Garside theory

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Abstract

These are very preliminary notes on Garside theory. One of their aims is possible use in a common project with Patrick Dehornoy, François Digne and Jean Michel.

Contents

Still to do: use the results of section 1 in the proofs of section 2. Use the same language in both sections.

1 Positive greedy forms

1.1 Axioms

Let C be a small category. We write C_0 for the set of objects of C and $C_1 = C(-,-)$ for the set of morphisms of C. If $x, y \in C_0$ then we write $C(x,-)$ for the set of morphisms in C with source object x and $C(-,x)$ for the set of morphisms in C with target object x. We put $C(x,y) = C(x,-) \cap C(-,y)$. The multiplication is written $C(x,y) \times C(y,z) \to C(x,z)$, $(f,g) \to fg$ (rather than gf). A product fg of two morphisms f, g is said to be *defined* if the target object of f is the source object of g.

In this section, we fix a small category C and a set of *simple* morphisms $\Sigma \subset C_1$.

Let Σ_∞ be the set of sequences of simple morphisms $(a_1, a_2, \ldots) \in \Sigma^\infty$ such that $a_n a_{n+1}$ is defined for all n, and only finitely many a_n are nontrivial. It follows that $a_1 a_2 \cdots$ is well-defined.

We define relations $\overset{i}{\circ\!\!\longrightarrow}$ $(i > 0)$ on Σ_∞ by:

$$(a_1, a_2, \ldots) \overset{i}{\circ\!\!\longrightarrow} (b_1, b_2, \ldots) \iff \left(\begin{array}{l} a_j = b_j \text{ whenever } j \notin \{i, i+1\}, \text{ and there} \\ \text{exists } c \in \Sigma \text{ such that } a_{i+1} = cb_{i+1},\ b_i = a_i c \end{array} \right). \tag{1}$$

Note that these imply that $a_1 a_2 \cdots = b_1 b_2 \cdots$.

We write $a \circ\!\!\longrightarrow b$ if and only if $a \overset{i}{\circ\!\!\longrightarrow} b$ for some i. Let $\overset{*}{\circ\!\!\longrightarrow}$ be the transitive closure of $\circ\!\!\longrightarrow$ and $\sim$ the equivalence relation generated by $\circ\!\!\longrightarrow$.

Let $\approx$ be the relation on Σ_∞ defined by $(a_1, a_2, \cdots) \approx (b_1, b_2, \cdots)$ if and only if there are isomorphisms h_i such that $b_i = h_{i-1}\, a_i\, h_i$ for all i, and h_0 is trivial. For example, if the right hand side of (1) holds and c is an isomorphism, then $a \approx b$.

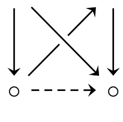
An element $a \in \Sigma_\infty$ is called *greedy* if for all b we have $a \circ\!\!\longrightarrow b \Rightarrow a \approx b$.

We have $a \sim b \Rightarrow a \approx b$. We call $\approx$ the *strong equivalence* and $\sim$ the *equivalence.*

If $a = (a_1, a_2, \ldots) \in \Sigma_\infty$ and a_i is trivial for all $i > n$, then we also write $a = [a_1, \ldots, a_n]$. Let Σ_n denote the set of all elements of Σ_∞ of the form $[a_1, \ldots, a_n]$. We say that $(a_1, \ldots, a_n)$ is greedy if $[a_1, \ldots, a_n]$ is.

In this section, we assume that the following are satisfied, except that at most one among (9), (10) may not be satisfied.

- Every isomorphism of C is in Σ. (2)
- Let $a, b \in \Sigma$ be such that ab is defined. If at least one of a, b is an isomorphism then $ab \in \Sigma$. (3)
- $\approx\ =\ \overset{*}{\circ\!\!\longrightarrow} \cap \overset{*}{\longleftarrow\!\!\circ}$. (4)
- The category C is presented by the generating set Σ and the relations $ab = c$ whenever true. [1] (5)
- The category C is connected, that is, $C(x, y) \neq \varnothing$ for all $x, y \in C_0$. (6)
- Let $\sim_2$ be the equivalence relation on Σ_2 generated by the pairs $(a, b) \in \Sigma_2 \times \Sigma_2$ such that $a \circ\!\!\longrightarrow b$. Let X be an $\sim_2$-class. Then X contains an element y such that $x \circ\!\!\longrightarrow y$ for all $x \in X$. (7)
- If the solid arrows in (8)

  are in Σ (and form a commutative diagram) then a dashed arrow as shown exists, which is again in Σ (and again makes the diagram commute).

