A Garside like structure on the framed mapping class group

Daan Krammer

June 22, 2007

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- We want to do something similar for the mapping class group. The role of S_n should be played by something like $\mathrm{Sp}(2g, \mathbb{Z})$.

- The ordering on S_n defined by

$$x \leq xy \iff \ell(xy) = \ell(x) + \ell(y)$$

(ℓ=length with respect to $\{(1,2),(2,3),\ldots,(n-1,n)\}$) is the well-known ***weak Bruhat*** ordering.

Plan

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(5) How good is this Garside structure?

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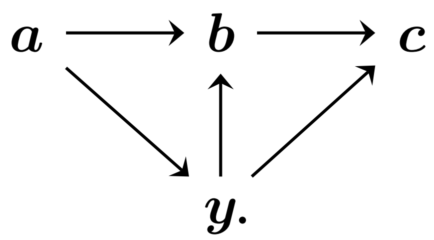

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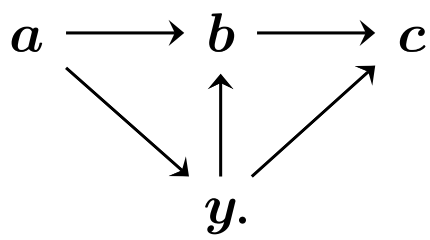

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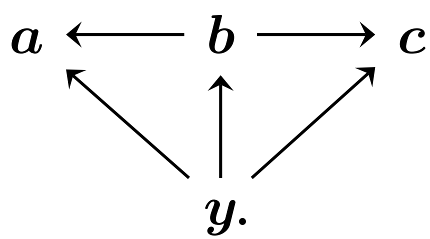

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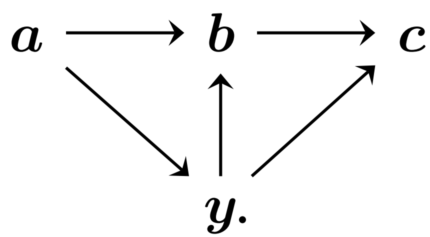

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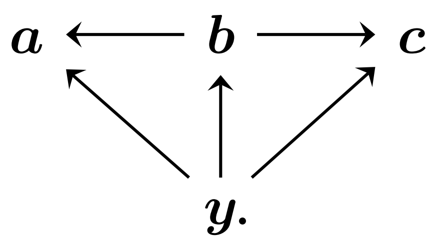

(7) Let $a \longrightarrow x \longleftarrow c$. Then there exists $b \in V$ such that $a \longleftarrow b \longrightarrow c$. □

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Definition. In (5) or (6) we call (a, b, c) and (c, b, a) ***distinguished paths*** (of length 2). We call $(a_0, \ldots, a_n)$ a distinguished path if (a_i, a_{i+1}, a_{i+2}) is for all i.

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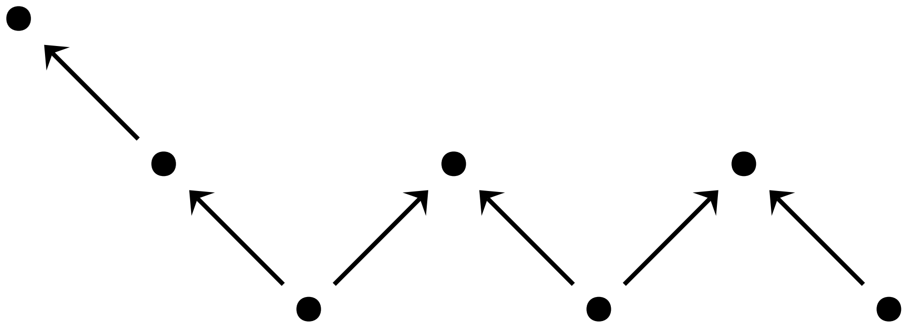
Proposition. *Any two vertices are connected by a distinguished path, which is unique up to repeating the endpoints.*

Example.



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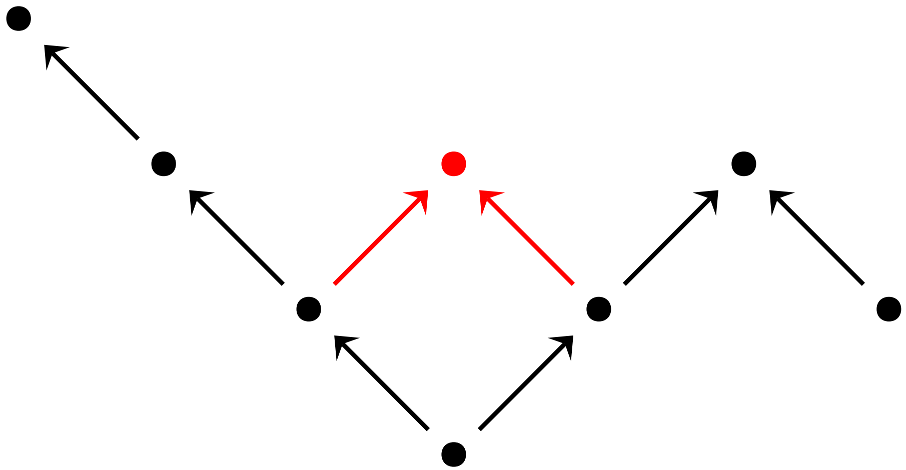

