Inverse cascades and conformally invariant curves

Antonio Celani

CNRS - INLN

CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

With: D. Bernard, LPT ENS
G. Boffetta, U.Torino
G. Falkovich, Weizmann

Questions

- Is 2D NS the only system displaying conformally invariant isolines ?
- Do other systems sharing this property, if any, fall into the same universality class (percolation) ?
- Can we predict the universality class from scaling arguments ?
- Are isolines conformally invariant in the direct cascade range of 2D NS as well ?

Surface Quasi-Geostrophic turbulence

- Temperature advection at the surface of a volume of constant potential vorticity (Held et al, JFM 1995)

$$\partial_t T + v \nabla T = \kappa \nabla^2 T + f - T/\tau$$

$$v = z \times \nabla \psi$$

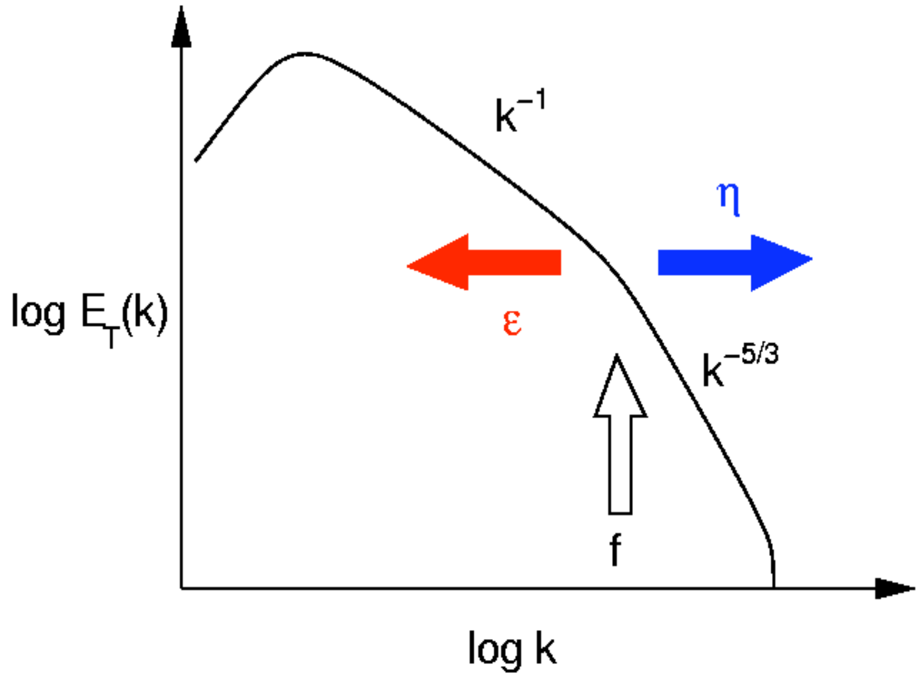
$$\psi(x,t) = \int |x-y|^{-1} T(y,t) \, dy$$

Phenomenology of SQG turbulence

Double cascade:

direct for $Z = \int T^2\, dx$

inverse for $E = \int \psi T\, dx$



Dimensional arguments:

