the Maskit embedding of the Teichmüller space of a once punctured torus.

What is this picture? This is a picture of the *Maskit embedding of the Teichmüller space of a once punctured torus*. Each point in the picture represents a complex number which defines a 2-generator group of Möbius transformations. The coloured region represents all those points for which the group is both free and discrete as a subset of $SL(2, \mathbb{C})$. Each such group defines a hyperbolic 3-manifold. The manifolds all have the same topology but varying geometries. The coloured lines are called *pleating rays*. The manifolds on a given ray have a special geometrical property in common: the boundary of their convex cores are 'bent' along a specific curve which is fixed along the ray. This is reflected in a special pattern of overlapping circles in their limit sets. These circles all overlap at an angle which increases from 0 (at infinity off the top of the picture) to π at the end of the ray. Beyond this, in the grey region, the group is no longer free and discrete.

Why is the picture interesting? A discrete subgroup of $SL(2, \mathbb{C})$ is called a *Kleinian group*. Kleinian groups are at the meeting point of several different parts of mathematics. Classically, Kleinian groups arose as the monodromy groups of Schwarzian equations, in modern terminology projective structures on Riemann surfaces. As Möbius maps, Kleinian groups act on the Riemann sphere, and thereby are closely linked in other ways with Riemann surfaces, complex dynamics and fractals. One of Poincaré's fundamental observations was that $SL(2, \mathbb{C})$ acts by isometries on hyperbolic 3-space $\mathbb{H}^3$. Thus if G is Kleinian, then $\mathbb{H}^3/G$ is a hyperbolic 3-manifold. And Thurston's revolutionary insights have made hyperbolic 3-manifolds central to 3-dimensional topology.

This picture illustrates one of the simplest possible examples of a holomorphically varying family of Kleinian groups. From the viewpoint of complex dynamics, it is a close analogue of the Mandelbrot set. From the viewpoint of quasiconformal mappings, it is a concrete realisation of a Teichmüller space; while from the 3-dimensional viewpoint it illustrates the continuously varying geometry of a family of hyperbolic 3-manifolds. Part of the fascination is that all these different viewpoints needed to be combined to make this picture.

Prior to this picture, noone had more than a very rudimentary idea of the location of the set of discrete groups in a given family as a subset of the ambient parameter space.

What are the pleating rays? The rays fill out the coloured region densely. There is one ray for each simple closed curve on $\mathbb{T}^*$. Such curves are indexed by the rational numbers. The picture actually shows all rays for curves $\gamma(p/q)$ with $q \leq N$: the colours just reflect the value of p/q. All groups along a given ray have a common geometric property, best understood by venturing into hyperbolic 3-space. All the closed geodesics in the manifold $\mathbb{H}^3/G$ are contained in a smallest convex set, called its *convex core*. The convex core is topologically $\mathbb{T}^* \times (0,1)$; its boundary $\partial\mathcal{C}$ is the union of $\mathbb{T}^*$ and a triply punctured sphere ($\mathbb{T}^*$ with one curve pinched to zero length).

$\partial\mathcal{C}$ is totally geodesic except along one geodesic along which it is 'bent'. The p/q-ray represents all groups for which the bending line is the fixed curve $\gamma(p/q)$. The length of $\gamma(p/q)$ decreases monotonically down the ray and the ray ends at the point where it has zero length. Beyond this, $G(\mu)$ is no longer both free and discrete. The ray can be computed by the polynomial condition $\operatorname{Tr}\gamma(p/q) \in \mathbb{R}$, although, crucially, this condition is not sufficient for a point to lie on the ray. A surprising feature is that the p/q-ray is asymptotic to the line $\Re\mu = 2p/q$ as $|\mu| \to \infty$.

This countable set of rays can be interpolated by rays along which the bending locus of $\mathcal{C}$ is a geodesic lamination on $\mathbb{T}^*$. Technically, each ray is the locus along which the bending measure of the convex core boundary is in a fixed projective class. In this form, the definition extends to any holomorphic family of Kleinian groups.

What is the group and what is the Maskit embedding of Teichmüller space of the once punctured torus? The picture represents part of the complex μ-plane, in which $\Im\mu > 0$ and $-1 \leq \Re\mu \leq 1$. It repeats under horizontal translation $\mu \mapsto \mu + 2$. A point μ in the picture represents the group $G = G(\mu)$ of Möbius transformations generated by the Möbius maps $z \mapsto z + 2$ and $z \mapsto \mu + 1/z$ acting on the complex z-plane. Consider a point z_0 and all of its images under $G(\mu)$. There are two possibilities: either there are no points $g(z_0), g \neq 1$ near z_0, or there are infinitely many points $g(z_0)$ arbitrarily close to z_0. In the first case z_0 is said to belong to the *regular set* of G, and in the second case it belongs to the *limit set*. The dynamics of G acting on the regular set is well behaved and 'reasonable', while the dynamics on the limit set is chaotic.

Each coloured point represents a Kleinian group for which the regular set Ω is non-empty[1] In this case, Ω/G makes sense and is a union of Riemann surfaces. Things are set up so that $\Omega/G(\mu)$ is the union of a torus with one missing point (a 'once punctured torus' $\mathbb{T}^*$); and a triply punctured sphere.

The Riemann surface structure on the triply punctured sphere is fixed. However as

[1]There are Kleinian groups for which $\Omega = \emptyset$; these lie actually on the boundary and their classification is the main object of Minsky's famous *ending lamination theorem*.

the parameter μ varies, there is exactly one coloured point for each possible Riemann surface structure on the punctured torus. Thus the coloured region represents the *Teichmüller space* of $\mathbb{T}^*$. The idea that one could create a holomorphically varying family of groups with this property originated with Bernie Maskit, hence *The Maskit embedding of Teichmüller space*. You can make a Maskit embedding for any orientable topological surface with negative Euler characteristic.

How is this related to my work? The family of groups in this picture was originally investigated by Mumford and Wright, who, as explained in our book *Indra's Pearls* [4], located the boundary between discrete and non-discrete groups by ingenious computer experiment. However rigorous justification remained open until the discovery of pleating rays and their properties by myself and Linda Keen [2]. Each ray is defined by a polynomial equation. The rays are dense in region of free discrete groups, so one can use them to locate the complicated boundary between the two regions. Unlike the rays, the boundary cannot be described by any analytic equations. It seems highly likely that it spirals almost everywhere, although to date this has not been completely proved.

Much of my work since our original discovery has involved developing the theory of these rays in general, so as to be able to study higher dimensional parameter spaces representing more complicated families of groups. It is remarkable that the rays in the Maskit picture don't contain any singularities. Together with my postdoc Young Choi [1], I proved in complete generality that the higher dimensional analogue of pleating rays are totally real sub-manifolds of the appropriate complex (and high dimensional) parameter space. My former student Sara Maloni and I used this to give a precise description of the Maskit embedding for any surface of negative Euler characteristic [3]. The asymptotic directions of the rays turned out to be governed by a simple formula involving the Dehn-Thurston coordinates of the bending lines, in a manner closely analogous to the one dimensional results. Knowing these directions is crucial for making graphics: some actual pictures of rays were computed for a twice punctured torus by Alex Austin as part of an undergraduate research project at Warwick.

I have also worked out all the details of the location and behaviour of rays for quasifuchsian groups [5]. I am currently studying the case of handlebodies and Schottky groups. This would solve an old problem about determining which 2-generator subgroups of $SL(2,\mathbb{C})$ are free and discrete. An interesting study of a special one complex dimensional slice is described in [6].

References

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