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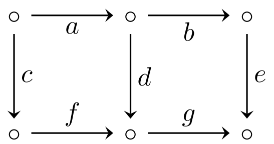
[1] More precisely, let C' be the category with the same object set as C, presented as follows. Its generating set is $\{r(a) \mid a \in \Sigma\}$ (a copy of Σ); here $r(a)$ has the same source and target objects that a has. The relations are $(ra)(rb) = rc$ whenever $ab = c$ and $r(a)$ is trivial if a is. Then the natural functor $C' \to C$ is an isomorphism.

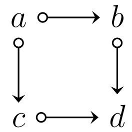
- Consider a commutative diagram (9)



  in which all arrows are simple morphisms of C. Then, if (a,b), (d,g) and (c,f) are greedy then so is (f,g).
- Consider again the commutative diagram of (9). If (f,g), (a,d) and (b,e) are greedy then so is (a,b). (10)

1.2 Uniqueness of greedy paths

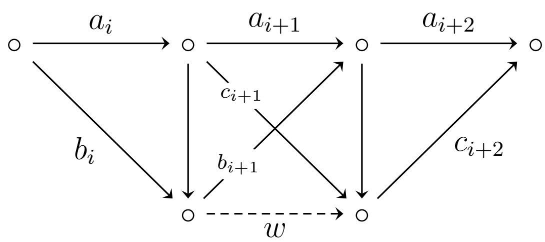
Lemma 11. *Let $a=(a_1,a_2,\ldots)$, $b=(b_1,b_2,\ldots)$, $c=(c_1,c_2,\ldots)$ be elements of Σ_∞ and assume $a \circ\!\!\longrightarrow b$, $a \circ\!\!\longrightarrow c$. Then there exists $d \in \Sigma_n$ such that $b \circ\!\!\longrightarrow d$ and $c \circ\!\!\longrightarrow d$.*



Proof. Suppose $a \overset{i}{\circ\!\!\longrightarrow} b$ and $a \overset{j}{\circ\!\!\longrightarrow} c$. If $|i-j|>1$ then it is clear.

Suppose next $i=j$. Let X be the $\sim$-class of (a_i,a_{i+1}). By (7), there exists $y=(d_i,d_{i+1}) \in X$ such that $x \circ\!\!\longrightarrow y$ for all $x \in X$. Put $d=(d_1,d_2,\ldots)$ where $d_j=a_j$ if $j \notin \{i,i+1\}$ and d_i, d_{i+1} are the entries of y. Then $b \circ\!\!\longrightarrow d$ and $c \circ\!\!\longrightarrow d$ as required.

Suppose finally that $j=i+1$. The assumptions $a \overset{i}{\circ\!\!\longrightarrow} b$ and $a \overset{j}{\circ\!\!\longrightarrow} c$ imply that at least the solid simple arrows of the commutative diagram



(12)

exist. Using (8) on the middle five solid arrows in (12) shows that there exists a dashed arrow w making (12) commutative. Now define $d=(d_1,d_2,\ldots)$ by $d_j=a_j$ if $j \notin \{i,i+1,i+2\}$ and $(d_i,d_{i+1},d_{i+2})=(b_1,w,c_3)$. Using (8) we find $b \circ\!\!\longrightarrow d$ and $c \circ\!\!\longrightarrow d$ as required. □

Lemma 13. (a). *Every equivalence class $X \subset \Sigma_\infty$ has finite upper bounds, that is, for all $u,v \in X$ there exists $w \in X$ with $u \overset{*}{\circ\!\!\longrightarrow} w$ and $v \overset{*}{\circ\!\!\longrightarrow} w$.*

(b). *Every greedy path is an upper bound of all equivalent elements. That is, if $a \sim b$ and b is greedy then $a \overset{*}{\circ\!\!\longrightarrow} b$.*

Proof. Part (b) follows immediately from (a). Proof of (a). Let $u,v \in \Sigma_\infty$ be equivalent, that is, there exist

$$u=u_0,\ u_1,\ \ldots,\ u_n=v, \qquad u_i \in \Sigma_\infty$$

such that $u_i \circ\!\!\longrightarrow u_{i+1}$ or $u_{i+1} \circ\!\!\longrightarrow u_i$ for all i. By induction on n, we shall prove that $\{u, v\}$ has an upper bound.

