A Garside type structure on the Torelli group

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Garside structures on B_n

- 1969, Garside. Word problem for B_n.
- Late 1990's, Dehornoy. Defines ***Garside groups***.
- 1998, Birman, Ko, Lee (BKL). New Garside structure on B_n (to be generalised today).
- Recently, weak versions of “Garside groups” were considered. We shall weaken much further.

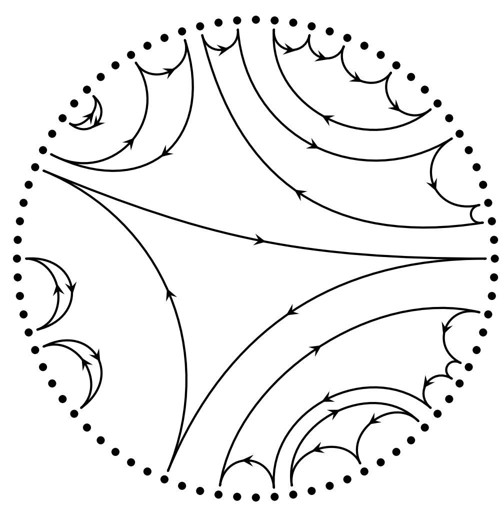
Garside structures on B_n

Plan of the talk:

- BKL Garside structure.
- Simple example.
- Crash course on Torelli groups.
- General set-up.
- Why does it generalise BKL?

Birman, Ko, Lee's structure on B_n

Definition. A ***(BKL) simple braid*** is a braid like the following example:



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Definition.

1. $\Omega := \{\text{simple braids}\}$.
2. $B_n^+ :=$ submonoid $\langle \Omega \rangle$ of B_n.
3. $\leq$ ordering on B_n: $x \leq xy \Leftrightarrow y \in B_n^+$.

Definition. A ***lattice*** is an ordered set such that for all $x, y \in L$ there is a least common upper bound or ***join*** $x \vee y$ and a greatest common lower bound or ***meet*** $x \wedge y$.

Theorem. *$(B_n, \leq)$ is a lattice.*

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Theorem. *Every $x \in B_n$ can uniquely be written in the **symmetric normal form** $x_k^{-1} \cdots x_1^{-1} y_1 \cdots y_\ell$ defined by the following properties:*

1. *x_i and y_j are nontrivial simple braids.*
2. *The greatest simple braid $\leq x_i x_{i+1}$ (which always exists) equals x_i. Same for y_i, y_{i+1}.*
3. *$x_1 \wedge y_1 = 1$.* □

Can we understand the symmetric normal form ***globally***? Yes!

- B_n is automatic (Thurston, 80's).
- Better than automaticity: Grid property to be explained next.

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Definition. Let $a, b \in B_n$. The ***distinguished path*** from a to b is $\{a_0, \ldots, a_r\} \subset B_n$ defined by:

1. $a_0 = a$, $a_r = b$.
2. There exists a symmetric normal form $(x_1, \ldots, x_r)$ such that $a_{i-1} x_i = a_i$ for all i.

Definition. A subset of B_n is ***convex*** if it contains an entire distinguished path as soon as it contains its endpoints.

Definition. The ***Cayley graph*** is the graph with vertex set B_n and edges $\{a, ax\}$ whenever $a \in B_n$ and x is a simple braid.

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Theorem (Grid property, K–Dehornoy). *The convex hull of three points in B_n is a planar graph. (This can and should be made more precise).*

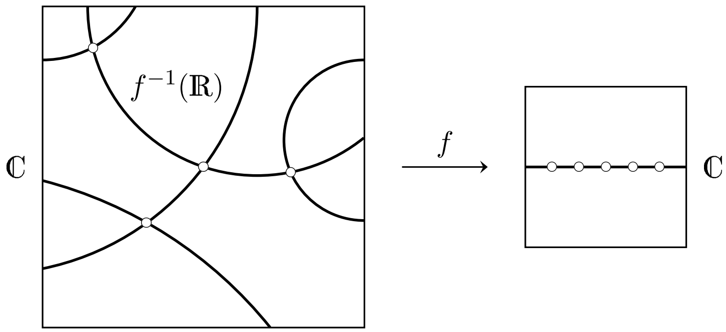
- The grid property is analogous to convex sets in $\mathbb{R}^n$.
- The grid property is the best advertisement for Garside groups.

We say that BKL discovered a ***Garside structure*** on B_n. All of the above properties are true for all Garside groups.

A simple example

Definition. Let $f\colon \mathbb{C} \to \mathbb{C}$ be a polynomial. If $f'(\alpha) = 0$ then α is called a ***critical point*** and $f(\alpha)$ a ***critical value***.

Definition. We define K to be the set of polynomials $f\colon \mathbb{C} \to \mathbb{C}$ of degree n such that all critical values are real, modulo $f \sim g$ if $f(x) = g(ax + b)$.

