# Constant Flux States in Models of Wave Turbulence

Colm Connaughton
Center for Nonlinear Studies
Los Alamos National Laboratory
U.S.A.

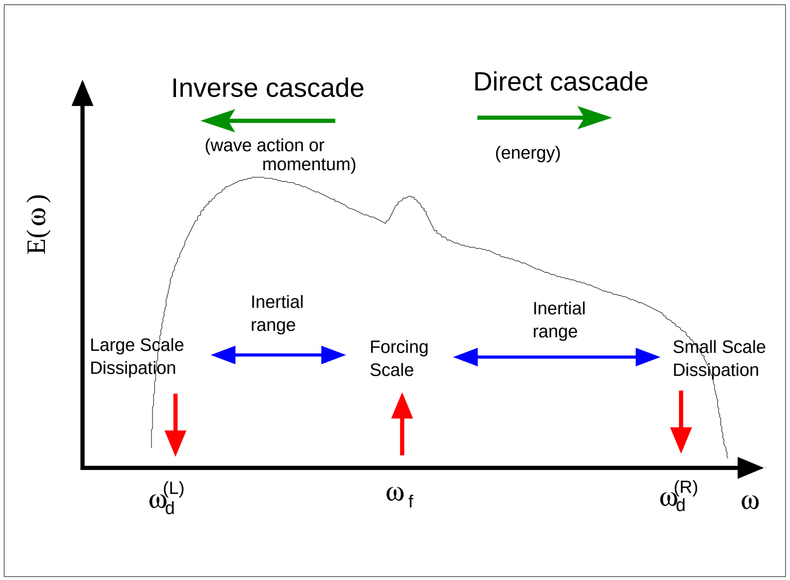
Joint work with:

- R. Rajesh (IMSc, India)
- O. Zaboronski (Univ. of Warwick, UK)

# Example 1 : “Weak” wave turbulence on a lake

# Example 2 : “Strong” wave turbulence on the ocean

# What is Wave Turbulence?




- Ensemble of interacting dispersive waves.
- Energy injected at some characteristic frequency, $\omega_f$'
- Waves are damped at very large and / or very small scales. Scale separation.
- Energy transferred by interaction between waves
- Concept of *inertial range*

In the limit of large inertial ranges, system often becomes scale invariant and exhibit a power law spectrum :$E(k) \sim k^{-x}$.

# Hamiltonian Model

Hamiltonian evolution for the complex Fourier amplitudes, $a_{\vec{k}}$, $a^*_{\vec{k}}$ :

$$\frac{\partial a_{\vec{k}}}{\partial t} = i\frac{\delta H}{\delta a^*_{\vec{k}}} + f_{\vec{k}} - \gamma_{\vec{k}} a_{\vec{k}}$$

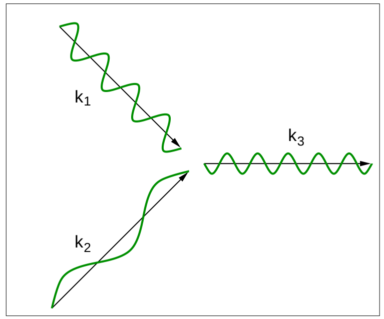
- Wave-vector, $\vec{k}$ is a $d$-dimensional vector.
- $H$ has a (linear) kinetic and (nonlinear) potential term

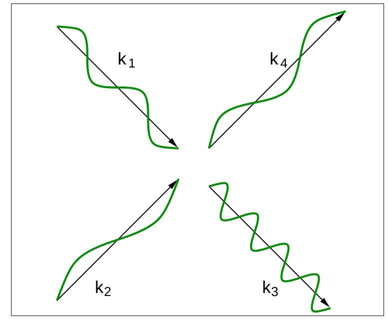
$$H = T + U = \int t(\mathbf{k}) d\mathbf{k} + \int u(k) d\mathbf{k}$$

where $t(\mathbf{k}) = \omega_{\vec{k}} a_{\vec{k}} a^*_{\vec{k}}$.

- Elementary solutions : $a(\mathbf{x}, t) = \int d\mathbf{k}\, a_{\mathbf{k}} e^{i(\mathbf{k}\cdot\mathbf{x} - \omega_{\mathbf{k}} t)}$.
- Forcing and dissipation functions are separated in $\vec{k}$-space.