For $n = 0$ there is nothing to prove. Assume that it is true for $n-1$. Then $\{u_0, u_1\}$ has an upper bound w. If $u_n \circ\!\!\longrightarrow u_{n+1}$ then w is an upper bound of u and v, so suppose $u_{n+1} \circ\!\!\longrightarrow u_n$.

Since $u_{n-1} \overset{*}{\circ\!\!\longrightarrow} w$ there exists a diagram as follows.

$$\begin{array}{ccccccc} u_{n-1} =: x_0 & \circ\!\!\longrightarrow & x_1 & \circ\!\!\longrightarrow \cdots \circ\!\!\longrightarrow & x_m := w \\ \circ\!\!\big\downarrow & & & & \\ v = u_n =: y_0 & & & & \end{array} \tag{14}$$

Using lemma 11 recursively we can extend (14) to a diagram as follows.

$$\begin{array}{ccccc} u_{n-1} =: x_0 & \circ\!\!\longrightarrow & x_1 & \circ\!\!\longrightarrow \cdots \circ\!\!\longrightarrow & x_m := w \\ \circ\!\!\big\downarrow & & \circ\!\!\big\downarrow & & \circ\!\!\big\downarrow \\ v = u_n =: y_0 & \circ\!\!\longrightarrow & y_1 & \circ\!\!\longrightarrow \cdots \circ\!\!\longrightarrow & y_m =: z \end{array}$$

So $v \overset{*}{\circ\!\!\longrightarrow} z$ and also $u = u_0 \overset{*}{\circ\!\!\longrightarrow} w \overset{*}{\circ\!\!\longrightarrow} z$. □

1.3 Existence of greedy paths

Lemma 15. *Every morphism in C has a greedy form.*

Proof. Recall that at least one of (9), (10) holds. We give the proof in case (9) is true, the other case being similar and left to the reader.

Let f be a morphism in C. By (5) there are simple morphisms $a_1, \ldots, a_n$ such that $f = a_1 \cdots a_n$.

Consider the element $a = [a_1, \ldots, a_n] \in \Sigma_\infty$. We need to prove that the equivalence class of a has a greatest element b, that is, $b \approx c$ whenever $b \circ\!\!\longrightarrow c$. We do this by induction on n.

For $n = 0$ this is clear. Suppose that $n > 0$ and it is true for $n-1$. By the induction hypothesis, a greedy form for $[a_1, \ldots, a_{n-1}]$ exists. By lemma 13(b) (uniqueness of greedy forms) it is an upper bound of $[a_1, \ldots, a_{n-1}]$ and hence is again in Σ_{n-1}. We replace $(a_1, \ldots, a_{n-1})$ by one of its greedy forms.

Then the top row in

$$\begin{array}{c} \circ \xrightarrow{a_1} \circ \xrightarrow{a_2} \circ \xrightarrow{a_3} \cdots \xrightarrow{a_{n-3}} \circ \xrightarrow{a_{n-2}} \circ \xrightarrow{a_{n-1}} \circ \\ \hfill \big\downarrow a_n \\ \hfill \circ \end{array}$$

is greedy. Working our way from right to left, we extend this to a diagram

$$\begin{array}{ccccccccccccc} \circ & \xrightarrow{a_1} & \circ & \xrightarrow{a_2} & \circ & \xrightarrow{a_3} \cdots \xrightarrow{a_{n-3}} & \circ & \xrightarrow{a_{n-2}} & \circ & \xrightarrow{a_{n-1}} & \circ \\ c_1 = d_1 \big\downarrow & & d_2 \big\downarrow & & d_3 \big\downarrow & & \big\downarrow d_{n-2} & & \big\downarrow d_{n-1} & & \big\downarrow d_n = a_n \\ \circ & \xrightarrow[c_2]{} & \circ & \xrightarrow[c_3]{} & \circ & \xrightarrow[c_4]{} \cdots \xrightarrow[c_{n-2}]{} & \circ & \xrightarrow[c_{n-1}]{} & \circ & \xrightarrow[c_n]{} & \circ \end{array} \tag{16}$$

by defining (d_{i-1}, c_i) to be a greedy form of (a_{i-1}, d_i) for all i, which we know to exist by (7). By an obvious induction from right to left, using (9), one proves that the bottom row in (16) is greedy. By construction, (c_1, c_2) is greedy and so $(c_1, \ldots, c_n)$ is a greedy form for f. $\square$