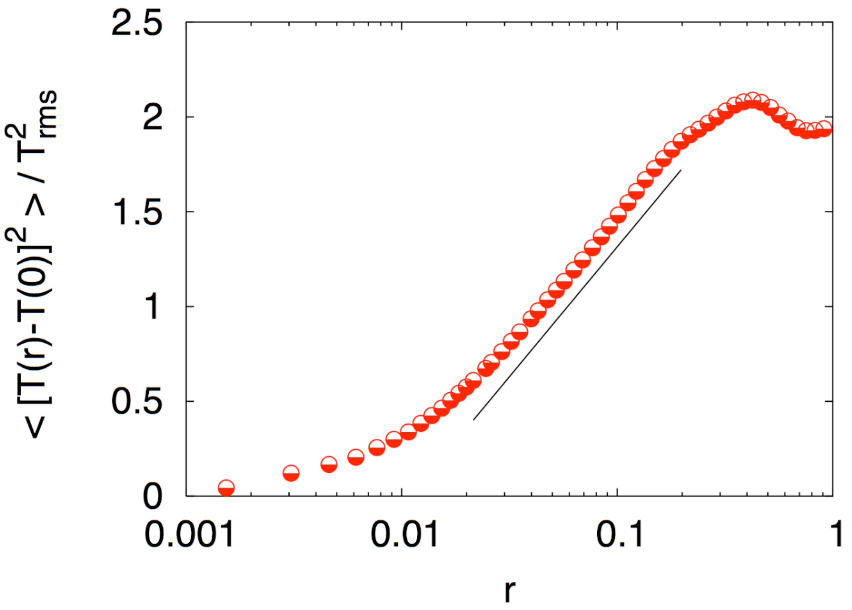
$T(x+r)-T(x) \sim r^{H}$

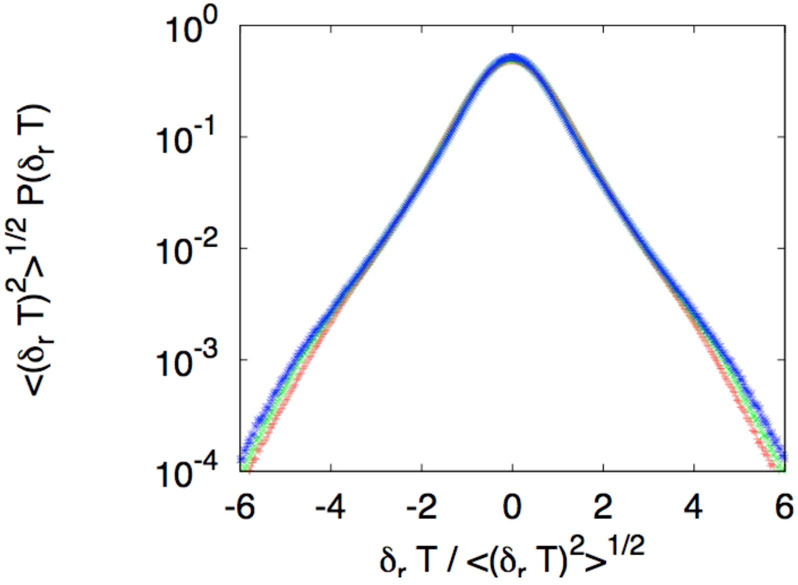
$H=2/3$ (direct)

$H=0$ (inverse)

DNS: Pierrehumbert *et al*, Chaos Sol Fract 1994, Held *et al* JFM 1995, Schorghofer PRE 2000, Smith *et al* JFM 2002, Celani *et al* NJP 2002