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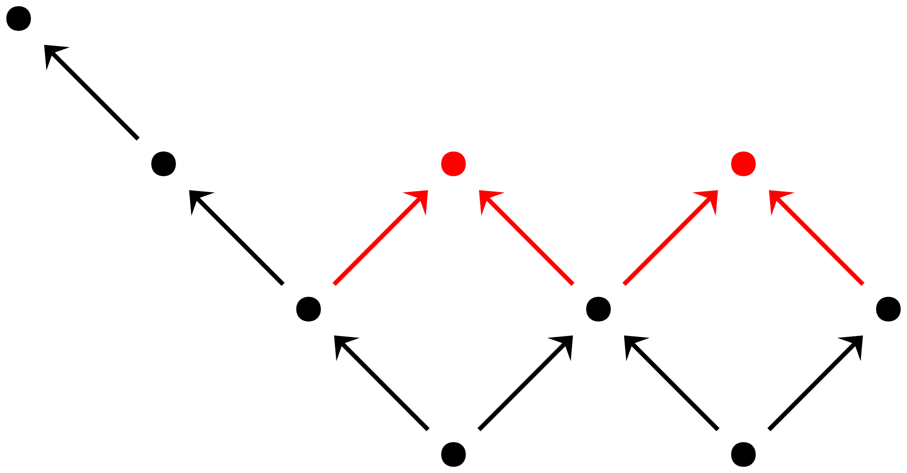

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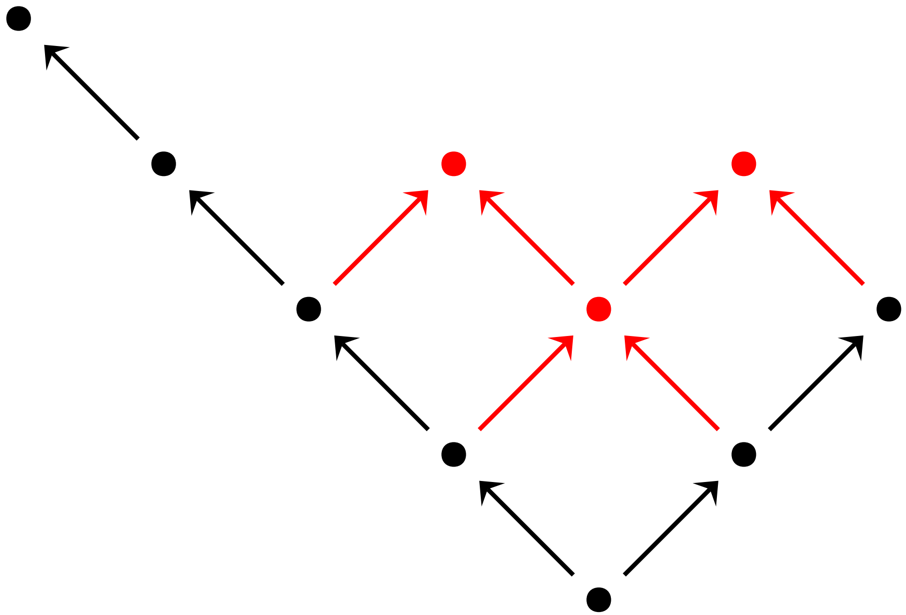

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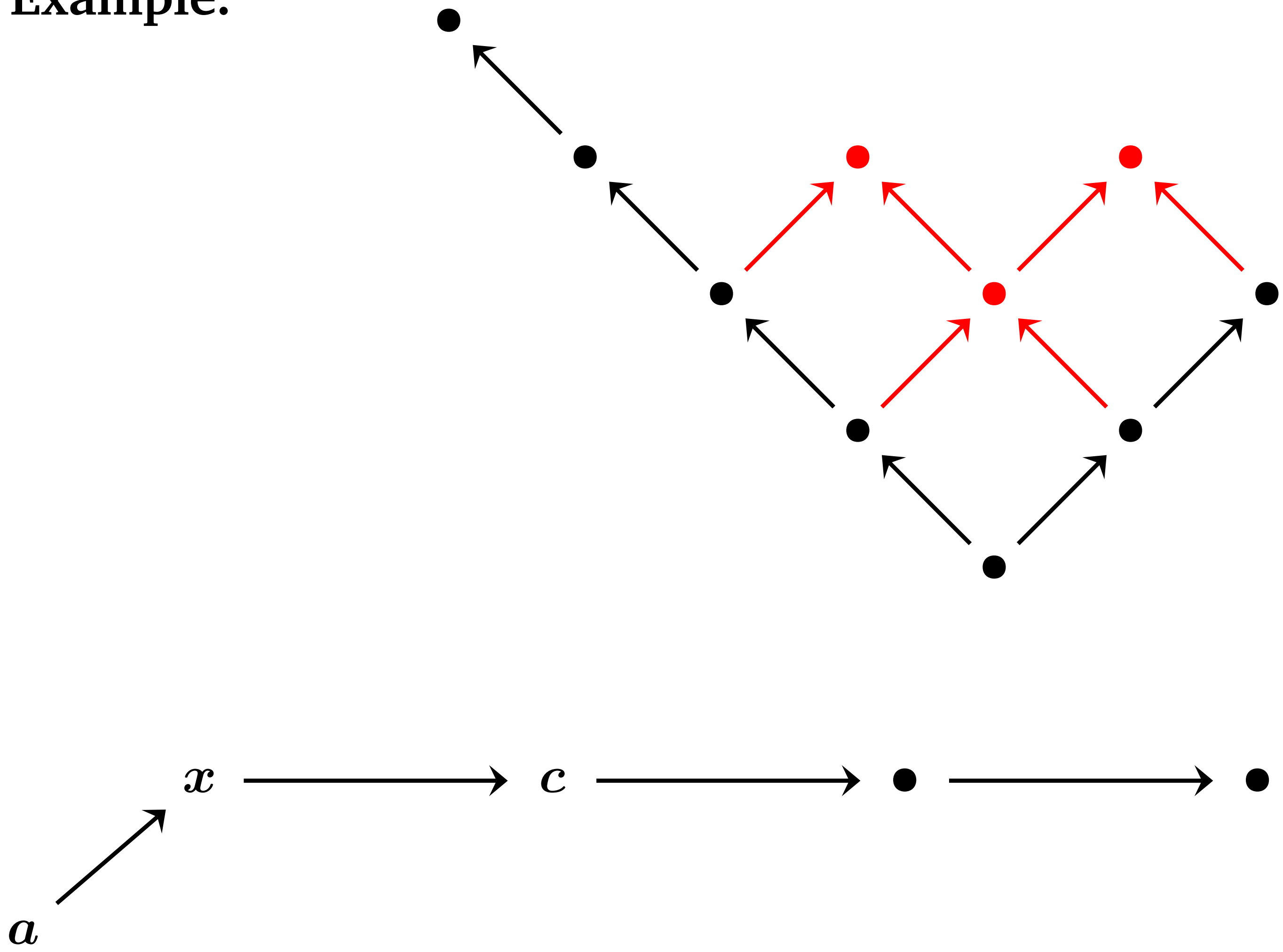