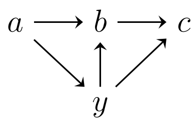
2 Greedy forms

2.1 Axioms

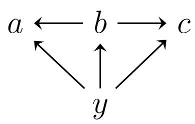
Definition 17. Let (V, E) be graph, that is, V is a set (of vertices) and $E \subset V \times V$ is a set of edges (or a binary relation). We write $x \to y$ for $(x, y) \in E$.

In (18)–(26) we list some properties that may or may not be satisfied.

- We have $(x, x) \in E$ for all $x \in V$. We call (x, x) a *loop*. (18)
- The graph is connected (as unoriented graph). (19)
- The fundamental group of the graph is generated by those conjugacy classes associated with closed (unoriented) paths of at most 3 edges. (20)
- There exists a (necessarily unique) ordering $\leq$ on V generated by E. (21)
- Let $a \to x \to c$. Then there exists $b \in V$ such that $a \to b \to c$ and if $y \in V$ is such that $a \to y \to c$ then $y \to b$. (22)



- Let $a \leftarrow x \to c$. Then there exists $b \in V$ such that $a \leftarrow b \to c$ and if $y \in V$ is such that $a \leftarrow y \to c$ then $y \to b$. (23)



We call (a, b, c) a *greedy path of type 1* (of length 2) if the conclusion of 22 holds. We also call (c, b, a) a *greedy path of type 2.* We call (a, b, c) (hence (c, b, a)) a *greedy path of type 3* if the conclusion of 23 holds.

It may happen that something is a greedy path of one type, not another, even though the arrows (are allowed to) point in the required direction. Here is the simplest example. Let $a \to b$ but $a \neq b$. Then $a \leftarrow a \to b$ and the triple (a, a, b) is greedy of type 3. At the same time we have $a \to a \to b$ and (a, a, b) is not greedy of type 1 (because that would be (a, b, b)). The conclusion is that a statement of the form 'this path is greedy' is only well-defined if the directions of the arrows have been specified.

- Let $a \to x \leftarrow c$. Then there exists y with $a \leftarrow y \to c$. (24)

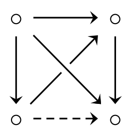
∘ In each of the 4 diagrams (25)

$$\begin{array}{ccccc} a & \longleftarrow & b & \longleftarrow & c \\ \downarrow & & \downarrow & & \downarrow \\ d & \longleftarrow & x & \longleftarrow & y \end{array} \quad \begin{array}{ccccc} a & \longleftarrow & b & \longrightarrow & c \\ \downarrow & & \downarrow & & \uparrow \\ d & \longleftarrow & x & \longleftarrow & y \end{array} \quad \begin{array}{ccccc} a & \longrightarrow & b & \longrightarrow & c \\ \downarrow & & \uparrow & & \uparrow \\ d & \longleftarrow & x & \longrightarrow & y \end{array} \quad \begin{array}{ccccc} a & \longrightarrow & b & \longrightarrow & c \\ \uparrow & & \uparrow & & \uparrow \\ d & \longrightarrow & x & \longrightarrow & y \end{array}$$

(25a) (25b) (25c) (25d)

the following holds. If abc, dxb, xyc are greedy (relative the arrows of the diagram) then so is dxy.

∘ If the solid arrows in the diagram (26)



exist, then so does the dashed arrow.

Remark 27. Axioms 25d is a consequence of the others among (18)–(26).

Definition 28. For $k, \ell \geq 0$, we write $X(k,\ell)$ for the set of those $(x_{-k}, x_{-k+1}, \ldots, x_{\ell-1}, x_\ell) \in V^{k+\ell+1}$ such that $x_{i-1} \to x_i$ for all $i \in \{1, \ldots, \ell\}$ and $x_{-i+1} \to x_{-i}$ for all $i \in \{1, \ldots, k\}$.

