

MA3E1 Groups and Representations
Assignment 3
Deadline: Monday 21 November 2011, 3pm

Write your name and student number on your solution sheet. Mention your department if it is not mathematics.

(3.1) Let ρ, σ be irreducible representations of a finite group G and let $h \in G$. Prove the **generalised orthogonality**

$$\sum_{g \in G} \chi_{\sigma}(g^{-1}) \chi_{\rho}(hg) = \begin{cases} 0 & \text{if } \rho \not\sim \sigma \\ \#G \frac{\chi_{\rho}(h)}{\chi_{\rho}(1)} & \text{if } \rho \sim \sigma. \end{cases}$$

Hint: modify the proof of theorem 105.

(3.2) Let G, H be finite groups. Let $I(G)$ denote the set of irreducible characters of G .

- (a) Prove that $I(G) \times I(H)$ and $I(G \times H)$ have equal cardinalities.
- (b) (Not for credit). You may assume that if χ, ϕ are characters of a group K then so is $\chi\phi$ defined by $(\chi\phi)(g) = \chi(g)\phi(g)$. This is true but beyond our scope. Prove that if p is a character of G and q a character of H then

$$p * q: G \times H \longrightarrow \mathbb{C} \\ (g, h) \longmapsto p(g)q(h)$$

is a character of $G \times H$.

- (c) Prove that the map $(p, q) \mapsto p * q$ defines a bijection $I(G) \times I(H) \rightarrow I(G \times H)$.

(3.3) Let

$$G = \langle a, b \mid a^3 = 1, b^4 = 1, bab^{-1} = a^{-1} \rangle \\ D_6 = \langle r, s \mid r^3, s^2, (rs)^2 \rangle, \quad C_4 = \langle c \mid c^4 \rangle.$$

- (a) Prove that there exist unique homomorphisms

$$f: G \rightarrow D_6: \quad f(a) = r, \quad f(b) = s, \\ h: G \rightarrow C_4: \quad h(a) = 1, \quad h(b) = c.$$

- (b) Prove $\#G = 12$ and $G = \{a^k b^\ell \mid 0 \leq k \leq 2 \text{ and } 0 \leq \ell \leq 3\}$. Hint: Prove that h is surjective and find the size of its kernel by using f .
- (c) Let us call two elements of G **weakly conjugate** if their images in D_6 as well as in C_4 are conjugate. Find all weak conjugacy classes. Then prove that the weak conjugacy classes are the conjugacy classes.
- (d) Calculate the character table for G . Justify your answers and show the intermediate steps in filling the character table.

Hints: You may wish to use proposition 125. Also, if $u: A \rightarrow B$ is a homomorphism of groups and χ a character of B then $\chi \circ u$ is clearly a character of A .

- (e) (Not for credit). Use the character table and a result from the lectures to find all normal subgroups of G .