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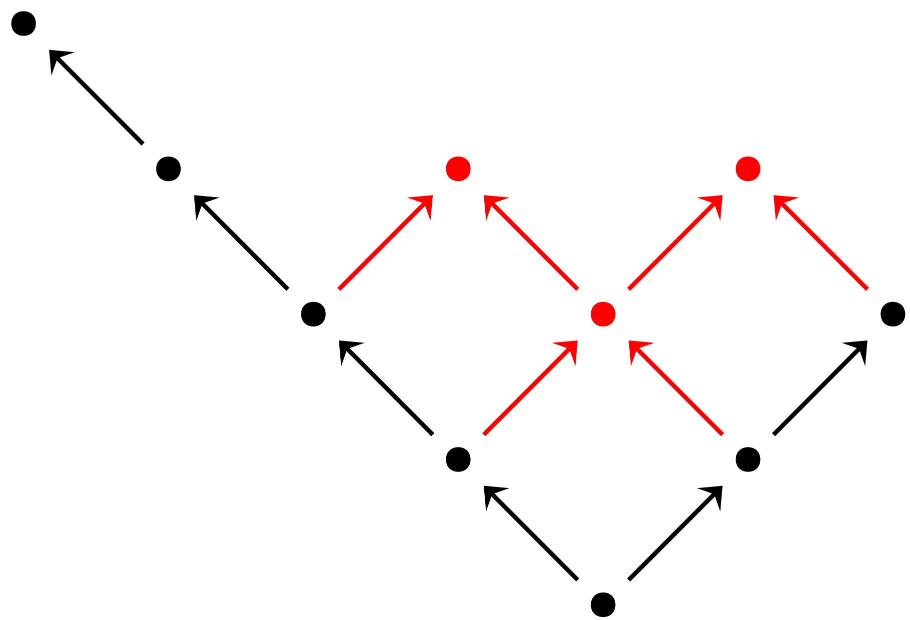