Inverse cascade in SQG turbulence

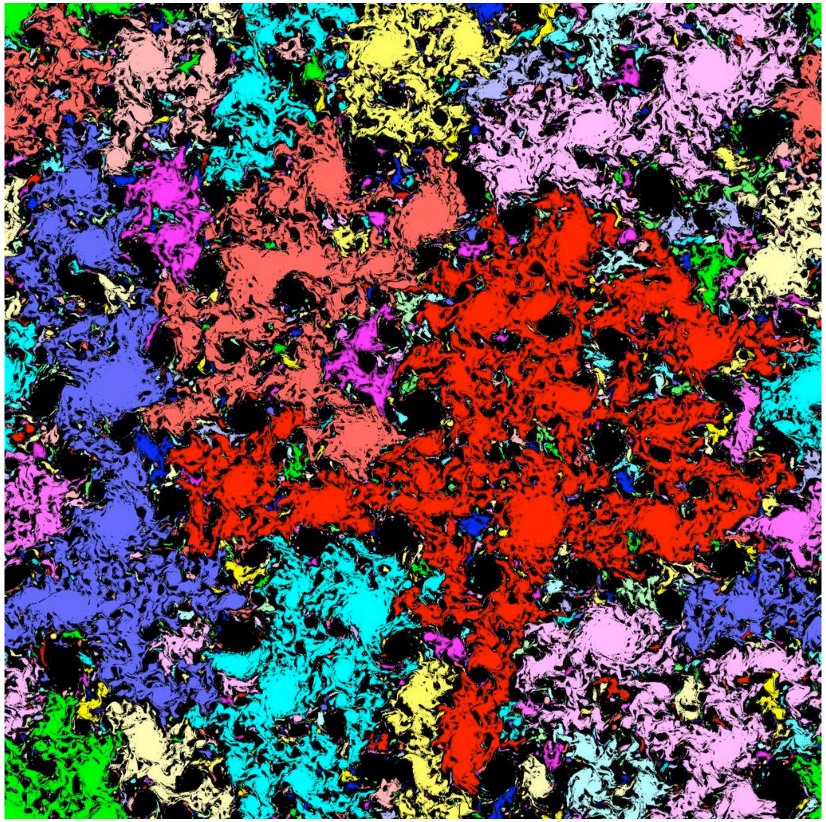


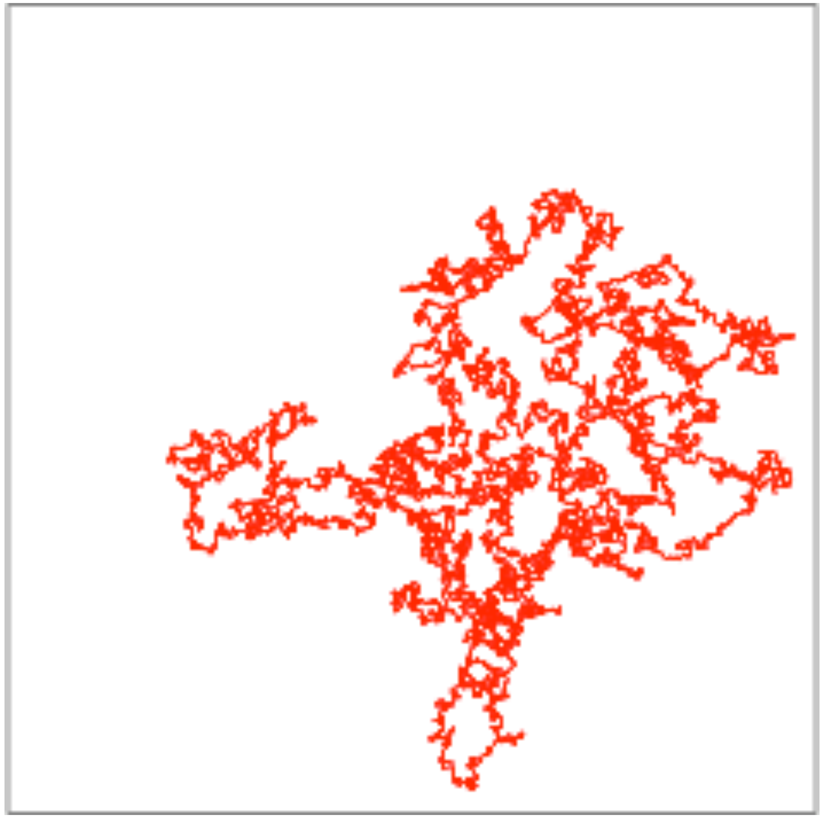


- Logarithmic correlations
- Self-similarity

Clusters and loops

- Clusters: connected domains of like-sign temperature
- Loops: isolevel lines topologically equivalent to circles





Cluster and loop statistics

- The temperature field is self-similar with H=0
- The fractal dimension of loops is 3/2

  [Cardy 1994, Saleur Duplantier PRL 1987

  D_F=(3-H)/2 for 0<H<1 Kondev & Henley PRL 1995]

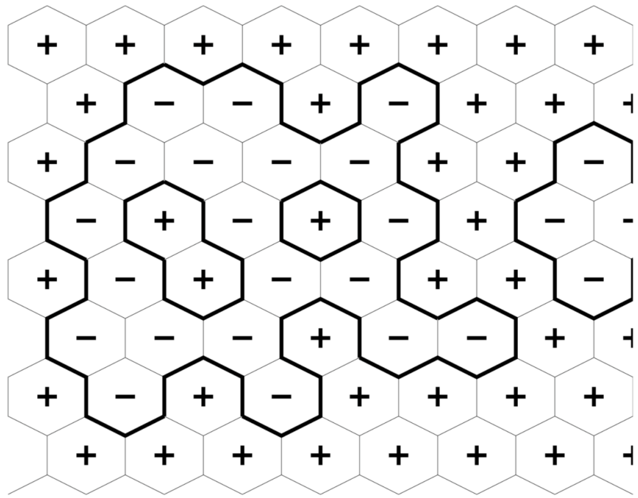
- Same as for O(2) and 4-state Potts model

→ Predictions for various fractal dimensions and probability density functions

O(n) model

- $Z = \mathrm{Tr}\,\prod_{rr'}(1 + x\, s^a_r s^b_{r'})$

  $a,b=1,\ldots,n \qquad \mathrm{Tr}\, s^a_r s^b_r = \delta^{ab}$
- Random graph representation
  $Z = \sum_{\mathrm{graphs}} x^{\mathrm{length}}\; n^{\mathrm{number\ of\ loop}}$
- n=1 Ising
- n=2 maps to SOS model
- n=0 SAW
- n=-2 LERW

