A Garside type structure on the Torelli group

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Garside structures on $B_n$

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- Recently, weak versions of “Garside groups” were considered. We shall weaken much further.
Plan of the talk:

- BKL Garside structure.
- Simple example.
- Crash course on Torelli groups.
- General set-up.
- Why does it generalise BKL?
Definition. A \textit{(BKL) simple braid} is a braid like the following example:
Birman, Ko, Lee’s structure on $B_n$

Definition.

1. $\Omega := \{\text{simple braids}\}$.
2. $B_n^+ := \text{submonoid } \langle \Omega \rangle \text{ of } B_n$.
3. $\leq$ ordering on $B_n$: $x \leq xy \leftrightarrow y \in B_n^+$.
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Definition. A lattice is an ordered set such that for all $x, y \in L$ there is a least common upper bound or join $x \lor y$ and a greatest common lower bound or meet $x \land y$.
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Theorem. $(B_n, \leq)$ is a lattice.
Birman, Ko, Lee’s structure on $B_n$ 6

**Theorem.** Every $x \in B_n$ can uniquely be written in the **symmetric normal form** $x_k^{-1} \cdots x_1^{-1} y_1 \cdots y_\ell$ defined by the following properties:

1. $x_i$ and $y_j$ are nontrivial simple braids.
2. The greatest simple braid $\cdot x_i x_i + 1$ (which always exists) equals $x_i$. Same for $y_i y_i + 1$.
3. $x_1^{-1} y_1 = 1$.

Can we understand the symmetric normal form globally? Yes! $B_n$ is automatic (Thurston, 80’s). Better than automaticity: Grid property to be explained next.
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- $B_n$ is automatic (Thurston, 80’s).
- Better than automaticity: Grid property to be explained next.
**Definition.** Let $a, b \in B_n$. The *distinguished path* from $a$ to $b$ is $\{a_0, \ldots, a_r\} \subset B_n$ defined by:

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**Definition.** A subset of $B_n$ is *convex* if it contains an entire distinguished path as soon as it contains its endpoints.

**Definition.** The *Cayley graph* is the graph with vertex set $B_n$ and edges $\{a, ax\}$ whenever $a \in B_n$ and $x$ is a simple braid.
Birman, Ko, Lee’s structure on $B_n$

**Theorem (Grid property, K–Dehornoy).** The convex hull of three points in $B_n$ is a planar graph. (This can and should be made more precise).
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Theorem (Grid property, K–Dehornoy). The convex hull of three points in $B_n$ is a planar graph. (This can and should be made more precise).

- The grid property is analogous to convex sets in $\mathbb{R}^n$.
- The grid property is the best advertisement for Garside groups.

We say that BKL discovered a Garside structure on $B_n$. All of the above properties are true for all Garside groups.
A simple example

**Definition.** Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial. If $f'(\alpha) = 0$ then $\alpha$ is called a *critical point* and $f(\alpha)$ a *critical value.*
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**Definition.** Let $f: \mathbb{C} \to \mathbb{C}$ be a polynomial. If $f'(\alpha) = 0$ then $\alpha$ is called a **critical point** and $f(\alpha)$ a **critical value**.

**Definition.** We define $K$ to be the set of polynomials $f: \mathbb{C} \to \mathbb{C}$ of degree $n$ such that all critical values are real, modulo $f \sim g$ if $f(x) = g(ax + b)$. 
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**A simple example**

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**Answer.**

\[
\begin{array}{c}
\bullet & \leftarrow & a \\
\bullet & \leftrightarrow & b \\
\bullet & \leftarrow & c \\
\bullet & \leftrightarrow & d \\
\end{array}
\]

\[\leq\]
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**Answer.**

![Diagram](image)

From now on, we will mostly *work upstairs*.

