New invariant for drift turbulence

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A new invariant for drift wave (or Rossby wave) turbulence in two cases, (1) in zonal flow and (2) in the large scale range, is discovered. This invariant is proved to be a unique additional invariant (besides energy and momentum). Thus the first examples of wave systems with a finite number of additional invariants are obtained. A new Kolmogorov-type spectrum with the flux of the additional invariant through the scales is derived and the structure of fluxes in the k-space of invariants is analyzed.

1. A lot of problems in the physics of atmosphere [1], in the physics of magnetized plasmas [2] and in astrophysics [3] leads to a study of drift-type waves, or Rossby waves, having the dispersion law

$$\omega_k = \frac{\beta k_x}{1+\rho^2 k^2} \tag{1}$$

($k=(k_x, k_y)$ is a wave vector, β, ρ are constants). The nonlinear interaction of the waves may vary [4,5], depending on a particular physical situation, e.g. in a number of cases it is described by the following equation [1,2],

$$\frac{\partial}{\partial t}(\rho^2\Delta\psi-\psi)-\beta\frac{\partial\psi}{\partial x}+\frac{\partial\psi}{\partial x}\frac{\partial\Delta\psi}{\partial y}-\frac{\partial\Delta\psi}{\partial x}\frac{\partial\psi}{\partial y}=0, \tag{2}$$

which differs from the equation of 2-D hydrodynamics only by the presence of terms $-\dot{\psi}-\beta\partial\psi/\partial x$. If $\beta\neq 0$ then under sufficiently small ψ the turbulence, described by eq. (2), may be represented as a system of dispersive weakly interacting waves. For the statistical description of such a turbulence one can use the kinetic equation for waves [6]:

$$\frac{\partial n_k}{\partial t}=4\pi\int |V_{kk_1k_2}|^2\delta(k-k_1-k_2)\times\delta(\omega_k-\omega_{k_1}-\omega_{k_2})\,[n_{k_1}n_{k_2}-n_k n_{k_1}\,\mathrm{sgn}(\omega_k\omega_{k_2})-n_k n_{k_2}\,\mathrm{sgn}(\omega_k\omega_{k_1})]\,\mathrm{d}k_1\,\mathrm{d}k_2, \tag{3}$$

where $n_k=\epsilon_k/\omega_k$ is the wave action spectrum, ϵ_k is the energy spectrum, $n_k=n_{-k}$; $V_{kk_1k_2}$ is the matrix element of the nonlinear interaction, e.g. in the case of eq. (2) it is of the form [7]

$$V_{kk_1k_2}=-\frac{\mathrm{i}}{4\pi}|k_x k_{1x}k_{2x}|^{1/2}\times\left(\frac{k_y}{1+\rho^2k^2}-\frac{k_{1y}}{1+\rho^2k_1^2}-\frac{k_{2y}}{1+\rho^2k_2^2}\right). \tag{4}$$

Note that all the following parts of this article do not depend on the form of the nonlinearity.

2. The time alteration of any integral

$$\phi=\tfrac{1}{2}\int\varphi_k(\mathrm{sgn}\,\omega_k)n_k\,\mathrm{d}k \tag{5}$$

is determined by the expression

$$\dot{\phi}=2\pi \int |V_{kk_1k_2}|^2 \delta(k-k_1-k_2)\delta(\omega_k-\omega_{k_1}-\omega_{k_2}) \times n_k n_{k_1} n_{k_2}(\varphi_k-\varphi_{k_1}-\varphi_{k_2})\left(\frac{\operatorname{sgn}\omega_k}{n_k}-\frac{\operatorname{sgn}\omega_{k_1}}{n_{k_1}}-\frac{\operatorname{sgn}\omega_{k_2}}{n_{k_2}}\right) \mathrm{d}k\,\mathrm{d}k_1\,\mathrm{d}k_2\,.$$