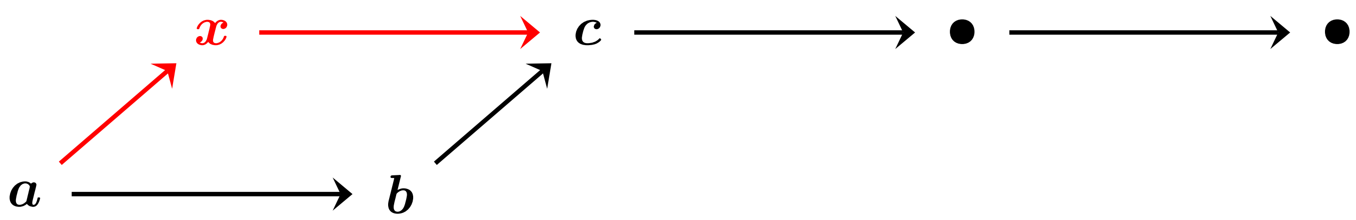

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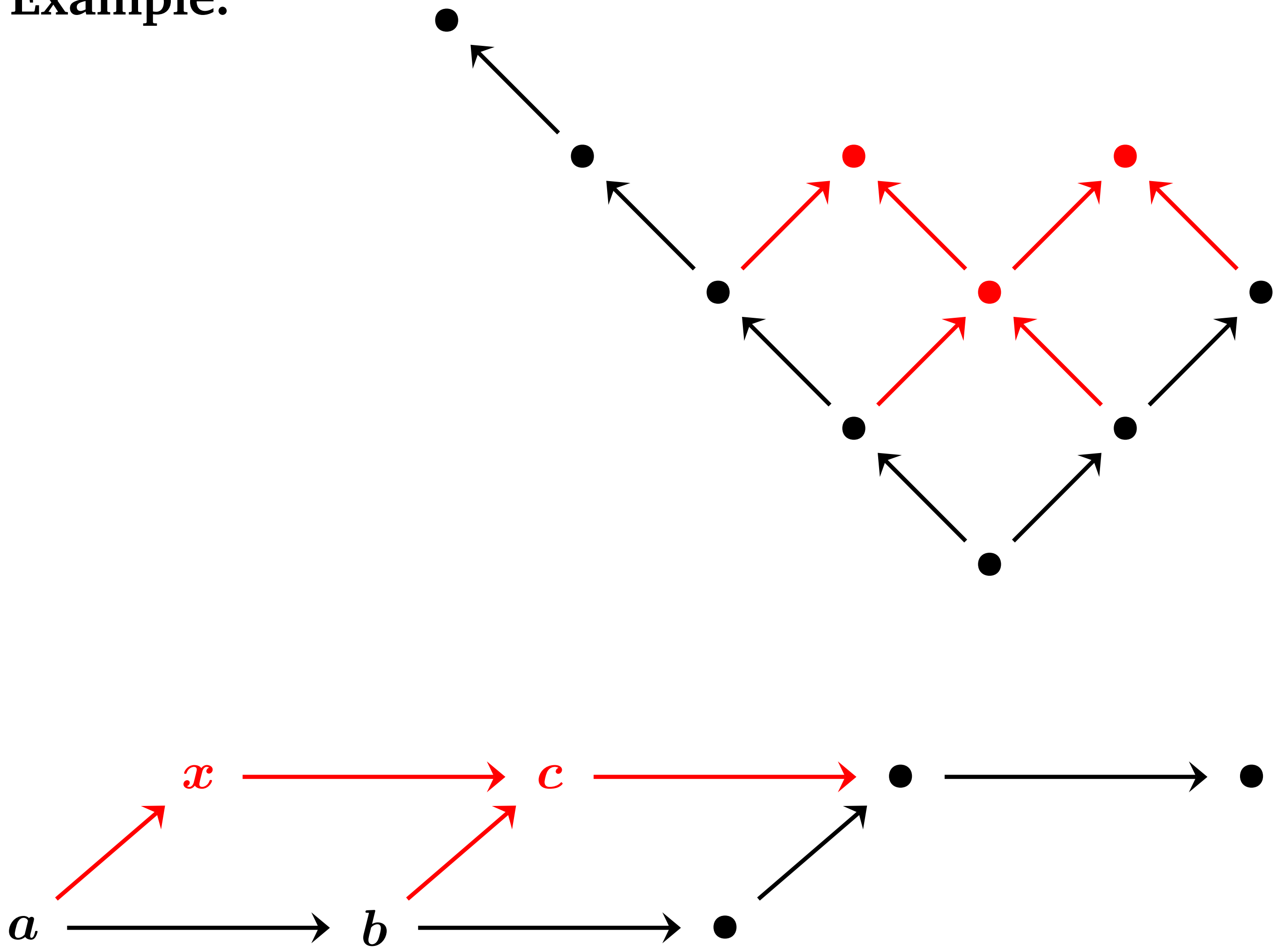

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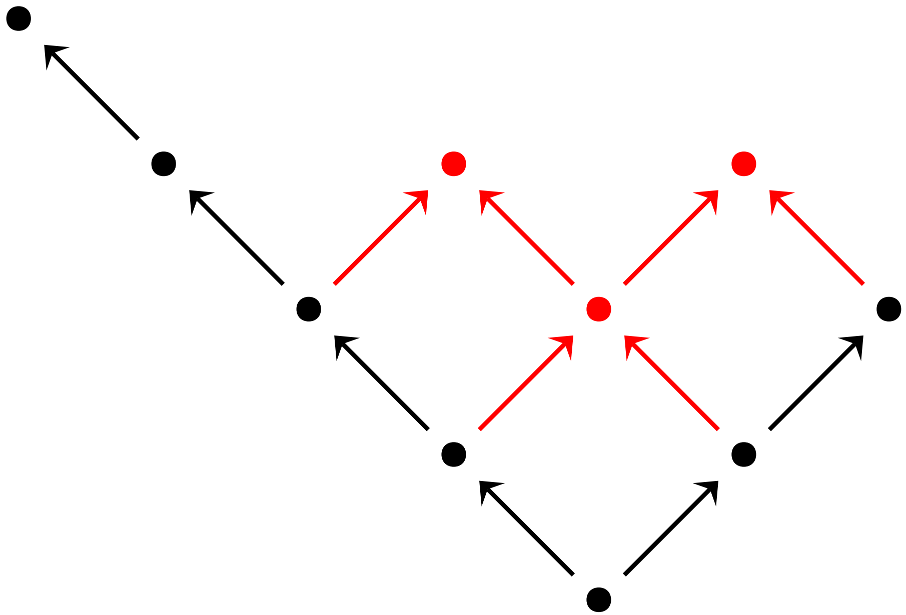