We will soon see examples of critical points bumping into each other or splitting.
A simple example

From now on, we mostly *work dually*. Critical points become 2-cells (called *regions*), and so on.
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Example of an element of $K$:

\[
\begin{align*}
\text{fixed bit} \\
\begin{array}{c}
\begin{array}{c}
\text{a} \\
\text{d} \quad \text{b} \\
\text{c} \quad \text{e}
\end{array}
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\{a, \ldots, e \in \mathbb{R} \\
\quad a \leq b \leq c \leq d \\
\quad e \leq b
\end{array}
\end{align*}
\]
A simple example

From now on, we mostly work dually. Critical points become 2-cells (called regions), and so on.

Example of an element of $K$:

Always modulo removing arcs between regions of equal heights:

$$
\begin{align*}
\{a, \ldots, e \in \mathbb{R} \mid a \leq b \leq c \leq d, e \leq b\}
\end{align*}
$$
A simple example

Example of a generator of the ordering:

\[
\begin{array}{ccc}
8 & 2 & 5 \\
6 & 5 & 3 \\
\end{array}
\]

\[
\begin{array}{ccc}
8 & 5 & 6 \\
5 & 3 & 6 \\
\end{array}
\]

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8 & 5 & 6 \\
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A simple example

Definition. A *semi-simple pair* is a pair \((f, g) \in K \times K\) where \(g\) is obtained from \(f\) by moving some critical values to the right by the same amount, keeping the others fixed.
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Example. Example of a semi-simple pair \((x, y) = (x \to y)\). We concentrate on a single region for \(x\).
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Example. Another example of a semi-simple pair \((x, y) = (x \to y)\), which is allowed even though \(5 > 4\):
Example. Example of a complement (relation):

\[
\begin{array}{c}
0 \\
\downarrow \\
0 \ 1 \ 0
\end{array}
\rightarrow
\begin{array}{c}
0 \\
\downarrow \\
9 \\
\downarrow \\
0
\end{array}
\]
A simple example

Example. Example of a complement (relation):
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Warning. It is not true that if $x \leq y \leq z$ and $(x, z)$ is semi-simple, then $(x, y)$ or $(y, z)$ is semi-simple.
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Explanation. Between any two distinct real numbers there is another real number. The semi-simple pairs behave like a preferred small set of positive normal forms.
A simple example

**Definition.** Let \((x, y) \in K \times K\) be a semi-simple pair. Its \textit{length} is the number of moving critical values (with multiplicities) times the common amount they move by.
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**Definition.** Let $(x, y) \in K \times K$ be a semi-simple pair. Its *length* is the number of moving critical values (with multiplicities) times the common amount they move by.

**Definition.** Let $x \in K$ and $a \in \mathbb{R}$. Then $(x, x\Delta^a)$ is the semi-simple pair where all critical values move to the right by $a$. 
A simple example

Theorem. 1. \((K, \leq)\) is a lattice.
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2. There is a largest metric \(d\) on \(K\) extending the length \(d(x, y)\) already defined for semi-simple pairs.
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2. There is a largest metric \(d\) on \(K\) extending the length \(d(x, y)\) already defined for semi-simple pairs.

3. Let \(x, y \in K\), \(x < y\), \(d(x, y) = a\). Consider the path from \(x\) to \(y\)

\[
\begin{align*}
[0, a] &\longrightarrow K \\
\{ \ & t \longmapsto x \Delta^t \land y. \}
\end{align*}
\]

There is a finite sequence \(x = x_0, \ldots, x_k = y\) such that \((x_i, x_{i+1})\) is semi-simple and the path passes through all \(x_i\).
A simple example

**Theorem.** 1. \((K, \leq)\) is a lattice.

2. There is a largest metric \(d\) on \(K\) extending the length \(d(x, y)\) already defined for semi-simple pairs.

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4. These paths satisfy a grid property. □
A simple example

**Suggestion.** Study *Garside spaces* instead of say CAT(0) spaces.
Definition. The mapping class group $\text{MCG}(S)$ of a surface $S$ is $H/H_0$ where $H$ is the topological group of (orientation preserving) self-homeomorphisms of $S$, preserving the boundary pointwise, and $H_0 \subset H$ is the connected component of 1 in $H$. 
Definition. The *mapping class group* $\text{MCG}(S)$ of a surface $S$ is $H/H_0$ where $H$ is the topological group of (orientation preserving) self-homeomorphisms of $S$, preserving the boundary pointwise, and $H_0 \subset H$ is the connected component of 1 in $H$.