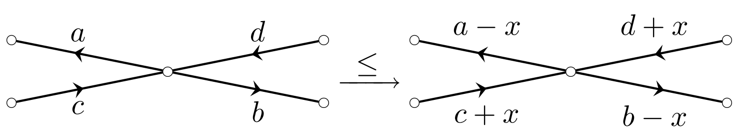


A simple example

Definition. The ***ordering*** $\leq$ on K is generated by moving one critical value to the right by some amount (but not further than its right neighbour).

Question. What does this look like upstairs?

Answer.



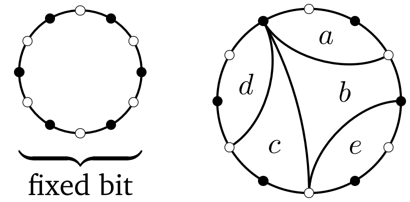
From now on, we will mostly ***work upstairs***.

We will soon see examples of critical points bumping into each other or splitting.

A simple example

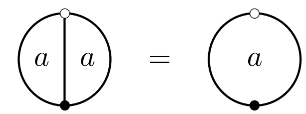
From now on, we mostly ***work dually***. Critical points become 2-cells (called ***regions***), and so on.

Example of an element of K:



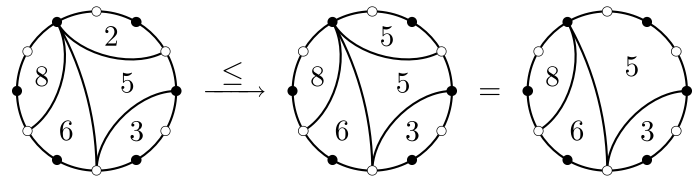
$$\begin{cases} a, \dots, e \in \mathbb{R} \\ a \leq b \leq c \leq d \\ e \leq b \end{cases}$$

Always modulo removing arcs between regions of equal heights:



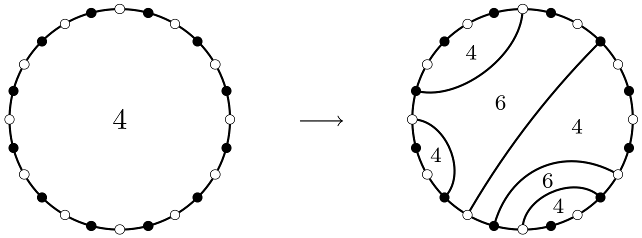
A simple example

Example of a generator of the ordering:

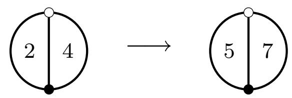


Definition. A ***semi-simple pair*** is a pair $(f, g) \in K \times K$ where g is obtained from f by moving some critical values to the right by the same amount, keeping the others fixed.

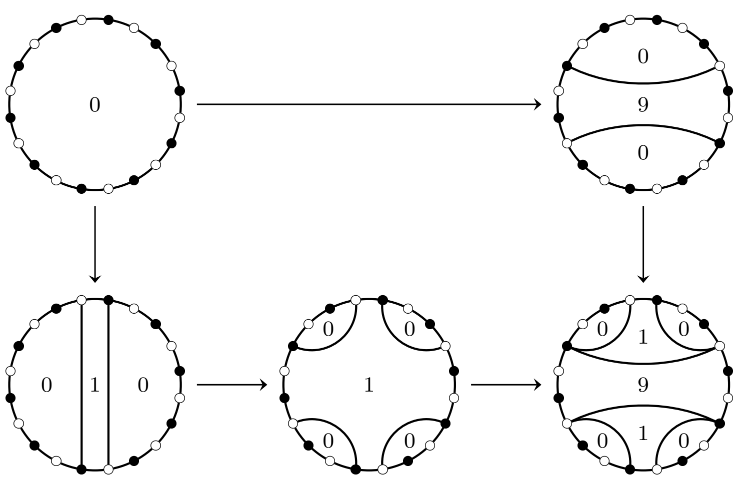
Example. Example of a semi-simple pair $(x, y) = (x \to y)$. We concentrate on a single region for x.



Example. Another example of a semi-simple pair $(x, y) = (x \to y)$, which is allowed even though $5 > 4$:



Example. Example of a complement (relation):



Warning. It is ***not*** true that if $x \leq y \leq z$ and (x, z) is semi-simple, then (x, y) or (y, z) is semi-simple.

Explanation. Between any two distinct real numbers there is another real number. The semi-simple pairs behave like a preferred small set of positive normal forms.

Definition. Let $(x, y) \in K \times K$ be a semi-simple pair. Its ***length*** is the number of moving critical values (with multiplicities) times the common amount they move by.

