MA3E1 Groups and Representations
Assignment 2
Deadline: Tuesday 6 November 2012, 3pm

Write your name and student number on your solution sheet. Mention your department if it is not mathematics.

(2.1) (Adopted from the 2011 exam.) Put $G = \langle x, y \mid x^2, y^3, (xy)^3 \rangle$ and consider the elements $a = (12)(34)$ and $b = (123)$ of the alternating group A_4.

(a) Prove that there exists a unique homomorphism $f\colon G \to A_4$ such that $f(x) = a$, $f(y) = b$.

(b) Consider the subgroup $H = \langle y \rangle$ of G and the set of cosets

$$A = \{H,\ xH,\ yxH,\ y^2xH\}.$$

Prove $xyxH \in A$.

(c) Justify that $zC \in A$ for all $z \in \{x, y\}$ and $C \in A$ by writing down, without proof, a table which for all $z \in \{x, y\}$ and $C \in A$ gives an element $g \in \{1, x, yx, y^2 x\}$ such that $zC = gH$.

(d) Prove $gH \in A$ for all $g \in G$.

(e) Prove that $f\colon G \to A_4$ is an isomorphism. You may assume that it is surjective.

(2.2) Let $(v_1, \ldots, v_n)$ be a basis of a complex vector space V with $n \geq 2$. For $s \in S_n$ (the symmetric group) and $a_1, \ldots, a_n \in \mathbb{C}$ we define

$$s\Big(\sum_{i=1}^{n} a_i\, v_i\Big) := \sum_{i=1}^{n} a_i\, v_{s(i)}.$$

(a) Prove that this makes V into a $\mathbb{C}S_n$-module. It is called the **permutation module**.

(b) Let $X = \left\{ \sum_{i=1}^{n} a_i\, v_i \in V \;\middle|\; \sum_{i=1}^{n} a_i = 0 \right\}$. Prove that X is a submodule of V.

(c) In order to prove that X is simple, assume from now on that Y is a nonzero submodule of X. Prove that there exists $y \in Y$ of the form

$$y = \sum a_i\, v_i, \qquad a_n = 1.$$

(d) Put $w = v_1 + \cdots + v_n$ and $S_{n-1} = \{g \in S_n \mid g(n) = n\}$. Define

$$z = \frac{1}{(n-2)!} \sum_{g \in S_{n-1}} gy.$$

Prove $gz = z$ for all $g \in S_{n-1}$. Prove $z = n\, v_n - w$.

(e) Prove $n v_i - w \in Y$ for all i.

(f) Prove $v_n - v_i \in Y$ for all i. Prove $Y = X$, that is, X is simple.

(g) Find an explicit submodule W of V such that $V = W \oplus X$. Prove that W is simple.

(2.3) An alternative proof of Maschke's theorem. Let G be a finite group and U a $\mathbb{C}G$-module. Let $V \subset U$ be a submodule and $p\colon U \to V$ any linear map such that $p(v) = v$ for all $v \in V$.

(a) (Not for credit). Why does such a p exist? Show by an example that p is not necessarily a homomorphism of $\mathbb{C}G$-modules (if p is chosen as above).

(b) Define $q\colon U \to V$ by

$$q(v) = \frac{1}{\#G} \sum_{g \in G} g^{-1}\, p\, g(v).$$

Prove that $q\colon U \to V$ is a homomorphism of $\mathbb{C}G$-modules.

(c) Prove that $q(v) = v$ for all $v \in V$.

(d) Let H be a group and $f\colon A \to B$ a homomorphism of $\mathbb{C}H$-modules. Prove that $\ker(f)$ is a submodule of A.

(e) Prove that there exists a submodule $W \subset U$ such that $U = V \oplus W$.

(f) (Not for credit). The present proof has many advantages over the proof from the lectures. List as many as you can think of.