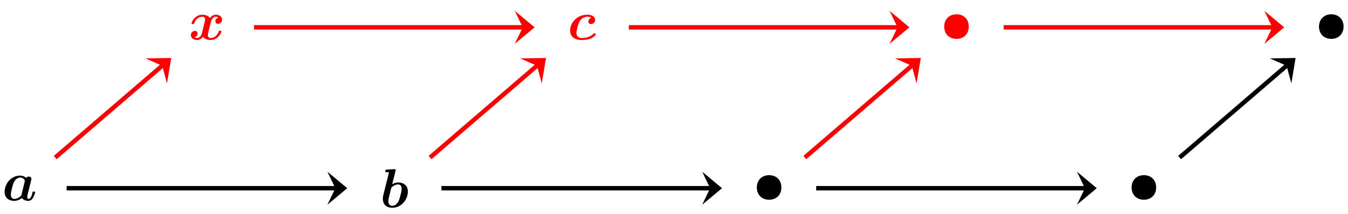

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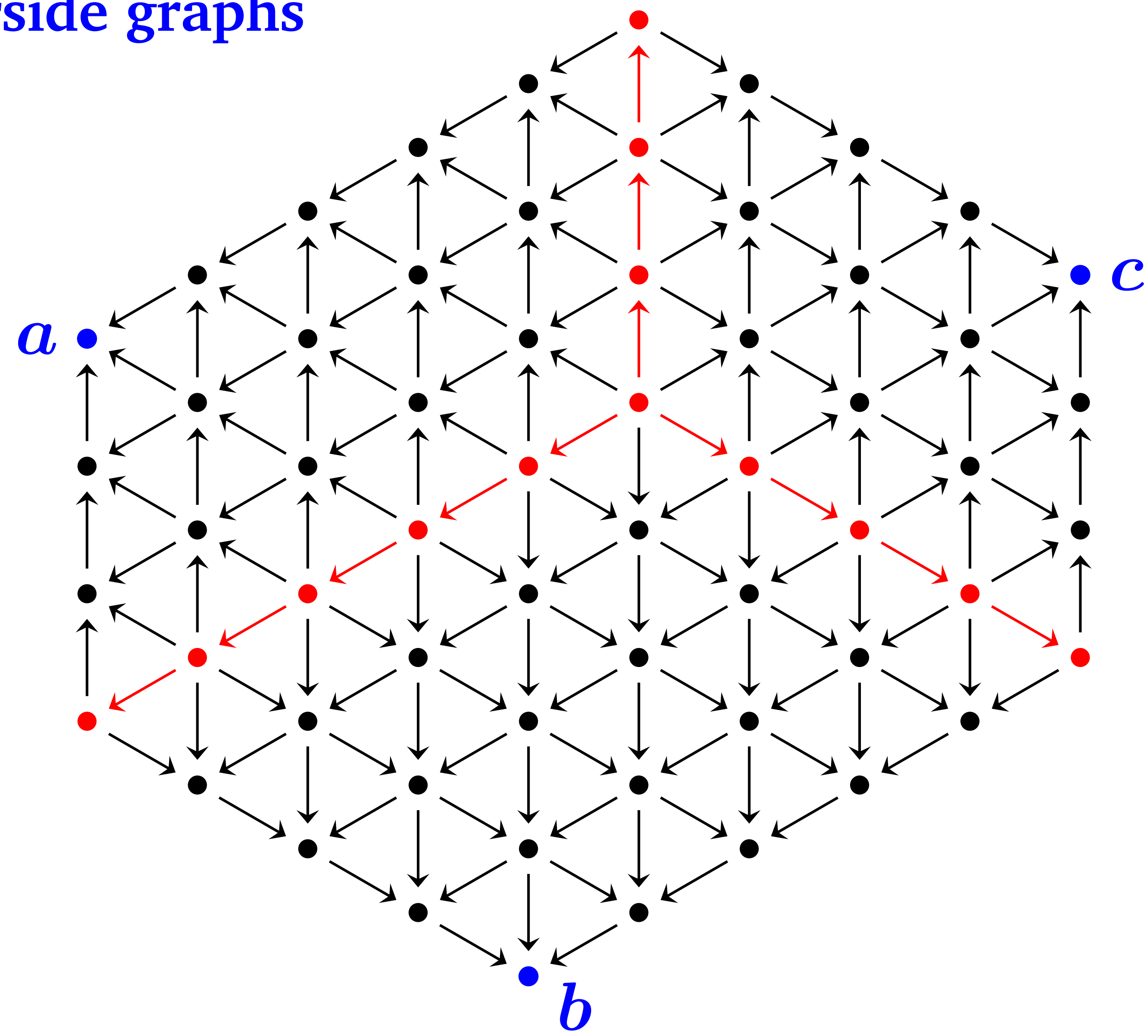
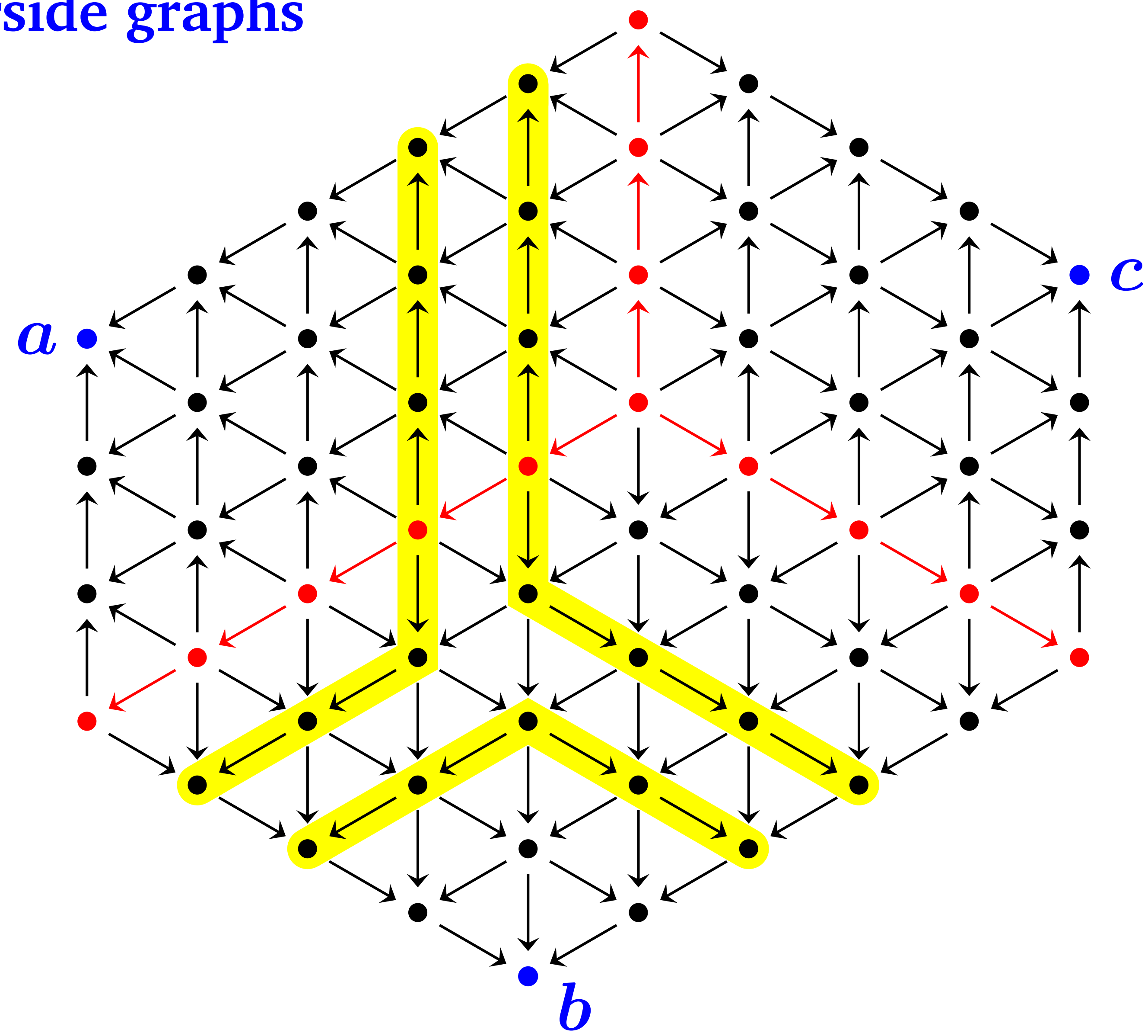
Proposition (Grid property). *The convex hull of any three* $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \boldsymbol{V}$ *looks as follows (up to vertex repetition).*