We define relations $\overset{i}{\circ\!\!\longrightarrow}$ $(-k < i < \ell)$ on $X(k,\ell)$ by:

$$(a_{-k}, \ldots, a_\ell) \overset{i}{\circ\!\!\longrightarrow} (b_{-k}, \ldots, b_\ell) \iff a_i \to b_i \text{ and } a_j = b_j \text{ for all } j \neq i.$$

Note that these imply $a_{-k} = b_{-k}$ and $a_\ell = b_\ell$. We write $a \circ\!\!\longrightarrow b$ if and only if $a \overset{i}{\circ\!\!\longrightarrow} b$ for some i. An element $a \in X(k,\ell)$ is called *greedy* if for all b we have $a \circ\!\!\longrightarrow b \Rightarrow a = b$. As noted above, an element of $X(k,\ell) \cap X(k',\ell')$ can be greedy in one not the other.

Question 29. Can you reduce the set of axioms?

From now on we assume that (V, E) satisfies (18)–(26).

2.2 Uniqueness of greedy paths

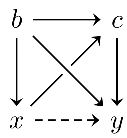
Lemma 30. *Let $u, v, w \in X(k,\ell)$ be such that $u \circ\!\!\longrightarrow v$ and $u \circ\!\!\longrightarrow w$. Then there exists $x \in X(k,\ell)$ such that $v \circ\!\!\longrightarrow x$ and $w \circ\!\!\longrightarrow x$.*

$$\begin{array}{ccc} u & \circ\!\!\longrightarrow & v \\ \circ\!\!\downarrow & & \circ\!\!\downarrow \\ w & \circ\!\!\longrightarrow & x \end{array}$$

Proof. Suppose $u \overset{i}{\circ\!\!\longrightarrow} v$ and $u \overset{j}{\circ\!\!\longrightarrow} w$. If $|i-j| > 1$ then it is clear.

Suppose next $i = j$. We may suppose $(k, i, \ell) \in \{(-1, 0, 1), (0, 1, 2)\}$. First suppose it is $(0, 1, 2)$. Write $u = (a, u_0, c)$. Define b as in (22). Then (22) proves that $x := (a, b, c)$ has the required properties. If $(k, i, \ell) = (-1, 0, 1)$ then the argument is similar and uses (23) instead of (22).

Suppose now $|i-j|=1$, say, $j=i+1$. Then we may suppose $(k,i,j,\ell)\in\{(-1,0,1,2),(0,1,2,3)\}$. First suppose it is $(0,1,2,3)$. Write $u=(a,b,c,d)$, $v=(a,x,c,d)$, $w=(a,b,y,d)$. Then the solid arrows of



exist. By (26), the dashed arrow exists. So $x:=(a,x,y,d)$ has the required properties. If $(k,i,j,\ell)=(-1,0,1,2)$ then the argument is similar. This finishes the proof. □

Definition 31. Let $\overset{*}{\circ\!\!\longrightarrow}$ be the transitive closure of $\circ\!\!\longrightarrow$. Let $\approx$ be the equivalence relation generated by $\circ\!\!\longrightarrow$.

It is clear that $\overset{*}{\circ\!\!\longrightarrow}$ is an ordering. An element $u\in X(k,\ell)$ is greedy if and only if it is maximal in this ordering.

Lemma 32. *(a). Every equivalence class $C\subset X(k,\ell)$ has finite upper bounds, that is, for all $u,v\in C$ there exists $w\in C$ with $u\overset{*}{\circ\!\!\longrightarrow}w$ and $v\overset{*}{\circ\!\!\longrightarrow}w$.*

(b). Every greedy path is an upper bound (with respect to $\overset{*}{\circ\!\!\longrightarrow}$) of all equivalent elements.

Proof. Part (b) follows immediately from (a). Proof of (a). Let $u,v\in X(k,\ell)$ be equivalent, that is, there exist

$$u=u_0,\ u_1,\ \ldots,\ u_n=v,\qquad u_i\in X(k,\ell)$$

such that for i one has $u_i\circ\!\!\longrightarrow u_{i+1}$ or $u_{i+1}\circ\!\!\longrightarrow u_i$. By induction on n, we shall prove that $\{u,v\}$ has an upper bound.

