MA3E1 Groups and Representations
Assignment 3
Deadline: Tuesday 20 November 2012, 3pm

Write your name and student number on your solution sheet. Mention your department if it is not mathematics.

(3.1) Let ρ, σ be irreducible representations of a finite group G and let $h \in G$. Prove the **generalised orthogonality**

$$\sum_{g \in G} \chi_\sigma(g^{-1})\, \chi_\rho(hg) = \begin{cases} 0 & \text{if } \rho \not\sim \sigma \\ \#G\, \dfrac{\chi_\rho(h)}{\chi_\rho(1)} & \text{if } \rho \sim \sigma. \end{cases}$$

Hint: modify the proof of theorem 107.

(3.2) Let G, H be finite groups. Let $I(G)$ denote the set of irreducible characters of G.

(a) Prove that $I(G) \times I(H)$ and $I(G \times H)$ have equal cardinalities.

(b) (Not for credit). You may assume that if χ, ϕ are characters of a group K then so is $\chi\phi$ defined by $(\chi\phi)(g) = \chi(g)\,\phi(g)$. This is true but beyond our scope. Prove that if p is a character of G and q a character of H then

$$\begin{aligned} p * q \colon G \times H &\longrightarrow \mathbb{C} \\ (g, h) &\longmapsto p(g)\, q(h) \end{aligned}$$

is a character of $G \times H$.

(c) Prove that the map $(p, q) \mapsto p * q$ defines a bijection $I(G) \times I(H) \to I(G \times H)$.

Hint: first prove $(p * q, r * s)_{G \times H} = (p, r)_G\, (q, s)_H$ for all $p, r \in I(G)$, $q, s \in I(H)$.

(3.3) Let

$$G = \langle a, b \mid a^3 = 1,\ b^4 = 1,\ bab^{-1} = a^{-1} \rangle$$
$$D_6 = \langle r, s \mid r^3,\ s^2,\ (rs)^2 \rangle, \qquad C_4 = \langle c \mid c^4 \rangle.$$

(a) Prove that there exist unique homomorphisms

$$f \colon G \to D_6 : \quad f(a) = r, \quad f(b) = s,$$
$$h \colon G \to C_4 : \quad h(a) = 1, \quad h(b) = c.$$

(b) Prove $\#G = 12$ and $G = \{a^k b^\ell \mid 0 \le k \le 2 \text{ and } 0 \le \ell \le 3\}$. Hint: Prove that h is surjective and find the size of its kernel by using f.

(c) Let us call two elements of G **weakly conjugate** if their images in D_6 as well as in C_4 are conjugate. Find all weak conjugacy classes. Then prove that the weak conjugacy classes are the conjugacy classes.

(d) Calculate the character table for G. Justify your answers and show the intermediate steps in filling the character table.

Hints: You may wish to use proposition 127. Also, if $u \colon A \to B$ is a homomorphism of groups and χ a character of B then $\chi \circ u$ is clearly a character of A.

(e) (Not for credit). Use the character table and a result from the lectures to find all normal subgroups of G.