Garside graphs





Yellow stripes are typical distinguished paths. They go straight ahead except for possibly once making one of the three turns shown.

□

Definition. Let K be an ordered field (for example, $K = \mathbb{R}$) and $n \geq 0$. A ***realisation*** of a Garside graph (V, E) is, for each vertex $v \in V$, a non-empty convex subset $C(v) \subset K^n$ with the following properties.

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• A realisation for B_n exists (covariant under a linear representation).

• A Garside graph is a combinatorial analog to (a convex subset of) a real vector space.

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(3) The set of laminations on a surface S is essentially independent of the hyperbolic metric on S. □

Main construction

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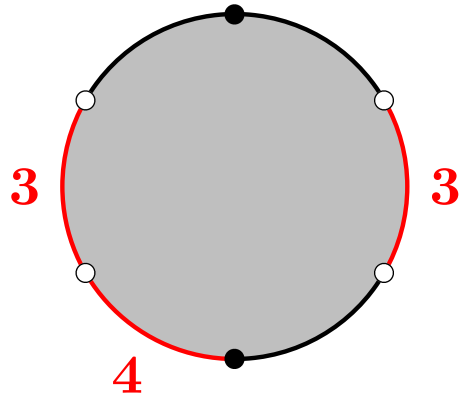

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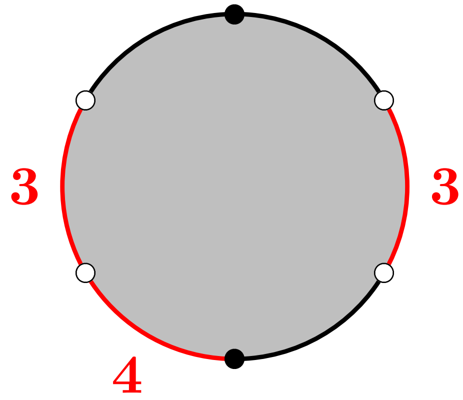

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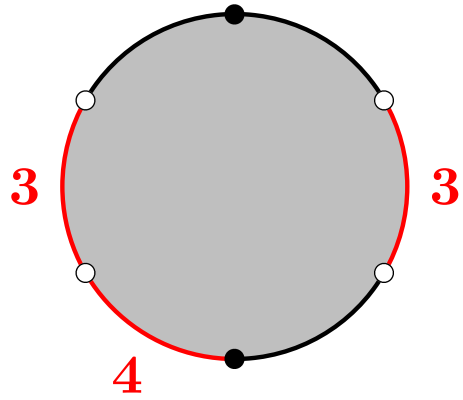

