$

$$G_{\alpha\beta}(\mathbf{x}_0, t_0|\mathbf{x}_1, \mathbf{x}_2, t) = \left\langle \frac{\delta w_\alpha(\mathbf{x}_0, t_0|\mathbf{x}_1, t)}{\delta f_\beta(\mathbf{x}_0, t_0|\mathbf{x}_2, t)} \right\rangle \quad (29)$$

- Main results of L'vov and Procaccia theory.
  - Individual diagrams in $\Sigma_{\alpha\beta}$ and $\Phi_{\alpha\beta}$ converge when $\ell_0 \to \infty$ and $\eta \to 0$.
  - Thus, to n-loop order approximation we obtain K41 prediction $\zeta_n = n/3$.
  - This is a generalization of Kraichnan's LHDIA theory.
  - Diagram locality and rigidity $\Longrightarrow$ Fusion Rules $\Longrightarrow$ Anomalous energy sink.
  - Intermittency emerges via a multi-interaction effect involving all diagrams
  - Scheme for perturbative calculation of scaling exponents ζ_n

Outline of my argument

- The L'vov-Procaccia theory aims to weaken the assumption of self-similarity used by Frisch (H2) while tolerating the other two assumptions: (H1) and (H3)
- A homogeneity assumption stronger than the assumption of local homogeneity, as envisioned by Frisch is required for
  - the elimination of the sweeping interactions
  - the derivation of the $4/5$-law
- The quasi-Lagrangian formulation to eliminate the sweeping interactions uses an even stronger homogeneity assumption which involves many-time correlations instead of one-time correlations.
- Local homogeneity is in fact a consistent framework provided that the sweeping interactions can be eliminated in a more rigorous manner.

Definitions of local homogeneity. I.

- The random velocity field $\mathbf{u}$, is a member of the homogeneity class $\mathcal{H}_m(\mathcal{A})$ where $\mathcal{A} \subseteq \mathbb{R}^d$ a region in $\mathbb{R}^d$, if and only if $\forall n \in \mathbb{N}^*, \forall \mathbf{x}_l, \mathbf{y}_k, \mathbf{y'}_k \in \mathcal{A}$ we have

$$\left(\sum_{l=1}^{m} \partial_{\alpha_l, \mathbf{x}_l} + \sum_{k=1}^{n} (\partial_{\beta_l, \mathbf{y}_l} + \partial_{\beta_l, \mathbf{y'}_l}) \right) \left\langle \left[\prod_{l=1}^{m} u_{\alpha_l}(\mathbf{x}_l, t) \right] \left[\prod_{k=1}^{n} w_{\beta_k}(\mathbf{y}_k, \mathbf{y'}_k, t) \right] \right\rangle = 0$$

- The random velocity field $\mathbf{u}$ is a member of the homogeneity class $\mathcal{H}^*_m(\mathcal{A})$ where $\mathcal{A} \subseteq \mathbb{R}^d$ a region in $\mathbb{R}^d$, if and only if $\forall n \in \mathbb{N}^*, \forall \mathbf{x}_l, \mathbf{y}_k, \mathbf{y'}_k \in \mathcal{A}$ we have

$$\left(\sum_{l=1}^{m} \partial_{\alpha_l, \mathbf{x}_l} + \sum_{k=1}^{n} (\partial_{\beta_l, \mathbf{y}_l} + \partial_{\beta_l, \mathbf{y'}_l}) \right) \left\langle \left[\prod_{l=1}^{m} u_{\alpha}(\mathbf{x}_l, t_l) \right] \left[\prod_{k=1}^{n} w_{\beta_k}(\mathbf{y}_k, \mathbf{y'}_k, t) \right] \right\rangle = 0$$

- We also write $\mathcal{H}_m \equiv \mathcal{H}_m(\mathbb{R}^d)$ and $\mathcal{H}^*_m \equiv \mathcal{H}^*_m(\mathbb{R}^d)$ and define

$$\mathcal{H}_\omega(\mathcal{A}) = \bigcap_{k \in \mathbb{N}} \mathcal{H}_k(\mathcal{A}) \quad \text{and} \quad \mathcal{H}^*_\omega(\mathcal{A}) = \bigcap_{k \in \mathbb{N}} \mathcal{H}^*_k(\mathcal{A}). \tag{30}$$

Definitions of local homogeneity. II.