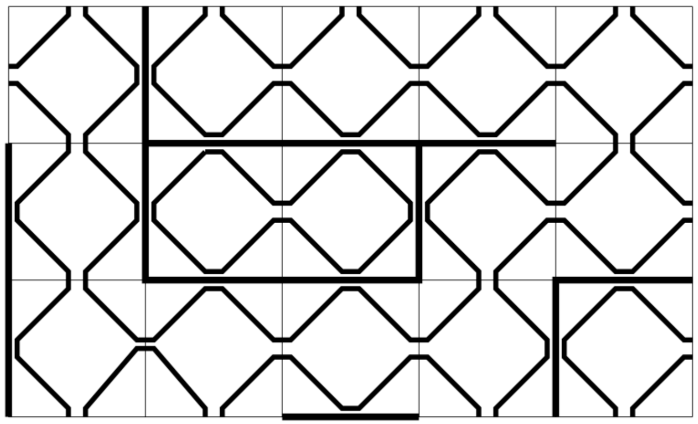


Potts model

- $Z = \mathrm{Tr} \exp [\beta J \sum_{rr'} \delta_{s_r, s_{r'}}] = \mathrm{Tr} \prod_{rr'} (1-p + p\, \delta_{s_r, s_{r'}})$

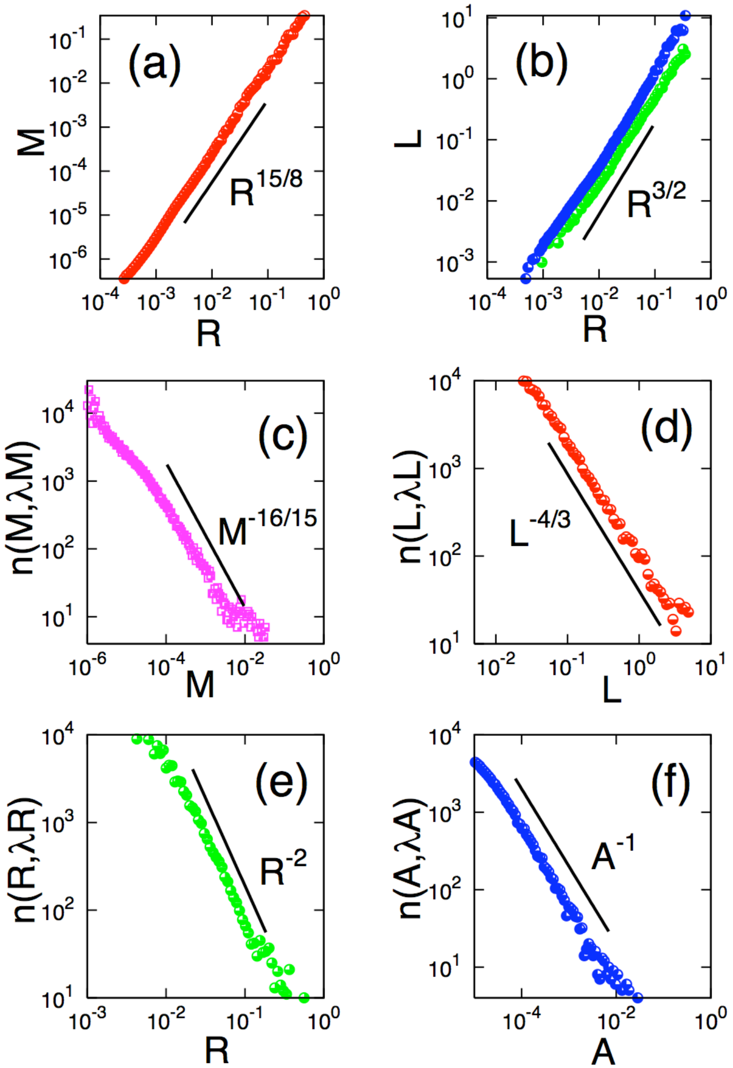
  $s=1,\ldots,Q \qquad \exp \beta J = (1-p)^{-1}$
- Random graph representation (Fortuin-Kasteleyn)

