THE UNIVERSITY OF WARWICK

THIRD YEAR EXAMINATION: June 2010

MA3E1 GROUPS AND REPRESENTATIONS

Time Allowed: **3 hours**

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

Calculators are not needed and are not permitted in this examination.

ANSWER 4 QUESTIONS.
If you have answered more than the required 4 questions in this examination, you will only be given credit for your 4 best answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

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1. Let A be an alphabet (a set). If u, v are words over A and $a \in A$, we say that uv is a **one-step reduction** of both $uaa^{-1}v$ and $ua^{-1}av$ if $a \in A$. We also write $u \to v$ if v is a one-step reduction of u.

Let $>$ be the reflexive transitive closure of $\to$.

A word over A is said to be **reduced** if there is no smaller word. The set of reduced words over A is written $F(A)$.

(a) Let u be a word over A. Prove that there exists a unique reduced lower bound of u, that is, a reduced word v over A such that $u \geq v$. It will be written $\mathrm{R}(u)$. **[11]**

(b) Let u, v be words over A. Prove: **[5]**

$$\mathrm{R}(\mathrm{R}(u)\, v) = \mathrm{R}(uv) = \mathrm{R}(u\, \mathrm{R}(v)). \qquad (1)$$

(c) If u, v are reduced words over A, we put $u * v := \mathrm{R}(uv)$. Prove that the pair $(F(A), *)$ is a group. **[5]**

(d) How many reduced words of length n are there if $n \geq 1$? Briefly justify your answer. **[4]**

 Question 2 continued overleaf

2. Let G be a finite group and V, W finite-dimensional $\mathbb{C}G$-modules. Let $f\colon V \to W$ be a homomorphism of $\mathbb{C}G$-modules.

(a) Define "homomorphism of $\mathbb{C}G$-modules". **[2]**

(b) Prove that the kernel of f is a submodule of V. **[3]**

(c) Suppose that V and W are simple. Prove that f is an isomorphism or zero. Also prove that if $V \cong W$ then f is a scalar. **[9]**

(d) Let $L\colon V \to W$ be a linear map and define $p\colon V \to W$ by **[5]**

$$p(x) = \sum_{g \in G} g^{-1}\, L\, g(x).$$

Prove that p is a homomorphism of $\mathbb{C}G$-modules.

(e) Suppose that V is simple of degree > 1 and $x \in V$. Prove that $\sum_{g \in G} gx = 0$. **[6]**

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3. Let G be a finite group and let $\mathrm{Irr}(G)$ denote the set of irreducible characters.

(a) Define the regular representation of G. **[2]**

(b) Let $f\colon G \to \mathbb{C}$ be a function such that $\sum_{g \in G} f(g) \cdot g(x) = 0$ for all $x \in V^{\mathrm{reg}}$. Prove that $f = 0$. **[3]**

(c) Compute the character χ^{reg} of the regular representation. **[4]**

(d) Prove: $\chi^{\mathrm{reg}} = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1) \cdot \chi$. **[4]**

(e) Prove $\#\mathrm{Irr}(G) = k(G)$ where $k(G)$ denote the number of conjugacy classes in G. You may assume $\#\mathrm{Irr}(G) \leq k(G)$. **[12]**

Hint: Let $f \in \mathrm{CF}(G)$ be a class function orthogonal to $\mathrm{Irr}(G)$. Show that if ρ is a representation of G then $\rho^f := \sum_{g \in G} f(g)\, \rho(g^{-1})$ is an intertwiner from ρ to itself.

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4. (a) For each conjugacy class of S_4 (the symmetric group) find its cardinality and give one element. Proofs are not necessary. **[4]**

(b) Find the character table of S_4. Justify your result and show intermediate steps in filling in the table. **[16]**

(c) Find an irreducible character of S_4 whose restriction to A_4 is not irreducible. You may use the table of conjugacy classes in A_4 given in question 5. **[5]**

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5. Let $H \subset G$ be finite groups.

(a) If p is a class function on G and q is a class function on H, define p_H (restriction) and q^G (induction). **[4]**

(b) Let $p \in \mathrm{CF}(G)$ and $q \in \mathrm{CF}(H)$ be class functions. Prove: $(p_H, q)_H = (p, q^G)_G$. **[7]**

(c) If K is a group, let 1_K denote the trivial character of degree 1 of K. Assume now $H = A_4$, $G = A_5$. Compute $(1_H)^G$. You may use the following tables which give the cardinality of each conjugacy class of A_4 or A_5 and one element of it. **[7]**

A_4:

1	(123)	(321)	(12)(34)
1	4	4	3

A_5:

1	(123)	(12)(34)	(12345)	(12354)
1	20	15	12	12

(d) Prove that $q := (1_H)^G - 1_G$ is a character of G. Hint: compute $(\chi, 1_G)_G$. **[4]**

(e) Prove that q is an irreducible character of G. **[3]**
