Random Planar Curves
and
Stochastic Loewner Evolution

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Outline

- the physical context: random curves in statistical mechanics
- planar curves, conformal mappings and Loewner evolution
- random curves and stochastic Loewner evolution (SLE)
- some properties of SLE
- a sample computation using SLE
- further results and extensions

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- the physical context: random curves in statistical mechanics
- planar curves, conformal mappings and Loewner evolution
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- nearly all the material in this talk is originally due to
  `G Lawler, O Schramm and W Werner`

Random curves in physical models

- many important physical processes give rise to self-similar random curves in the plane, e.g.
- percolation
- cluster boundaries in the Ising model
- self-avoiding walks

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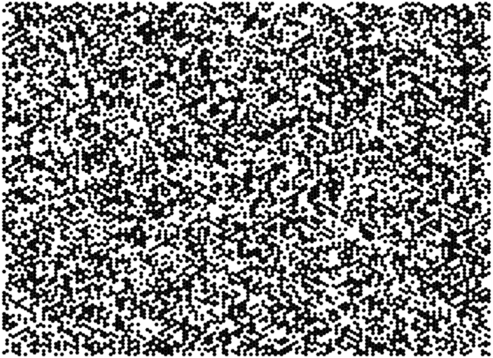
- turbulence?

Critical Percolation

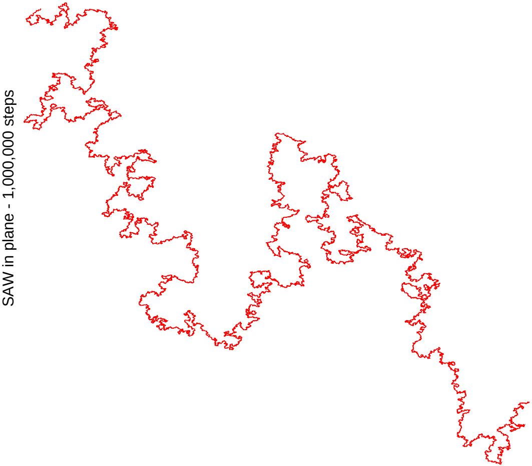
- cells of a honeycomb lattice independently coloured black/white with equal probability



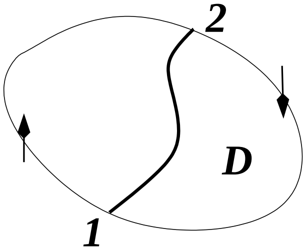
- cluster distribution
- cluster boundaries
- *statistical* scale invariance
- conformal invariance
- is there a crossing?
- Ising interactions

Self-avoiding Walk

- all non-intersecting walks of fixed length given equal weight

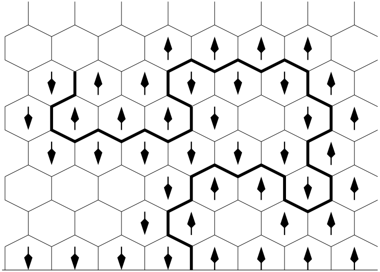


- in the continuum limit (at critical point) these become a set of fractal curves - what is the measure on this set?
- or, what is the measure on just one of them?
- specify conditions on the boundary of a simple connected domain $\mathcal{D}$ such that there is always a single open curve from r_1 to r_2:



Curves as growth processes

- such curves can be ‘grown’ on the lattice by a discrete exploration process:



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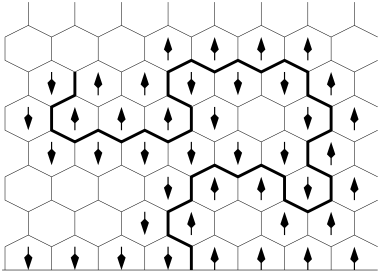
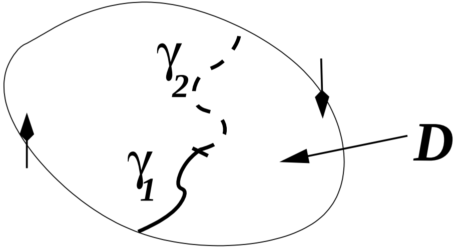
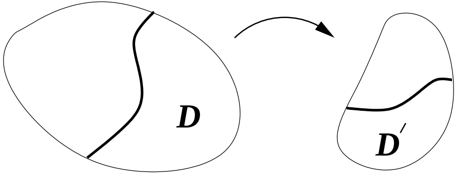