  $Z = \sum_{\mathrm{graphs}} p^{\mathrm{edges}} (1-p)^{\mathrm{edges\ in\ the\ complement}}\ Q^{\mathrm{connected\ components}}$



- Boundaries of critical FK clusters correspond to loops in the dense phase of the O(n) model with $n=Q^{1/2}$

Clusters and loops in SQG turbulence



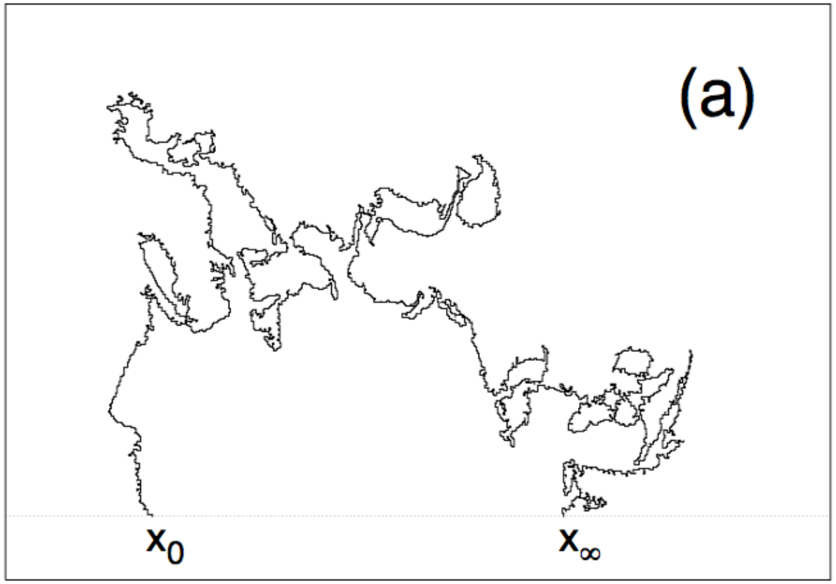
- Fractal dimension of clusters and loops
- Number of clusters of given mass
- Number of loops of given length, radius of gyration, area

Are loops conformally invariant curves ?

- Conjecture: locally equivalent to SLE_4
- Verification: requires an algorithm to extract the driving function from candidate SLE traces
- Difficulties: loops, no locality