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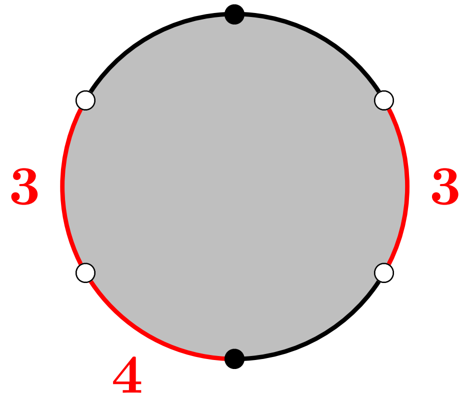

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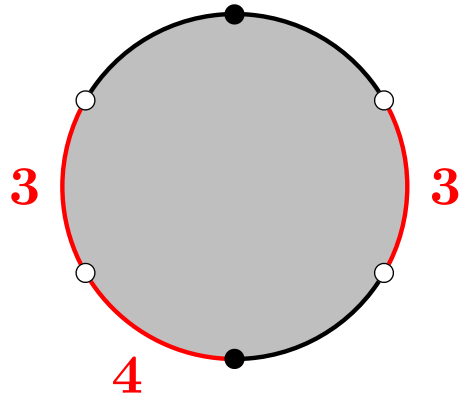

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(5) G is the mapping class group of all of the above. □

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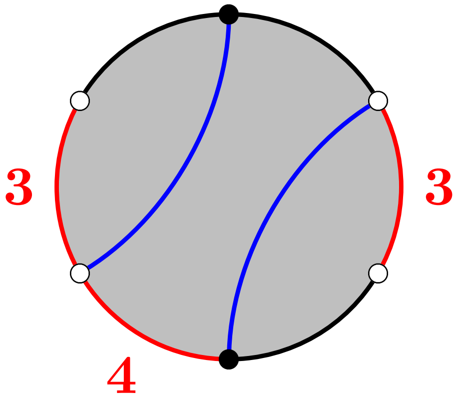
Example.



Black and red are fixed. Blue is a choice of a lamination in W. □

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Note that if S is simply connected then $W = W_0$ and W is finite.

Note: $W \neq \varnothing \iff$

- For all $e \in \pi_0(\partial S \smallsetminus Q)$ one has that ∂e is single-coloured if and only if $e \in R$ and $w(e)$ is odd; and

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- some numerical condition coming from the Euler characteristic.

From now we assume these conditions to hold.

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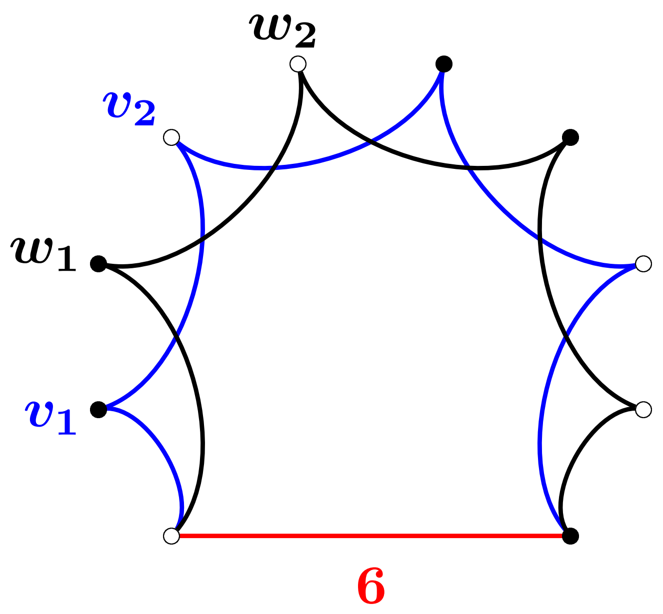
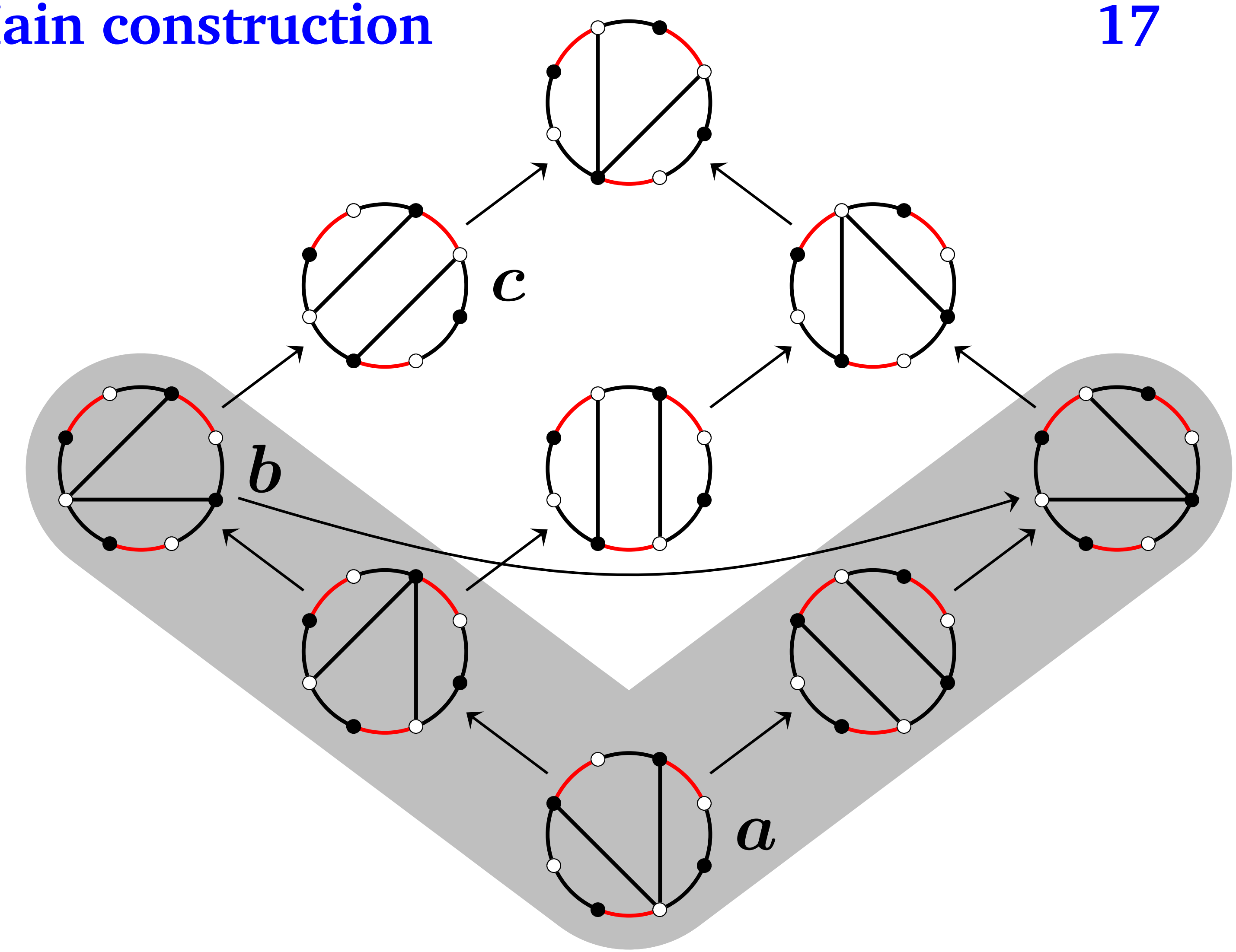