- SLE describes the continuous version of this

The postulates of SLE



Property 1. Conditional measure on γ_2 in domain $\mathcal{D}$, given γ_1, is the same as the unconditional measure on γ_2 in the modified domain $\mathcal{D} \setminus \gamma_1$.

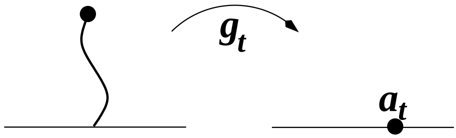


Property 2 (conformal invariance.)

- Let Φ be a conformal mapping of $\mathcal{D}$ to $\mathcal{D}'$
- scaling limit of a lattice model in $\mathcal{D}$ gives a measure on curves γ in $\mathcal{D}$
- this gives a measure on the image curves $\Phi(\gamma)$ in $\mathcal{D}'$
- this is the same as that given by the scaling limit of the same lattice model defined in $\mathcal{D}'$

Loewner evolution (1923)

- choose $\mathcal{D}$ = upper half plane $\mathbb{H}$
- let γ_t be the curve at time t



- let $g_t(z)$ be the conformal mapping which sends $\mathbb{H} \setminus \gamma_t$ to $\mathbb{H}$, hydrodynamic normalisation:

$$g_t(z) \sim z + 0 + \frac{2t}{z} + \cdots \qquad (\text{as } z \to \infty)$$

- g_t sends the growing tip into a_t on the real axis

Example: $\gamma_t = (a, a + ib)$

$$g_t(z) = a + \left((z-a)^2 + 4t\right)^{1/2}$$

where $b = 2t^{1/2}$.

- more generally

$$g_{t+\delta t}(z) \approx a_t + \left((g_t(z) - a_t)^2 + 4\delta t\right)^{1/2}$$

- the evolution of g_t satisfies the Loewner equation

$$\frac{dg_t(z)}{dt} = \frac{2}{g_t(z) - a_t}$$

- if curve is continuous so is a_t
- so instead of thinking about a measure on curves we can think about a measure on continuous functions a_t

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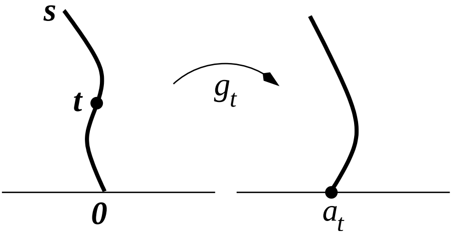
Theorem `[Schramm]`: *if Properties 1-2 hold then a_t is proportional to a standard Brownian motion.*

That is

$$a_t = \sqrt{\kappa} B_t$$

Idea of proof



- evolve for time t then for a further time $s - t$
- image under g_t is same as evolving for time $s - t$ starting from a_t
- $a_s - a_t$ given a_t has the same law as a_{s-t} given a_0
- $a_{(n+1)\delta t} - a_{n\delta t}$ are i.i.d. random variables for all $\delta t > 0$
- Brownian motion

$$\mathbb{E}\left[(a_s - a_t)^2\right] = \kappa |s - t|$$

Universality classes

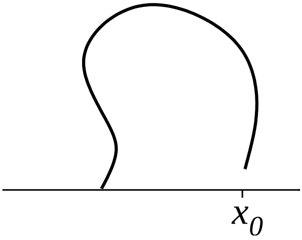
- different values of κ are conjectured (in some cases proved) to correspond to different universality classes of 2d equilibrium critical behaviour, eg:
- $\kappa = \frac{8}{3}$ self-avoiding walks
- $\kappa = 3$ Ising domain walls
- $\kappa = 4$ level lines at roughening transition
- $\kappa = 6$ percolation cluster boundaries