- The homogeneity classes are hierarchically ordered, according to the following relations

$$\mathcal{H}_\omega(\mathcal{A}) \subseteq \mathcal{H}_k(\mathcal{A}),\ \forall k \in \mathbb{N}, \tag{31}$$

$$\mathcal{H}^*_\omega(\mathcal{A}) \subseteq \mathcal{H}^*_k(\mathcal{A}),\ \forall k \in \mathbb{N}, \tag{32}$$

$$\mathcal{H}_a(\mathcal{A}) \subseteq \mathcal{H}_b(\mathcal{A}) \wedge \mathcal{H}^*_a(\mathcal{A}) \subseteq \mathcal{H}^*_b(\mathcal{A}),\ \forall a, b \in \mathbb{N} : a > b, \tag{33}$$

$$\mathcal{H}_a(\mathcal{A}) \subseteq \mathcal{H}^*_a(\mathcal{A}),\ \forall a \in \mathbb{N}. \tag{34}$$

- Local homogeneity, in the sense of Frisch: $\mathbf{u} \in \mathcal{H}_0(\mathcal{A})$.
- The homogeneity condition sufficient to
  - eliminate the sweeping interactions over the domain $\mathcal{A}$: $\mathbf{u} \in \mathcal{H}_1(\mathcal{A})$.
  - prove the $4/5$-law over the domain $\mathcal{A}$: $\mathbf{u} \in \mathcal{H}_1(\mathcal{A})$.
  - employ the Belinicher-L'vov quasi-Lagrangian transformation: $\mathbf{u} \in \mathcal{H}^*_\omega$.

Balance equations and sweeping. I.

- The clearest way to understand the sweeping interactions is by employing the balance equations introduced by L'vov and Procaccia (1996).
- The Navier-Stokes equations, where the pressure term has been eliminated, read

$$\frac{\partial u_\alpha}{\partial t} + \mathcal{P}_{\alpha\beta}\partial_\gamma(u_\beta u_\gamma) = \nu\nabla^2 u_\alpha + \mathcal{P}_{\alpha\beta} f_\beta, \tag{35}$$

where $\mathcal{P}_{\alpha\beta} = \delta_{\alpha\beta} - \partial_\alpha\partial_\beta\nabla^{-2}$ is the projection operator

- The Eulerian generalized structure function is defined as

$$F_n^{\alpha_1\alpha_2\cdots\alpha_n}(\{\mathbf{x},\mathbf{x}'\}_n, t) = \left\langle \left[\prod_{k=1}^{n} w_{\alpha_k}(\mathbf{x}_k, \mathbf{x}'_k, t)\right]\right\rangle, \tag{36}$$

where $\{\mathbf{x},\mathbf{x}'\}_n$ is shorthand for a list of n position vectors.

Balance equations and sweeping. II.

- The balance equations are obtained by differentiating the definition of F_n with respect to time t and substituting the Navier-Stokes equations:

$$\frac{\partial F_n}{\partial t} + D_n = \nu J_n + Q_n, \tag{37}$$

where D_n represents the contributions from the nonlinear term and

$$Q_n^{\alpha_1\alpha_2\cdots\alpha_n}(\{\mathbf{X}\}_n, t) = \sum_{k=1}^{n} \left\langle \left[\prod_{l=1, l\neq k}^{n} w_{\alpha_l}(\mathbf{x}_l, \mathbf{x'}_l, t) \right] \mathcal{P}_{\alpha_k\beta}(f_\beta(\mathbf{x}_k, t) - f_\beta(\mathbf{x'}_k, t)) \right\rangle . \tag{38}$$

$$J_n^{\alpha_1\alpha_2\cdots\alpha_n}(\{\mathbf{X}\}_n, t) = \mathcal{D}_n F_n^{\alpha_1\alpha_2\cdots\alpha_n}(\{\mathbf{X}\}_n, t) \tag{39}$$

$$= \sum_{k=1}^{n} (\nabla^2_{\mathbf{x}_k} + \nabla^2_{\mathbf{x'}_k}) F_n^{\alpha_1\alpha_2\cdots\alpha_n}(\{\mathbf{X}\}_n, t), \tag{40}$$

Balance equations and sweeping. III.