Hence the quantity ϕ is an integral of motion of eq. (3) if and only if for all wave vectors k, k_1, k_2 bound by the resonance relations

$$k=k_1+k_2\,,\quad \omega_k=\omega_{k_1}+\omega_{k_2}\,, \tag{6}$$

the identity

$$\varphi_k=\varphi_{k_1}+\varphi_{k_2} \tag{7}$$

is fulfilled. It is clear that eq. (3) always conserves energy ($\mathscr{E}$) and momentum ($\mathscr{P}$):

$$\mathscr{E}=\tfrac{1}{2}\int |\omega_k|\,n_k\,\mathrm{d}k\,,\quad \mathscr{P}=\tfrac{1}{2}\int k(\operatorname{sgn}\omega_k)n_k\,\mathrm{d}k\,.$$

If the kinetic equation (3) has some more integrals of motion linearly independent of $\mathscr{E}$ and $\mathscr{P}$ (such a situation is exceptional) then the dispersion law is called degenerate [8,9]. To the integral of motion ϕ of the kinetic equation corresponds an adiabatic (approximate) invariant of the original equation. This invariant contains a quadratic (with respect to the wave amplitudes) part equal to ϕ (regardless of the form of nonlinearity), and its time derivative has fourth order of smallness with respect to the wave amplitudes [9].

3. We will show that for the turbulence of drift waves in two limiting cases, (1) in the case of zonal flow ($|k_y|\gg|k_x|$) and (2) in the case of large scale turbulence ($\rho^2k^2\ll 1$), the dispersion law is degenerate and there exists one additional invariant of the form (5).

The turbulence in the region of zonal flow and in the region of large scales plays a particularly important role in drift turbulence. This is confirmed by some experimental data, e.g. the observations of the Jupiter atmosphere [10], and the measurements of plasma turbulence in the F-layer of the equatorial ionosphere [11]. Computer experiments also show a strong concentration of the spectrum in the zonal flow and in the large scale range [12–14]; such a process can be regarded as the result of inverse energy transfer in the case of absence of effective dissipation in large scales. It is shown in ref. [15] that the evolution of drift turbulence can be determined only by its interaction with the zonal flow and large scale range (but not by close scales). Moreover the system tends to the state when just the mentioned regions contain the main amount of energy.

4. In the case of zonal flow ($|k_y|\gg|k_x|$) the dispersion law of Rossby waves has the form

$$\omega_k=\frac{\beta k_x}{1+\rho^2k^2}\,. \tag{8}$$

Now we shall show that the system of such waves possesses an additional invariant (5) with

$$\varphi_k=\omega_k^3/k_y^2\,. \tag{9}$$

For this sake we will twice use the following easy verifiable fact: if some quantities x, x_1, x_2, y, y_1, y_2 are bound by the relations

$$x+x_1+x_2=0\,,\quad y+y_1+y_2=0\,,$$

then

$$xy^2+x_1y_1^2+x_2y_2^2\equiv yy_1y_2\left(\frac{x}{y}+\frac{x_1}{y_1}+\frac{x_2}{y_2}\right).$$

Supposing $x=\omega_k$, $y=k_y$, $x_i=-\omega_{k_i}$, $y_i=-k_{iy}$ ($i=1$, 2) and taking into account that $\omega_k k_y^2\equiv(\beta k_x-\omega)\rho^{-2}$ we obtain that for any k, k_1, k_2 complying with relations (6) the following equality,

$$\frac{\omega_k}{k_y}+\frac{\omega_{k_1}}{k_{1y}}+\frac{\omega_{k_2}}{k_{2y}}\equiv 0\,,$$

is fulfilled. Now supposing $y=\omega_k/k_y$, $y_i=\omega_{k_i}/k_{iy}$ ($i=1$, 2) and leaving the quantities x, x_1, x_2 unchanged, we obtain that on the resonance manifold (6) the following identity,

$$\frac{\omega_k^3}{k_y^2}-\frac{\omega_{k_1}^3}{k_{1y}^2}-\frac{\omega_{k_2}^3}{k_{2y}^2}=0\,,$$

is valid, so that the quantity (5) with spectral density (9) is invariant indeed.