Some properties of SLE

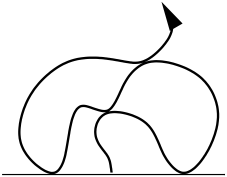
- let $\tilde{x}_t = x_t - a_t$
- points on the real axis evolve according to Bessel process

$$d\tilde{x}_t = \frac{2dt}{\tilde{x}_t} - \sqrt{\kappa} dB_t$$

- if $\kappa > 4$, $\tilde{x}_t \to 0$ (almost surely)
- the curve swallows a whole region:



- for $\kappa \leq 4$ the curve is simple
- for $4 < \kappa < 8$ it has double points (on all length scales):

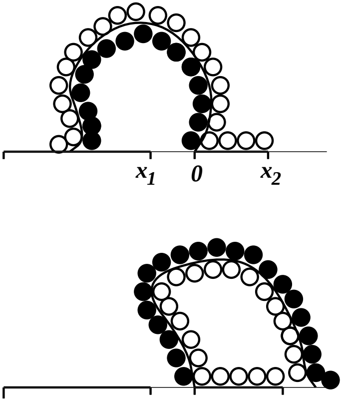


- for $\kappa \geq 8$ it is space-filling
- fractal dimension for $\kappa \leq 8$:

$$d_f = 1 + \kappa/8$$

Sample calculation: the crossing formula for percolation

- is there a left-right crossing of the rectangle?
- conformally map to $\mathbb{H}$



- is there a crossing on white from $(0, x_2)$ to $(-\infty, x_1)$?
- happens iff x_1 gets swallowed by the SLE before x_2

$$\text{Let} \qquad P(x_1, x_2) = \Pr(\tilde{x}_{1t} \to 0 \text{ before } \tilde{x}_{2t})$$

- evolve for time dt

$$P(x_1, x_2) = \mathbb{E}\left[P\left(x_1 + \frac{2dt}{x_1} - \sqrt{\kappa}dB_t,\ x_2 + \frac{2dt}{x_2} - \sqrt{\kappa}dB_t\right)\right]$$

- equate terms $O(dt)$, using $\mathbb{E}[dB_t] = 0$ and $\mathbb{E}\big[(dB_t)^2\big] = dt$

$$\left(\frac{2}{x_1}\frac{\partial}{\partial x_1} + \frac{2}{x_2}\frac{\partial}{\partial x_2} + \frac{\kappa}{2}\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)^2\right) P(x_1, x_2) = 0$$

- P depends only on $\eta = -x_2/x_1 \Rightarrow$ hypergeometric equation, solution ($\kappa = 6$):

$$P = \frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{4}{3})\Gamma(\frac{1}{3})}\eta^{1/3}{}_2F_1(\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{4}{3}; \eta)$$

Further results and extensions

- most of the previously known and conjectured critical exponents of 2d critical behaviour can be derived by asking the right questions in SLE
- generalisation to N curves

$$\frac{dg_t(z)}{dt} = \sum_{j=1}^{N} \frac{2}{g_t(z) - a_{jt}}$$

  - the a_{jt} satisfy Dyson's Brownian motion
- measures on closed loops can also be constructed (CLE)
- deep connection with conformal field theory

Some References

- papers by Lawler, Schramm, Werner on arXiv:math.PR

For physicists:

- W. Kager and B. Nienhuis, *A guide to stochastic Loewner evolution and its applications*, `math-ph/0312251`
- J. Cardy, *SLE for theoretical physicists*, `cond-mat/0503313`
- M. Bauer and D. Bernard, *2D growth processes: SLE and Loewner chains*, `math-ph/0602049`