- L'vov and Procaccia (1996) showed that the contribution of the nonlinear term D_n can be rewritten as $D_n = \mathcal{O}_n F_{n+1} + I_n$ where $\mathcal{O}_n$ is a linear integrodifferential operator, and I_n is given by

$$I_n^{\alpha_1\alpha_2\cdots\alpha_n}(\{\mathbf{X}\}_n, t) = \sum_{k=1}^{n} (\partial_{\beta,\mathbf{x}_k} + \partial_{\beta,\mathbf{x}'_k}) \left\langle \mathcal{U}_\beta(\{\mathbf{X}\}_n, t) \left[\prod_{l=1}^{n} w_{\alpha_l}(\mathbf{X}_l, t)\right] \right\rangle, \tag{41}$$

where $\mathcal{U}_\beta(\{\mathbf{X}\}_n, t)$ is defined as

$$\mathcal{U}_\alpha(\{\mathbf{X}\}_n, t) = \frac{1}{2n} \sum_{k=1}^{n} \left(u_\alpha(\mathbf{x}_k, t) + u_\alpha(\mathbf{x}'_k, t)\right). \tag{42}$$

- The second term, I_n, represents exclusively the effect of the sweeping interactions.
- To set $I_n = 0$ we need the homogeneity assumption $\mathbf{u} \in \mathcal{H}_1(\mathcal{A})$.

Dropping sweeping: The $4/5$-law proof

- If we assume $\mathbf{u} \in \mathcal{H}_1(\mathcal{A})$ and set $I_2 = 0$, then one can calculate ζ_3 is from the solvability condition of the homogeneous equation $\mathcal{O}_2 F_3 = 0$, as shown by L'vov and Procaccia (1996).
- Use the conservation of energy to show that

$$\mathcal{O}_2 F_3(\mathbf{x}_1, \mathbf{x}'_1, \mathbf{x}_2, \mathbf{x}'_2) = \frac{1}{2}\frac{d[S_3(r_{12}) - S_3(r_{12'})]}{dr_1} + \frac{1}{2}\frac{d[S_3(r_{1'2'}) - S_3(r_{1'2})]}{dr_{1'}}$$
$$= A[r_{12}^{\zeta_3 - 1} - r_{12'}^{\zeta_3 - 1} + r_{1'2'}^{\zeta_3 - 1} - r_{1'2}^{\zeta_3 - 1}]. \qquad (43)$$

  where $r_{12} = \|\mathbf{x}_1 - \mathbf{x}_2\|$, etc.
- It follows that $D_n \approx \mathcal{O}_2 F_3 = 0 \iff \zeta_3 = 1$
- Dropping I_2 cannot be justified under $\mathbf{u} \in \mathcal{H}_0(\mathcal{A})$, i.e. local homogeneity in the sense of Frisch.

Dropping sweeping: The multifractal formalism.

- The homogeneous equations $\mathcal{O}_n F_{n+1} = 0$ are invariant with respect to the following group of transformations

$$\mathbf{r} \mapsto \lambda \mathbf{r}, \quad F_n \mapsto \lambda^{nh+\mathcal{Z}(h)} F_n. \tag{44}$$

- Thus, in an inertial range, solutions $F_{n,h}$ that satisfy the self-similarity property

$$F_{n,h}(\{\lambda \mathbf{x}_k, \lambda \mathbf{x}'_k\}_{k=1}^n, t) = \lambda^{nh+\mathcal{Z}(h)} F_{n,h}(\mathbf{x}_k, \mathbf{x}'_k\}_{k=1}^n, t), \tag{45}$$

  are admissible.

- The correct solution is the linear combination of these solutions, given by

$$F_n = \int d\mu(h) F_{n,h}. \tag{46}$$

- This conclusion also needs the assumption $\mathbf{u} \in \mathcal{H}_1(\mathcal{A})$.