# Structure of the interaction




cubic nonlinearity quartic nonlinearity

3 wave systems :

$$u(\mathbf{k}_1) = \int V_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}\left(a_{\mathbf{k}_1}a_{\mathbf{k}_2}a^*_{\mathbf{k}_3} + a^*_{\mathbf{k}_1}a^*_{\mathbf{k}_2}a_{\mathbf{k}_3}\right)\delta(\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3)\,d\mathbf{k}_2 d\mathbf{k}_3$$

4 wave systems :

$$\begin{aligned} u(\mathbf{k}_1) &= \int T_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4}\left(a_{\mathbf{k}_1}a_{\mathbf{k}_2}a^*_{\mathbf{k}_3}a^*_{\mathbf{k}_4} + a^*_{\mathbf{k}_1}a^*_{\mathbf{k}_2}a_{\mathbf{k}_3}a_{\mathbf{k}_4}\right) \\ &\qquad \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4)\,d\mathbf{k}_2 d\mathbf{k}_3 d\mathbf{k}_4 \end{aligned}$$

# Conservation Laws and Scale Invariance

Scale invariance: Many interesting cases possess scale invariance :

- Dispersion relation : $\omega_{h\mathbf{k}} = h^{\alpha}\omega_{\mathbf{k}}$.
- Nonlinear interactions :

$$
\begin{aligned}
V_{h\mathbf{k}_1 h\mathbf{k}_2 h\mathbf{k}_3} &= h^{\beta} V_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} \text{ 3-wave} \\
T_{h\mathbf{k}_1 h\mathbf{k}_2 h\mathbf{k}_3 h\mathbf{k}_4} &= h^{\beta} T_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4} \text{ 4-wave}
\end{aligned}
$$

The parameters $\alpha$, $d$ and $\beta$ determine the scaling properties of the system.

Conservation laws:

- Energy, $H$.
- Momentum : $\vec{P} = \int \mathbf{k} a_{\vec{k}} a^*_{\vec{k}} d\mathbf{k}$. (but often, $\vec{P} = 0$)
- Wave-action : $N = \int a_{\vec{k}} a^*_{\vec{k}} d\mathbf{k}$. (4-wave only)

# A starting point...

Hamilton's equations give an equation governing for the spectrum. For the 3-wave case :

$$
\begin{aligned}
\frac{\partial n(\mathbf{k}_1)}{\partial t} &= 2\int V_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}\mathrm{Im}\langle a^*_{k_1}a_{\mathbf{k}_2}a_{\mathbf{k}_3}\rangle\delta(\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)d\mathbf{k}_2d\mathbf{k}_3 \\
&- 2\int V_{\mathbf{k}_2\mathbf{k}_3\mathbf{k}_1}\mathrm{Im}\langle a^*_{k_2}a_{\mathbf{k}_3}a_{\mathbf{k}_1}\rangle\delta(\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_1)d\mathbf{k}_2d\mathbf{k}_3 \\
&- 2\int V_{\mathbf{k}_3\mathbf{k}_1\mathbf{k}_2}\mathrm{Im}\langle a^*_{k_3}a_{\mathbf{k}_1}a_{\mathbf{k}_2}\rangle\delta(\mathbf{k}_3-\mathbf{k}_1-\mathbf{k}_2)d\mathbf{k}_2d\mathbf{k}_3
\end{aligned}
$$

Similarly the 4-wave case ...but now what? Closure problem.

# The weak turbulence limit

In weak turbulence, potential energy is small compared to the kinetic energy.

$$\frac{u(k)}{t(k)} = \epsilon \ll 1$$

- $\epsilon$ provides a small parameter to do perturbation theory.
- Nonlinearity becomes localised on resonant triads or quartets. Resonances transfer energy over long timescales ($1/\epsilon^2$)
- Roughly :

$$\begin{aligned} \mathrm{Im}\langle a^*_{k_1} a_{\mathbf{k}_2} a_{\mathbf{k}_3}\rangle \quad &\sim \quad V_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}(n_2 n_3 - n_1 n_2 - n_1 n_3)\delta(\omega_1 - \omega_2 - \omega_3) \\ & \qquad \delta(\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) \end{aligned}$$

- Lowest order of perturbation theory gives a wave kinetic theory for the slow evolution of $n(\mathbf{k})$.