Definition. Let $x \in K$ and $a \in \mathbb{R}$. Then $(x, x\Delta^a)$ is the semi-simple pair where all critical values move to the right by a.

Theorem. *1.* $(K, \leq)$ *is a lattice.*

2. There is a largest metric d *on* K *extending the length* $d(x, y)$ *already defined for semi-simple pairs.*

3. Let $x, y \in K$*,* $x < y$*,* $d(x, y) = a$*. Consider the path from* x *to* y

$$\begin{cases} [0, a] \longrightarrow K \\ \quad t \longmapsto x\Delta^t \wedge y. \end{cases}$$

There is a finite sequence $x = x_0, \ldots, x_k = y$ *such that* (x_i, x_{i+1}) *is semi-simple and the path passes through all* x_i*.*

4. These paths satisfy a grid property. □

Suggestion. Study ***Garside spaces*** instead of say CAT(0) spaces.

Definition. The ***mapping class group*** $\mathrm{MCG}(S)$ of a surface S is H/H_0 where H is the topological group of (orientation preserving) self-homeomorphisms of S, preserving the boundary pointwise, and $H_0 \subset H$ is the connected component of 1 in H.

Fact 1. *A compact connected oriented surface is determined by its genus and the number of boundary components.* □

Fact 2. *The mapping class group of the disk with* $n + 1$ *holes is essentially* B_n*.* □

Fact 3. *If* S *is compact then* $\mathrm{MCG}(S)$ *is finitely presented.* □

Fact 4. *Let* $g \geq 0$*. The set of isomorphism classes of genus* g *Riemann surfaces can be given the structure of an algebraic orbifold (moduli space); its fundamental group is* $\mathrm{MCG}(S)$*.* □

Definition. The ***Torelli group*** $I(S)$ is the group of mapping classes which act trivially on $H_1(S)$.

So there is an exact sequence

$$1 \longrightarrow I(S) \longrightarrow \mathrm{MCG}(S) \longrightarrow \mathrm{Aut}\, H_1(S).$$

The quotient $\mathrm{MCG}(S)/I(S)$ is an arithmetic group and infinite in general.

The general set-up 21

- S: a compact connected oriented surface with non-empty boundary.
- Q_0, Q_1: finitely many boundary points. They meet every boundary component, and do so alternatingly. (White, respectively, black dots).
- $\phi\colon \pi_1 S \to \mathbb{R}$ a homomorphism. If $x \in \pi_1 S$ is representable by a simple closed curve and is not in the commutator subgroup of $\pi_1 S$ then $\phi(x) \neq 0$.

Example 1. $\ker \phi = (\pi_1 S)'$.

Example 2. $S = \mathbb{C} \smallsetminus \{\alpha_1, \ldots, \alpha_n\}$,

$$\phi(\gamma) = \frac{1}{2\pi i} \oint_\gamma \sum_i \mathrm{dlog}(x - \alpha_i).$$

So $\mathrm{im}\, \phi = \mathbb{Z}$!.

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Definition. An element of K is a cell decomposition of S with $Q_0 \cup Q_1$ for vertices (as before); the height function is now a function

$$h\colon \left\{\text{regions of } \widetilde{S}\right\} \longrightarrow \mathbb{R}$$

such that, for all regions R and all $\gamma \in \pi_1 S$:

$$h(\gamma R) = h(R) + \phi(\gamma). \qquad \square$$

Equivalently:

The general set-up 23

Equivalent definition. An element of K is an isomorphism class of a Riemann surface S minus finitely many points and a nonzero holomorphic 1-form ω on S such that

1. The missing points of S are poles of ω.
2. For every (non-closed) path γ starting and ending in critical points of ω, one has $\int_\gamma \omega \in \mathbb{R}$.
3. For every closed path γ we have $\int_\gamma \omega = \phi(\gamma)$.

Also, a universal cover $\widetilde{S}$ of S and a function $f\colon \widetilde{S} \to \mathbb{C}$ such that $df = \omega$ and $f(p) \in \mathbb{R}$ for one (hence all) critical points p. $\square$

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Observation. The Torelli group $I(S)$ acts on K.

K looks a lot like Teichmüller space T (:=orbifold universal cover of moduli space=space of homotopy classes of complex structures on S); K and T are both simply connected spaces on which $I(S)$ act.

Definition/Theorem. The remaining definitions and theorems are as in the special case of $S = \mathbb{C}$. $\square$

Why did we generalise BKL?

- It may not be clear yet why our construction generalises BKL.
- Consider the space of degree n polynomials $f \in \mathbb{C}[x]$ such that critical values have modulus 1. Its fundamental group is B_n. Write the universal cover K.
- The ordering on K is generated by moving critical values around the unit circle in positive direction.
- We can replace $\mathbb{R}$ by any totally ordered abelian group. In our case, $2\pi\mathbb{Z}$ suffices.