The Bottom Line

If we assume $\mathbf{u} \in \mathcal{H}_1(\mathcal{A})$,
then we can simply “exterminate” the sweeping term,
with no further worries.

Quasi-Langrangian to Eulerian. I. The claim

- Consider the definitions

$$F_n^{\alpha_1\alpha_2\cdots\alpha_n}(\{\mathbf{x},\mathbf{x}'\}_n,t) = \left\langle \left[\prod_{k=1}^{n} w_{\alpha_k}(\mathbf{x}_k,\mathbf{x'}_k,t) \right] \right\rangle , \qquad (47)$$

$$\mathcal{F}_n^{\alpha_1\alpha_2\cdots\alpha_n}(\mathbf{x}_0,t_0|\{\mathbf{x},\mathbf{x}'\}_n,t) = \left\langle \left[\prod_{k=1}^{n} W_{\alpha_k}(\mathbf{x}_0,t_0|\mathbf{x}_k,\mathbf{x'}_k,t) \right] \right\rangle . \qquad (48)$$

- The claim of L'vov and Procaccia was that it can be shown that

$$\mathcal{F}_n(\mathbf{x}_0,t_0|\{\mathbf{x},\mathbf{x}'\}_n,t) = F_n(\{\mathbf{x},\mathbf{x}'\}_n,t), \forall n \in \mathbb{N}^* \qquad (49)$$

- The claim can be rewritten equivalently as

$$W_\alpha(\mathbf{x}_0,t_0|\mathbf{x},\mathbf{x}',t) \overset{\mathbf{x},\mathbf{x}'}{\sim} w_\alpha(\mathbf{x},\mathbf{x}',t). \qquad (50)$$

Quasi-Langrangian to Eulerian. II. Proofs

- Proof was re-examined recently by Gkioulekas.
  - The claim holds if and only if

$$W_\alpha(\mathbf{x}_0, t_0 + \Delta t | \mathbf{x}, \mathbf{x}', t) \overset{\mathbf{x},\mathbf{x}'}{\sim} W_\alpha(\mathbf{x}_0, t_0 | \mathbf{x}, \mathbf{x}', t), \ \forall \Delta t \in \mathbb{R} - \{0\} \tag{51}$$

  - If $u \in \mathcal{H}_\omega^*$, and u_α is incompressible, then

$$W_\alpha(\mathbf{x}_0, t_0 + \Delta t | \mathbf{x}, \mathbf{x}', t) \overset{\mathbf{x},\mathbf{x}'}{\sim} W_\alpha(\mathbf{x}_0, t_0 | \mathbf{x}, \mathbf{x}', t). \tag{52}$$

- $u \in \mathcal{H}_\omega^*$ is a sufficient but perhaps not necessary assumption
- However, the assumption $\mathbf{u} \in \mathcal{H}_1(\mathcal{A})$ is **not** sufficient
- The artifact introduced by the quasi-Lagrangian formulation is that the turbulent velocity field is being perceived from the viewpoint of an arbitrary fluid particle whose own motion is also stochastic.

Quasi-Langrangian to Eulerian. III. Another proof

- Introduce the conditional correlation tensor defined as

$$\mathcal{F}_n(\mathbf{x}_0, t_0, \mathbf{y}|\{\mathbf{X}\}_n, t) = \left\langle \prod_{k=1}^{n} W_{\alpha_k}(\mathbf{x}_0, t_0|\mathbf{x}_k, \mathbf{x}'_k, t) \middle| \rho(\mathbf{x}_0, t_0|t) = \mathbf{y} \right\rangle \quad (53)$$

$$= \left\langle \prod_{k=1}^{n} w_{\alpha_k}(\mathbf{x}_k + \mathbf{y}, \mathbf{x}'_k + \mathbf{y}, t) \middle| \rho(\mathbf{x}_0, t_0|t) = \mathbf{y} \right\rangle \quad (54)$$

- The random velocity field $\mathbf{u}$ is a member of the homogeneity class $\mathcal{H}_0^c$, if and only if $\forall \mathbf{y} \in \mathbb{R}^d, \forall \mathbf{x}_k, \mathbf{x}'_k \in \mathbb{R}^d$ we have

$$\sum_{k=1}^{n} (\partial_{\beta_k, \mathbf{y}_k} + \partial_{\beta, \mathbf{y}'_k}) \mathcal{F}_n(\mathbf{x}_0, t_0, \mathbf{y}|\{\mathbf{X}\}_n, t) = 0, , \forall n \in \mathbb{N}, n > 1 \quad (55)$$

- The condition $\mathbf{u} \in \mathcal{H}_0^c$ implies **The claim**

Elimination of the sweeping interactions. I.