5. Let us show that the system of large scale drift waves with the dispersion law

$$\omega_k = \beta k_x (1 - \rho^2 k^2) \tag{10}$$

has an additional invariant (5), where

$$\varphi_k = \frac{k_x^3}{k_y^2 - 3k_x^2}. \tag{11}$$

In fact, in the variables

$$p = k_y + \sqrt{3}\, k_x , \quad q = k_y - \sqrt{3}\, k_x ,$$

the quantity (11) acquires the form

$$\varphi = \frac{p^2}{q} - \frac{q^2}{p}, \tag{12}$$

and relations (6) (when the dispersion has the form (10)) can be presented in the following way,

$$p = p_1 + p_2 , \quad q = q_1 + q_2 , \quad p p_1 p_2 = q q_1 q_2$$
$$(p_i = p(k_i) , q_i = q(k_i) , \quad i = 1, 2) .$$

Under these conditions we have

$$\frac{p^2}{q} - \frac{p_1^2}{q_1} - \frac{p_2^2}{q_2} = \frac{p_1 q_2 - p_2 q_1}{q q_1 q_2}$$
$$= \frac{q_1 p_2 - q_2 p_1}{p p_1 p_2} = \frac{q^2}{p} - \frac{q_1^2}{p_1} - \frac{q_2^2}{p_2},$$

so that the quantity (12) satisfies identity (7).

6. The existence of an additional invariant is of fundamental significance for the integrability theory of nonlinear equations [9]. So far it was assumed that all wave systems either have no additional invariants (besides energy, momentum, wave action) or have an infinite number of additional invariants as, e.g., the well-known Korteweg–de Vries, Kadomtsev–Petviashvili, Davey–Stewartson equations, and the nonlinear Schrödinger equation. It can be shown (the proof will be published apart), that in the systems considered in this paper the obtained invariant is the only additional invariant (quadratic with respect to the wave amplitudes). Thus the first examples of wave systems with a finite number of additional invariants are discovered.

7. The availability of the additional invariant (5) leads to a more general form of the thermodynamical equilibrium spectrum:

$$n_k = \frac{T}{\omega_k + (k, v) + \mu\varphi_k} \operatorname{sgn} \omega_k \tag{13}$$

(T, $v = (v_x, v_y)$, μ are arbitrary constants) and involve the existence of a new Kolmogorov-type (see ref. [16]) spectrum, which is determined by the flux of the new invariant through the scales. This spectrum we succeeded in finding only for the case of zonal flows – and moreover; either for long wave turbulence ($\rho^2 k_y^2 \gg 1$) or for short wave turbulence ($\rho^2 k_y^2 \ll 1$); in these situations the matrix element is usually scale invariant with respect to the components of the wave vectors with some vector exponent $\beta = (\beta_x, \beta_y)$,

$$V(\lambda_x k_x, \lambda_y k_y; \lambda_x k_{1x}, \lambda_y k_{1y}; \lambda_x k_{2x}, \lambda_y k_{2y})$$
$$= \lambda_x^{\beta_x} \lambda_y^{\beta_y} V(k, k_1, k_2)$$

(λ_x, λ_y are arbitrary positive numbers). Then the Kolmogorov-type spectrum with flux R of invariant ϕ is of the form

$$n_k = C R^{1/2} k^{-\nu} ,$$
$$\nu = \beta + (2, -1) , \quad \rho^2 k_y^2 \ll 1 ,$$
$$= \beta + (2, -2) , \quad \rho^2 k_y^2 \gg 1 \tag{14}$$

(C is a dimensionless constant). For example in the case (2), (4): for long wave turbulence $\beta = (\frac{3}{2}, 3)$, $\nu = (\frac{7}{2}, 2)$, for short wave turbulence $\beta = (\frac{3}{2}, -1)$, $\nu = (\frac{7}{2}, -3)$. The Kolmogorov-type spectrum (14) along with the already known Kolmogorov-type spectra with fluxes of energy and enstrophy [17,18], as well as the thermodynamical spectrum (13) are exact stationary solutions of the kinetic equation (3).