# The Wave Kinetic Equation

Isotropic 3-wave kinetic equation in frequency space is:

$$
\begin{aligned}
\frac{\partial n_1}{\partial t} &= \epsilon^2 \int U_{\omega_1\omega_2\omega_3}(n_2n_3 - n_1n_2 - n_1n_3)\delta(\omega_1 - \omega_2 - \omega_3)d\omega_2 d\omega_3 \\
&- \epsilon^2 \int U_{\omega_2\omega_3\omega_1}(n_3n_1 - n_2n_3 - n_2n_1)\delta(\omega_2 - \omega_3 - \omega_1)d\omega_2 d\omega_3 \\
&- \epsilon^2 \int U_{\omega_3\omega_1\omega_2}(n_1n_2 - n_3n_1 - n_3n_2)\delta(\omega_3 - \omega_2 - \omega_1)d\omega_2 d\omega_3
\end{aligned}
$$

4wave kinetic equation in frequency space :

$$
\begin{aligned}
\frac{\partial n_1}{\partial t} &= \epsilon^2 \int U_{\omega_1\omega_2\omega_3\omega_4}(n_2n_3n_4 + n_1n_3n_4 - n_1n_2n_4 - n_1n_2n_3) \\
&\quad \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4)d\omega_2 d\omega_3 d\omega_4
\end{aligned}
$$

# Kolmogorov-Zakharov Solutions of the Kinetic Equation

Stationary scaling solutions, $n(k) = c\, k^{-x}$, can be found exactly :

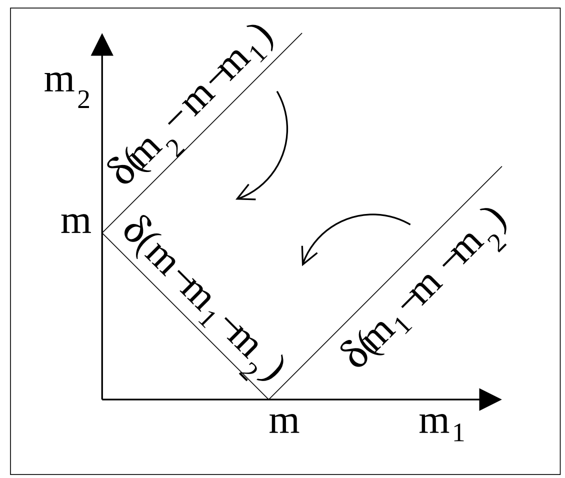
$$(\omega_2, \omega_3) \rightarrow (\frac{\omega_1 \omega_2'}{\omega_3'}, \frac{\omega_1^2}{\omega_3'}), \qquad (\omega_2, \omega_3) \rightarrow (\frac{\omega_1^2}{\omega_2'}, \frac{\omega_1 \omega_3'}{\omega_2'}).$$

3W case : *kinetic* energy cascade :

$$n(k) = c_3^{(P)} \sqrt{P} k^{-\beta - d}$$

4W case : Two cascades - *kinetic* energy and wave action :

$$n(k) = c_4^{(P)} P^{1/3} k^{-(2\beta+3d)/3}$$
$$n(k) = c_4^{(Q)} Q^{1/3} k^{-(2\beta-3d+\alpha)/3}$$



# What about the strong turbulence case?

- In the strong turbulence case, potential energy may be as larger or larger than the kinetic energy.
- No perturbative expansion.
- However, conservation laws still hold but it is *total* energy rather than kinetic energy which is conserved. For 4W case, wave action remains conserved.
- More like hydrodynamic turbulence case.
- Conservation laws should still provide constraints on the inertial range statistics, even for strong turbulence in the spirit of Kolmogorov's 4/5 Law.