$$v_1 \leq w_1 < v_2 \leq w_2 < \cdots .$$

□



Example. A full (W, E') which is Garside. We only show the ***indecomposable*** arrows. The gray region is the set of vertices x such that $a \longrightarrow x$. An example of a distinguished path is abc.

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Flips are precisely the indecomposable arrows (see previous page).

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Proof. What makes this doable is that W is finite. So $\leq$ on W is generated by the flips. The desired result is known to be equivalent to a condition on triples of flips (Dehornoy) which one checks by brute force. □

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Proof. What makes this doable is that W is finite. So $\leq$ on W is generated by the flips. The desired result is known to be equivalent to a condition on triples of flips (Dehornoy) which one checks by brute force. □

In general however, W is locally infinite and the ordering on W is not generated by the flips.

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Also $A(x)$ is the set of $\pi_1(S)$-invariant elements of the lattice $\lim_{x_0} A(x_0)$ hence $A(x)$ is itself a lattice. □

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What is H?

Definition. Let S be a surface with non-empty boundary.

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(4) The ***framed mapping class group*** is defined to be $F = \mathrm{Stab}_G(\text{one framing})$. □

Theorem/Conjecture. *We have* $\boldsymbol{H} = \boldsymbol{F}$*, that is,*
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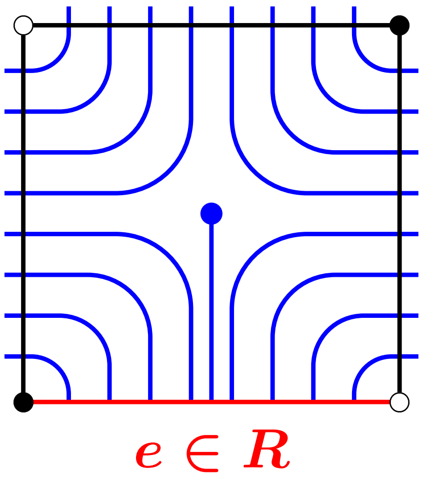

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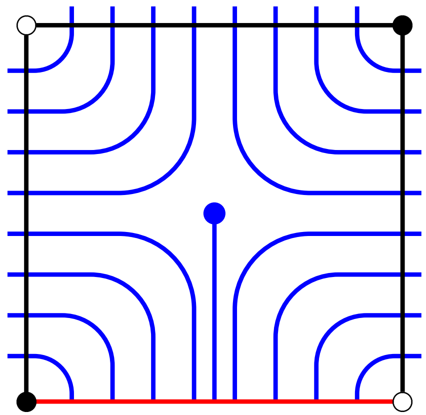

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Proof of $\supset$. ? □

Suggestion. Find a presentation for the framed mapping class group.

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Conjecture. *We have* $\operatorname{diam} V/H < \infty$. □

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Proposition. *Let* $(x, y), (x, z) \in E \cap (V_1 \times V_1)$. *If* $y \neq z$ *then there is no* g *in the Torelli group such that* $gy = z$ *(as promised).* □