8. The presence of the additional invariant (5) enables us to make essential conclusions about the structure of drift turbulence and in particular about the directions of fluxes in the k-space of various invariants [19]. This is analogous to the situation in 2D-hydrodynamics, where the presence of the additional (in comparison with 3D-hydrodynamics) invariant of enstrophy forbids energy to flow to small scales. In figs. 1–3 we have schematically shown the picture of the fluxes of the invariants $\mathscr{E}$, $Q = \beta\mathscr{P}_x - \mathscr{E}$, ϕ, $\mathscr{P}_y$ in stationary turbulence, when the regions of dissipation and growth rate considerably differ in

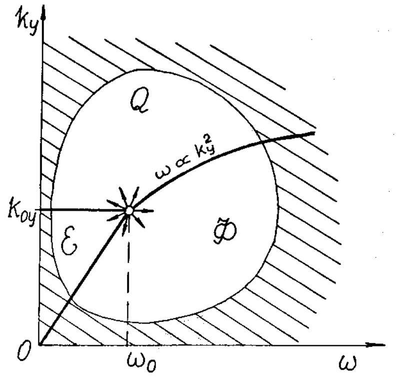

Fig. 1. Domains restricting the directions of the fluxes of invariants $\mathscr{E}$, Q, ϕ in the case of zonal flow.

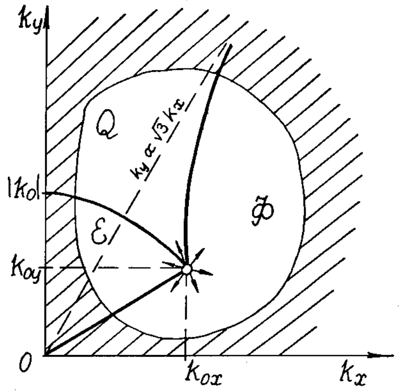

Fig. 2. Domains restricting the directions of the fluxes of the invariants $\mathscr{E}$, Q, ϕ in the case of large scale turbulence under the condition $|k_{0y}|<\sqrt{3}\,|k_{0x}|$.

scales, the growth rate is characterized by some scales k_{0x}, k_{0y} and the region of possible dissipation is marked by touches (we used coordinates (ω_k, k_y) instead of (k_x, k_y)). Let us consider for example how the invariant ϕ restricts the directions of the flux of energy $\mathscr{E}$.

In the case of zonal flow (see (8), (9)) the invariant ϕ is positive definite, so an essential amount of energy cannot dissipate in the region

$$\left|\frac{\omega_k}{\varphi_k}\right| \ll \left|\frac{\omega_{k0}}{\varphi_{k0}}\right| ,$$

because otherwise a too large amount of the invariant ϕ would dissipate in that region. Therefore almost all of the energy transfers in the region limited by the condition

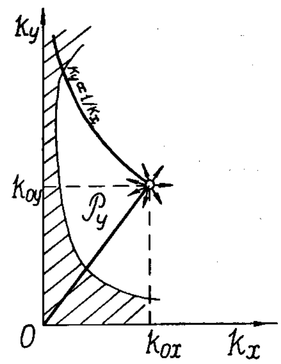

Fig. 3. Domain restricting the directions of the flux of y-momentum $\mathscr{P}_y$ in the case of zonal flow; in the case of large scales this domain is qualitatively the same.

$$\left|\frac{\omega_k}{\varphi_k}\right| > \left|\frac{\omega_{k0}}{\varphi_{k0}}\right| \quad \Leftrightarrow \quad \left|\frac{\omega_k}{k_y}\right| < \left|\frac{\omega_{k0}}{k_{0y}}\right|$$

(see fig. 1).

In the case of large scale turbulence (see (10), (11)) the invariant ϕ is not of fixed sign, and in general it does not restrict the fluxes of other invariants. However, it does if the growth rate region is located in the domain $|k_y|<\sqrt{3}\,|k_x|$. In fact, the enstrophy (x-momentum) restricts the flux of the invariant ϕ by the domain

$$\left|\frac{\varphi_k}{k_x}\right| > \left|\frac{\varphi_{k0}}{k_{0x}}\right| \quad \Leftrightarrow \quad \left|\frac{k_y}{k_x}\right| < \left|\frac{k_{y0}}{k_{x0}}\right|$$

(see fig. 2). If $|k_{0y}|<\sqrt{3}\,|k_{0x}|$ then almost all of the invariant ϕ remains in the domain $|k_y|<\sqrt{3}\,|k_x|$ and the invariant ϕ can be considered to be of fixed sign, and during that time it restricts the direction of energy flux by the domain $|\omega_k/\varphi_k|>|\omega_{k0}/\varphi_{k0}|$ (see fig. 2). Similarly other restrictions on fluxes of invariants are obtained (figs. 1–3).

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