# Exact Conservation Law for Wave Action

Consider 4-wave wave-action cascade. In the inertial range:

$$\frac{\partial N_{k_1}}{\partial t} \equiv \frac{\partial Q_{k_1}}{\partial k_1} = \int \prod_{i=2,3,4} (k_i^{d-1} dk_i) \left[L_{1,2,3,4} + L_{2,1,3,4} - L_{3,4,1,2} - L_{4,3,1,2}\right]$$

where

$$N_{k_1} = \int n(\mathbf{k}_1) k_1^{d-1} d\Omega_1, \qquad L_{1,2,3,4} = T_{k_1,k_2,k_3,k_4} K_{k_1,k_2,k_3,k_4}$$

$$K_{k_1,k_2,k_3,k_4} = \int \prod_{i=1,2,3,4} d\Omega_i \mathrm{Im}\langle a^*_{\mathbf{k}_1} a^*_{\mathbf{k}_2} a_{\mathbf{k}_3} a_{\mathbf{k}_4}\rangle \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4)$$

This is true whether nonlinearity is weak or strong.

# Constant Flux Relation for Wave Action

Assume $K_{k_1,k_2,k_3,k_4}$ is a scaling function of degree, $h$, with the obvious symmetries.

$$K_{ak_1,ak_2,ak_3,ak_4} = a^h K_{k_1,k_2,k_3,k_4}$$

One can still apply the Zakharov Transformation even without the $\omega$ delta functions, to obtain:

$$\frac{\partial N_{k_1}}{\partial t} = \int \prod_{i=2,3,4} (k_i^{d-1} dk_i) L_{1,2,3,4} \left[k_1^y + k_2^y - k_3^y - k_4^y\right]$$

where $y = -h - \beta - 4d$. RHS clearly vanishes for $h = \beta + 4d$, corresponding to a constant flux of $n(\mathbf{k})$ in the inertial range.
Weak limit :

$$K_{k_1,k_2,k_3,k_4} \sim T(k) n^3 \delta(k) \delta(\omega) \sim k^{\beta - 3*(2\beta+3d-\alpha)/3+d-\alpha} \sim k^{-\beta-4d}$$

# Exact conservation law for total energy

Problem: if the nonlinearity is strong then it is the total energy density, $h(\mathbf{k}) = t(\mathbf{k}) + u(\mathbf{k})$, rather than just the kinetic energy density whose flux should be constant in the stationary state. The following statement is exact :

$$\frac{\partial h_{k_1}}{\partial t} \equiv \frac{\partial J_{k_1}}{\partial k_1} = 4 \int \prod_{i=2,3} (k_i^{d-1} dk_i) \left[L_{1,2,3} - L_{2,1,3}\right]$$

$$h_{k_1} = \int (t(\mathbf{k}_1) + u(\mathbf{k}_1))\, k_1^{d-1} d\Omega_1, \qquad L_{1,2,3} = V_{k_1,k_2,k_3} K_{k_1,k_2,k_3}$$

$$K_{k_1,k_2,k_3} = \int \prod_{i=1,2,3} d\Omega_i \mathrm{Re}\langle a^*_{\mathbf{k}_1} \dot{a}^*_{\mathbf{k}_2} a_{\mathbf{k}_3}\rangle \delta(\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3)$$

# Constant Flux Relation for Energy

Again, if $K_{k_1,k_2,k_3}$ is a scaling function of degree, $h$, the Zakharov transformation can be applied to give

$$\frac{\partial J_{k_1}}{\partial k_1} = 4 \int \prod_{i=2,3} (k_i^{d-1} dk_i) L_{1,2,3} \left[ 1 - \left( \frac{k_1}{k_2} \right)^{3d+\beta+h} \right]$$

Clearly we have a constant flux when $h = -\beta - 3d$.
This is different from previously. The CFR correlation function $\mathrm{Re}\langle a^*_{\mathbf{k}_1} \dot{a}^*_{\mathbf{k}_2} a_{\mathbf{k}_3} \rangle$ contains time derivatives.
Corresponding result for 4-wave energy cascade is :

$$\int \prod_{i=1,2,3,4} d\Omega_i \mathrm{Re}\langle a^*_{\mathbf{k}_1} \dot{a}^*_{\mathbf{k}_2} a_{\mathbf{k}_3 a_{\mathbf{k}_4}} \rangle \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4) \sim k^{-\beta-4d}$$

# The Assumption of Locality

In deriving the CFR scaling, we implicitly assumed convergence of the exact collision integral when the flux-carrying correlation function takes the scaling form. For example, for the 3-wave case, we can deduce that

$$K_{k_1,k_2,k_3} = (k_2 k_3)^{\frac{h}{2}} \Phi\left(\frac{k_2}{k_3}\right)$$

We require that the scaling function, $\Phi$ decays fast enough at 0 and $\infty$ to ensure convergence. This is to say that the transfer of energy is local. Our arguments at present do not tell anything about $\Phi$.