From loops to SLE traces

- loop surgery



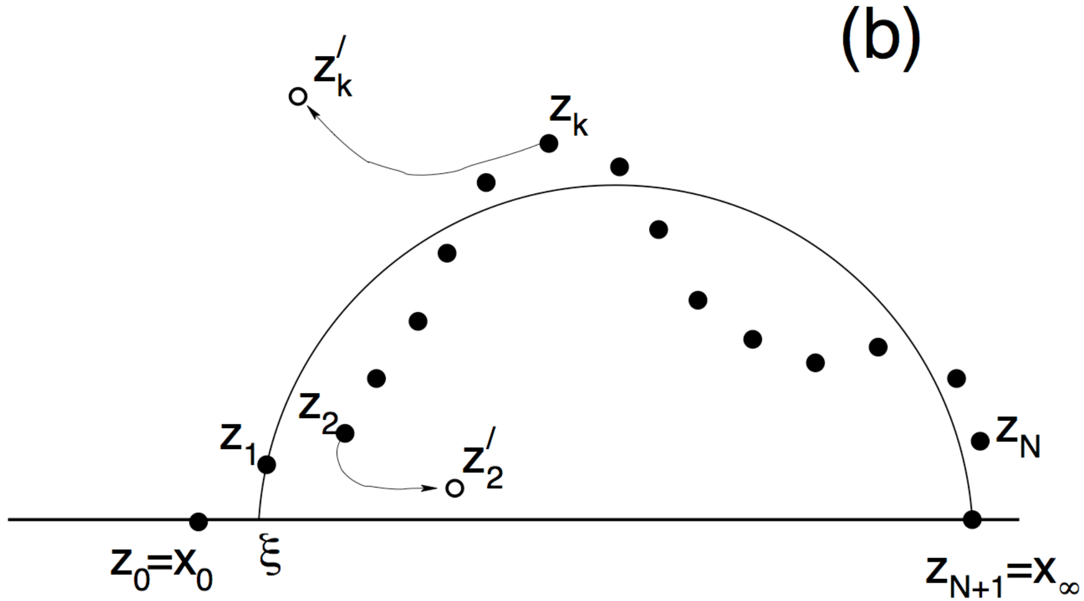
From loops to SLE traces

Chordal SLE from x_0 to x_∞ in the upper half-plane

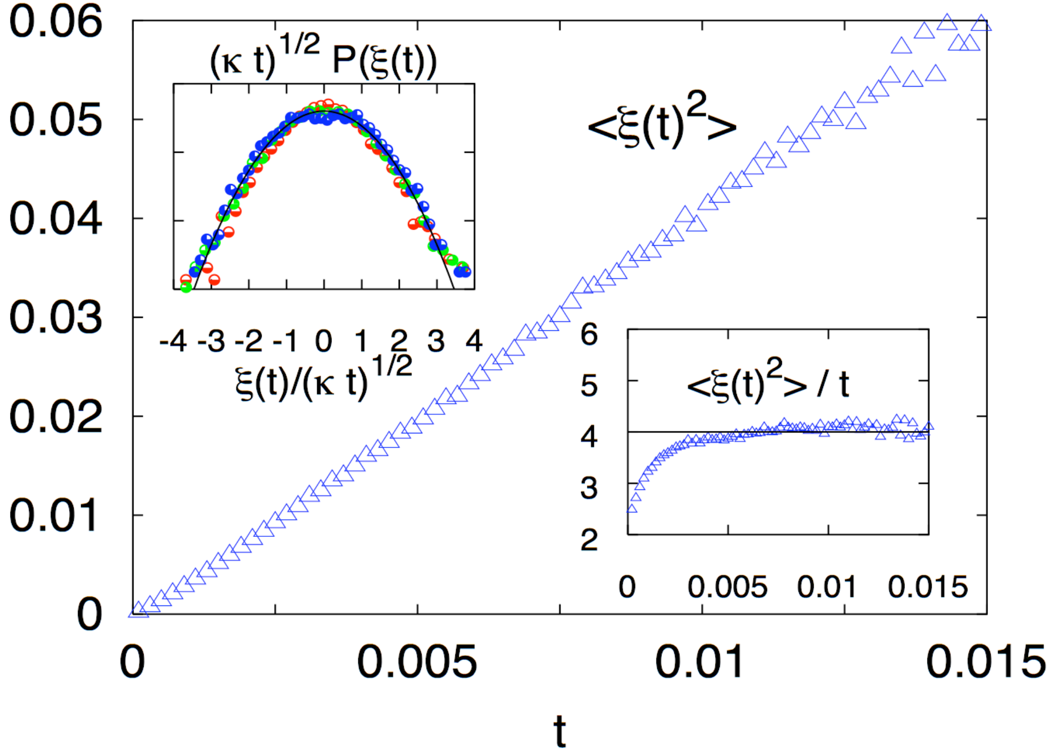
$$\partial_t g = 2 / \{\varphi(g_t)[\varphi(g_t) - \xi_t]\}$$