**Fact 1.** A compact connected oriented surface is determined by its genus and the number of boundary components. □
Crash course on Torelli group

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**Fact 3.** If $S$ is compact then $\text{MCG}(S)$ is finitely presented.
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Definition. The \textbf{mapping class group} $\text{MCG}(S)$ of a surface $S$ is $H/H_0$ where $H$ is the topological group of (orientation preserving) self-homeomorphisms of $S$, preserving the boundary pointwise, and $H_0 \subset H$ is the connected component of 1 in $H$.

Law 1. A compact connected oriented surface is determined by its genus and the number of boundary components.

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Fact 3. If $S$ is compact then $\text{MCG}(S)$ is finitely presented.

Fact 4. Let $g \geq 0$. The set of isomorphism classes of genus $g$ Riemann surfaces can be given the structure of an algebraic orbifold (moduli space); its fundamental group is $\text{MCG}(S)$.
Definition. The **Torelli group** $I(S)$ is the group of mapping classes which act trivially on $H_1(S)$. 
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So there is an exact sequence

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So there is an exact sequence

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The quotient $\text{MCG}(S)/I(S)$ is an arithmetic group and infinite in general.
The general set-up

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- $\phi: \pi_1 S \to \mathbb{R}$ a homomorphism. If $x \in \pi_1 S$ is representable by a simple closed curve and is not in the commutator subgroup of $\pi_1 S$ then $\phi(x) \neq 0$. 

Example 1. $\ker \phi = (\pi_1 S)_0$.

Example 2. $S = r f \circ \cdots \circ n g$, $\phi(\circ X) = 1 \phi_i I \circ X_i d \log(x - \circ_i)$. 

So $\text{im} \phi = \mathbb{R}$. 

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Example 1. $\ker \phi = (\pi_1 S)'$.

Example 2. $S = \mathbb{C} \setminus \{\alpha_1, \ldots, \alpha_n\}$,

$$\phi(\gamma) = \frac{1}{2\pi i} \oint_\gamma \sum_i d\log(x - \alpha_i).$$

So $\text{im} \phi = \mathbb{Z}!$. 
The general set-up

**Definition.** An element of $K$ is a cell decomposition of $S$ with $Q_0 \cup Q_1$ for vertices (as before); the height function is now a function

$$h: \left\{ \text{regions of } \tilde{S} \right\} \longrightarrow \mathbb{R}$$

such that, for all regions $R$ and all $\gamma \in \pi_1 S$:

$$h(\gamma R) = h(R) + \phi(\gamma).$$
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**Equivalently:**
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**Equivalent definition.** An element of $K$ is an isomorphism class of a Riemann surface $S$ minus finitely many points and a nonzero holomorphic 1-form $\omega$ on $S$ such that

1. The missing points of $S$ are poles of $\omega$.
2. For every (non-closed) path $\gamma$ starting and ending in critical points of $\omega$, one has $R\gamma \omega = \omega$.
3. For every closed path $\gamma$, we have $R\gamma \omega = \omega(\gamma)$.

Also, a universal cover $eS$ of $S$ and a function $f : eS \to \Omega$ such that $df = \omega$ and $f(p) = p$ for one (hence all) critical points $p$. 
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Daan Krammer – p.23/25
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$K$ looks a lot like Teichmüller space $T$ (:=orbifold universal cover of moduli space=space of homotopy classes of complex structures on $S$); $K$ and $T$ are both simply connected spaces on which $I(S)$ act.
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Definition/Theorem. The remaining definitions and theorems are as in the special case of $S = \mathbb{C}$. □
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- The ordering on $K$ is generated by moving critical values around the unit circle in positive direction.
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• The ordering on $K$ is generated by moving critical values around the unit circle in positive direction.

• We can replace $\mathbb{R}$ by any totally ordered abelian group. In our case, $2\pi\mathbb{Z}$ suffices.