For weak turbulence, locality can be checked a posteriori. For hydrodynamic turbulence no such problem exists (the entire argument is local in $x$.) Here, it is an assumption.

# The MMT Model

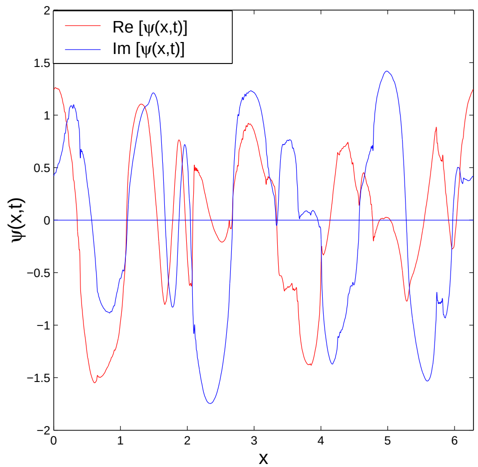
The MMT model is the simplest real wave turbulence model :

$$H = \int w_{k_1} \phi_{k_1} \phi^*_{k_1} dk_1 + \int T_{k_1 k_2 k_3 k_4} \phi^*_{k_1} \phi^*_{k_2} \phi_{k_3} \phi_{k_4} \delta(k_1 + k_2 - k_3 - k_4) dk_{1,2,3,4}$$

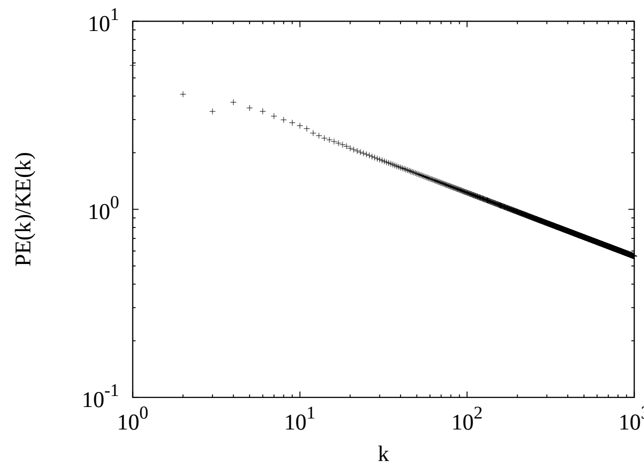
with $\omega_k = k^\alpha$ and $T_{k_1 k_2 k_3 k_4} = (k_1 k_2 k_3 k_4)^{\frac{\beta}{4}}$.

- Original model is one dimensional but it can clearly be written down in higher dimensions too.
- Does not generally exhibit weak turbulence due to resonance sparsity in 1-d.
- Exhibits direct and inverse cascades.
- A good candidate to check CFR?

# A Tentative Numerical Result for the MMT Model




Snapshot of $\psi(x,t)$

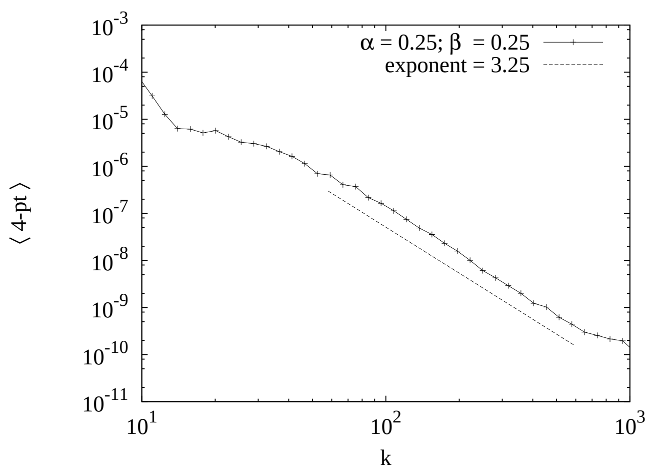



Energy density ratio, $t(k)/u(k)$.