$$\varphi(z) = x_0 + (x_\infty - x_0)(z - x_0)/(x_\infty - z)$$

Discretization



Statistics of the driving function

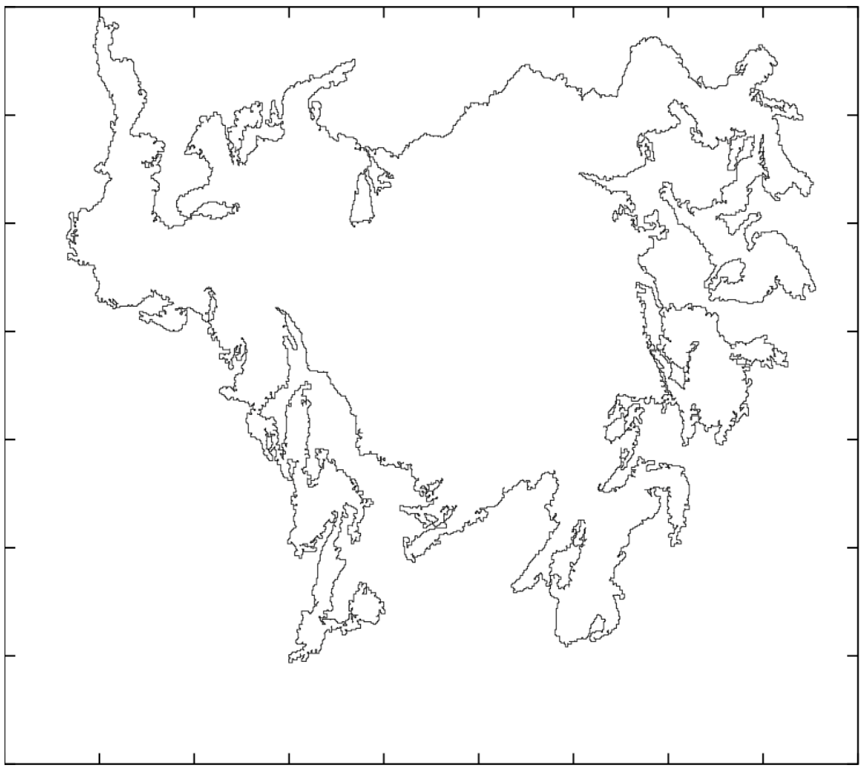


- $\xi(t)$ is an effective diffusion process with $\kappa=4$

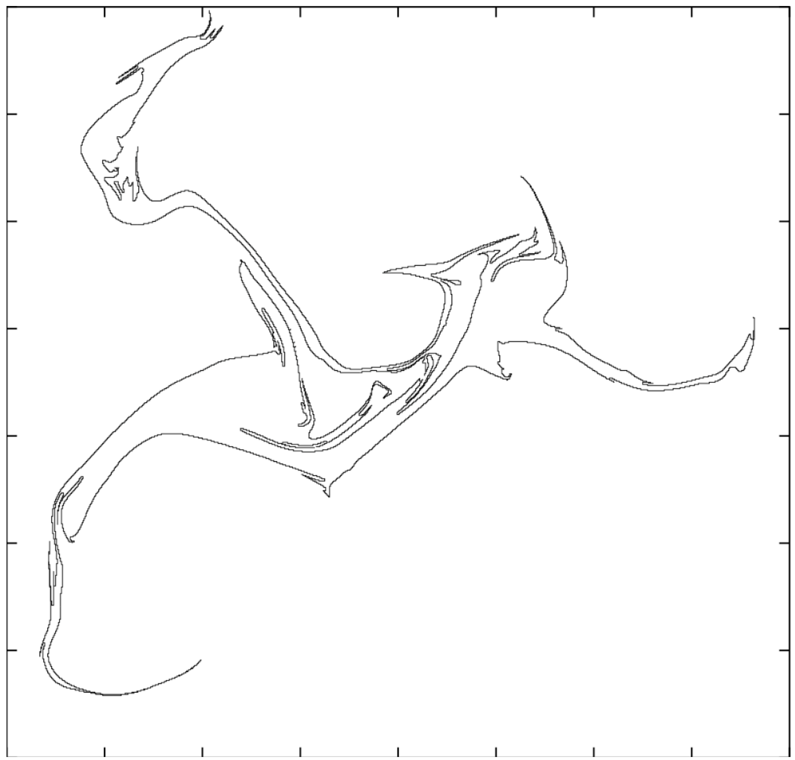
→ loops are statistically equivalent to SLE_4

Inverse *vs* direct cascades

SQG turbulence
inverse cascade



2D NS turbulence
direct cascade



Conclusions

- Evidence for conformal invariance of isolevel lines in Surface Quasi-Geostrophic turbulence
- Loops are locally SLE_4
- Loops in NS 2D direct cascade are not SLE

Perspectives

Active scalar models (Pierrehumbert *et al*, Chaos Sol Fract 1994)

$$\partial_t T + v \nabla T = \kappa \nabla^2 T + f$$

$$v = z \times \nabla \psi$$

$$\psi(x,t) = \int |x-y|^{\alpha-2} T(y,t)\, dy$$

$\alpha = 1$: Surface Quasi-Geostrophic

$\alpha \rightarrow 2$: 2D Navier-Stokes

$\alpha = -2$: Charney-Hasegawa-Mima

Perspectives: active scalar

- Dimensional arguments for the inverse cascade: $H = (2-2\alpha)/3$
- Fractal dimension of loops

  $(3-H)/2 = (7+2\alpha)/6$

  (in the range $0 < H < 1$ corresponding to $1 > \alpha > -1/2$)
- Conjecture: isolevel lines are SLE_κ with $\kappa = 4(1 + 2\alpha)/3$

Open issues

- CFT for inverse cascades ?
- Are correlation functions conformally invariant ?
- Markov domain property ?
- Relationship to stat mech models (point vortices) ?