- One may conjecture that the sweeping interactions act as a large-scale forcing term whose effect is forgotten in the inertial range.
- It is possible to use the theoretical work based on the quasi-Lagrangian transformation in a way that requires only the assumption $\mathbf{u} \in \mathcal{H}_1(\mathcal{A})$.
  - The quasi-Lagrangian formulation modifies the Navier-Stokes equations by redefining the material derivative.
  - The modified equation remains mathematically equivalent to the Navier-Stokes equation if the velocity field is reinterpreted from an Eulerian field into a quasi-Lagrangian field.
  - This reinterpretation necessitates the stronger assumption $\mathbf{u} \in \mathcal{H}_\omega^*$ to enable a return back to the Eulerian representation.
  - If we accept the hypothesis that the sweeping interactions act as a large-scale forcing, we can just modify the equation of motion in precisely the same way without interpreting the velocity field as quasi-Lagrangian, but rather as Eulerian.

Elimination of the sweeping interactions. II.

- To prove that the sweeping interactions act as a large-scale forcing it is sufficient to calculate the scaling exponent Δ_n associated with the ratio

$$\frac{I_n(R\{\mathbf{X}\}_n)}{(\mathcal{O}_n F_{n+1})(R\{\mathbf{X}\}_n)} \sim \left(\frac{R}{\ell_0}\right)^{\Delta_n}, \tag{56}$$

- Then, provided that one starts with the assumption $\mathbf{u} \in \mathcal{H}_0$, proving $\Delta_n > 0$ is also a proof that $\mathbf{u} \in \mathcal{H}_1(\mathcal{A})$ which is sufficient to eliminate the sweeping interactions.
- If we assume that the generalized structure functions $F_n(R\{\mathbf{X}\}_n)$ satisfy the fusion rules, then the scaling exponent of $\mathcal{O}_n F_{n+1}(R\{\mathbf{X}\}_n)$ is $\zeta_{n+1} - 1$ and it follows that $\Delta_n = \lambda_n - (\zeta_{n+1} - 1)$.
- The problem of calculating the scaling exponents λ_n needs to be investigated primarily with numerical simulations and the analysis of experimental data.

Elimination of the sweeping interactions. III.

- Commit the following crimes against reality:
  - Assume that the velocity field $u_\alpha(\mathbf{x}, t)$ can be modeled as a random gaussian delta-correlated (in time) stochastic field acting at large scales.
  - Assume that the velocity field $u_\alpha(\mathbf{x}, t)$ has an effect on the velocity differences $w_\alpha(\mathbf{x}, \mathbf{x}', t)$ via the sweeping interactions
  - Disregard that $u_\alpha(\mathbf{x}, t)$ and $w_\alpha(\mathbf{x}, \mathbf{x}', t)$ are obviously constrained by the definition of $w_\alpha(\mathbf{x}, \mathbf{x}', t)$.
- Using the multifractal formulation, the contribution that supports the Holder exponent h gives $\zeta_n = nh + \mathcal{Z}(h)$, which gives the following evaluation:

$$\Delta_n(h) = -2h + \lambda + 2 \tag{57}$$

- The window for positive scaling exponents Δ_n covers the entire range $h \in (0, 1)$ of local scaling exponents.
- The real challenge is to determine what happens in reality

Conclusion

- Analytical theories are an extension of the Frisch reformulation of K41
- The main stumbling block is the elimination of the sweeping interactions
- Lagrangian methods do not prove that the sweeping interactions are negligible in the inertial range.
- The open question: prove that sweeping is negligible in the inertial range.