For $n=0$ there is nothing to prove. Assume that it is tue for $n-1$. Then $\{u_0,u_1\}$ has an upper bound w. If $u_n\circ\!\!\longrightarrow u_{n+1}$ then w is an upper bound of u and v, so suppose $u_{n+1}\circ\!\!\longrightarrow u_n$.

Since $u_{n-1}\overset{*}{\circ\!\!\longrightarrow}w$ there exists a diagram as follows.

$$\begin{array}{ccccccc} u_{n-1}=:x_0 & \circ\!\!\longrightarrow & x_1 & \circ\!\!\longrightarrow\cdots\circ\!\!\longrightarrow & x_m:=w \\ \downarrow & & & & \\ v=u_n=:y_0 & & & & \end{array}\tag{33}$$

Using lemma 30 recursively we can extend (33) to a diagram as follows.

$$\begin{array}{ccccccc} u_{n-1}=:x_0 & \circ\!\!\longrightarrow & x_1 & \circ\!\!\longrightarrow\cdots\circ\!\!\longrightarrow & x_m:=w \\ \downarrow & & \downarrow & & \downarrow \\ v=u_n=:y_0 & \circ\!\!\longrightarrow & y_1 & \circ\!\!\longrightarrow\cdots\circ\!\!\longrightarrow & y_m=:z \end{array}$$

So $v\overset{*}{\circ\!\!\longrightarrow}z$ and also $u=u_0\overset{*}{\circ\!\!\longrightarrow}w\overset{*}{\circ\!\!\longrightarrow}z$. □

2.3 Existence of greedy paths

Lemma 34. *Every two vertices can be connected by a greedy path.*

Proof. Let $x, y \in V$. By (24) there exist $k, \ell \in \mathbb{Z}_{\geq 0}$ and $(a_{-k}, \ldots, a_\ell) \in X(k, \ell)$ such that $(x, y) = (a_{-k}, a_\ell)$. By induction on $m := k + \ell$ we prove that it is equivalent to a greedy path. For $m = 0$ this is trivial. Suppose $m > 0$ and it is true for $m - 1$. After interchanging k with ℓ if necessary we may suppose $k > 0$. By the induction hypothesis, there exists a greedy path $(b_{1-k}, \ldots, b_\ell)$ from $a_{1-k} = b_{1-k}$ to $a_\ell = b_\ell$.

$$\begin{array}{ccccccccccccc}
b_{1-k} & \longleftarrow & b_{2-k} & \longleftarrow & \cdots & \longleftarrow & b_0 & \longrightarrow & \cdots & \longrightarrow & b_{\ell-1} & \longrightarrow & b_\ell = y \\
\downarrow & & & & & & & & & & & & \\
x & & & & & & & & & & & &
\end{array}$$

Using (22) and (23) we can uniquely extend this to a diagram

$$\begin{array}{ccccccccccccccc}
b_{1-k} & \longleftarrow & b_{2-k} & \longleftarrow & \cdots & \longleftarrow & b_0 & \longrightarrow & b_1 & \longrightarrow & \cdots & \longrightarrow & b_{\ell-1} & \longrightarrow & b_\ell = y \\
\downarrow & & \downarrow & & & & \downarrow & & \uparrow & & & & \uparrow & & \uparrow \\
x = c_{-k} & \longleftarrow & c_{1-k} & \longleftarrow & \cdots & \longleftarrow & c_{-1} & \longleftarrow & c_0 & \longrightarrow & \cdots & \longrightarrow & c_{\ell-2} & \longrightarrow & c_{\ell-1}
\end{array}$$

such that (b_t, c_{t-1}, c_{t-2}) is greedy (relative the above orientations of the arrows) whenever $2 - k \leq t \leq \ell$. Then (c_t, c_{t+1}, c_{t+2}) is greedy (relative the above orientations of the arrows) whenever $-k \leq t \leq \ell - 2$: if $-k \leq t \leq -3$ this follows from (25a); if $t = -2$ this follows from (25b); if $t = -1$ this follows from (25c); if $0 \leq t \leq \ell - 3$ this follows from (25d); and if $t = \ell - 2$ this follows from the construction.

Then $(c_{-k}, \ldots, c_\ell)$ is a greedy path from x to y. $\square$