Measured following contribution to the flux :

$$\langle \dot{a}_k a_{-k-2} \bar{a}_{k-1} \bar{a}_{-k-1} \rangle$$

$$\alpha = 0.25,$$
$$\beta = 0.25$$




4-point function.

# A "Shell" Model of 3-wave Turbulence

What is the simplest possible system which should obey a non-trivial CFR?

- Discrete waves (0-dimensional)
- 3-wave interactions
- Positive wave vectors only.
- "Local" interactions only : $k + k \rightarrow 2k$. So only have powers of 2 : $k_n = 2^n$.

These simplifications lead one to write down the following minimal Hamiltonian:

$$H = \sum_n k_n^\alpha a_n a_n^* + k_{n-1}^\beta (a_n^* a_{n-1}^2 + a_n a_{n-1}^{*2})$$

Really a system of oscillators, strongly coupled in a special way.

# Momentum CFR for the “Shell” Model

Equation of motion :

$$\frac{\partial a_n}{\partial t} = i\omega_n a_n + igk_{n-1}^{\beta} a_{n-1}^2 + 2igk_n^{\beta} a_{n-1}\bar{a}_n$$

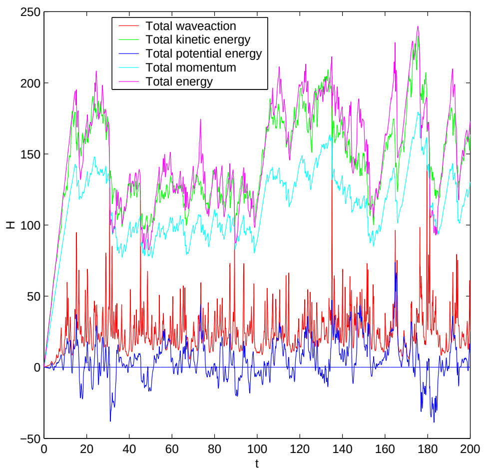
- Minimal model conserves $H$ and momentum $P = \sum_n k_n a_n a_n^*$.
- Momentum cascade is inverse and energy cascade is direct.
- Force and damp different modes to generate “turbulence”.
- Model is exactly solvable for the stationary states. Momentum CFR :

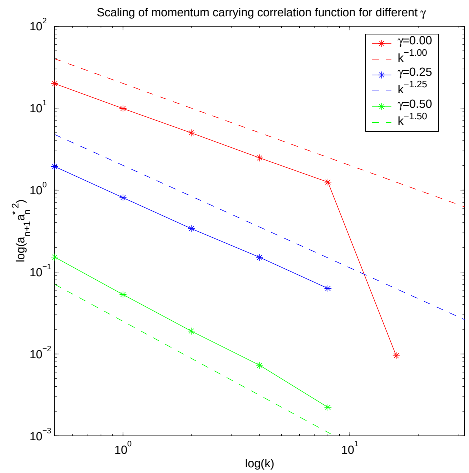
$$\mathrm{Im}\langle a_{n+1} a_n^{*2}\rangle = -\frac{J_P}{4} k_n^{-\beta-1}$$

where $J_P$ is the momentum flux.

# Numerical Confirmation of Momentum CFR for the “Shell” Model

Numerical integration of the minimal model with forcing and damping confirm the CFR prediction for momentum :




Time evolution for $\beta = 0$ $\quad \mathrm{Im}\langle a_{n+1} a_n^{*2} \rangle$ for different $\beta$.

Corresponding energy cascade is still a work in progress.

# Conclusions and Ongoing Work

## Conclusions:

- Conservation laws in wave turbulence can be used to fix the scaling of the flux-carrying correlation function in the inertial range, even in the case of strong wave turbulence.
- Energy conservation in wave turbulence, (assuming that our results are borne out by numerics), is of a different character to hydrodynamic turbulence. It links correlation functions of different order.

## Current work:

- Verify energy cascade in the minimal model.
- Observe CFR scaling for more realistic models, especially MMT.
- Physical predictions in real space?