MA3E1 Groups and Representations
Assignment 1
Deadline: Friday 22 October 2011, 3pm

Write your name and student number on your solution sheet. Mention your department if it is not mathematics.

(1.1) We say that a square matrix X is upper triangular, if all entries below the main diagonal in X are zero. For example $\left(\begin{smallmatrix} 2i & 1 \\ 0 & -3 \end{smallmatrix}\right)$ is upper triangular. Let B_n be the set of upper triangular matrices in $\mathrm{GL}(n, \mathbb{C})$. Prove that B_n is a subgroup of $\mathrm{GL}(n, \mathbb{C})$.

(1.2) Let $\rho\colon G \to \mathrm{GL}(n, \mathbb{C})$ be a representation of a group G, and let $\sigma\colon G \to \mathrm{GL}(n, \mathbb{C})$ be given by $\sigma(x) = \rho(x)^2$ for all $x \in G$.

(a) Prove that σ is again a representation if G is abelian.

(b) Prove that σ is again a representation if $n = 1$.

(c) Give an example showing that, in general, σ is not again a representation.

(d) Give an example where σ is again a representation but not equivalent to ρ.

(1.3) Let A be a subset of a group G and $B = \{gag^{-1} \mid g \in G,\ a \in A\}$. Prove that $\langle\!\langle A \rangle\!\rangle = \langle B \rangle$.

(1.4) Let G, H be groups. Suppose we have a map $H \times G \to H\colon (x, a) \mapsto x^a$ which is an **action** (that is, $x^{ab} = (x^a)^b$ for all $x \in H$, $a, b \in G$) by **group automorphisms** (that is, $(xy)^a = x^a y^a$ for all $x, y \in H$, $a \in G$). On the set $G \times H$ we define the binary operation

$$(a, x)(b, y) = (ab, x^b y).$$

(a) Prove that this binary operation makes $G \times H$ into a group. It is called an **external semi-direct product** and written $G \ltimes H$.

Let G, H be subgroups of a group P and write $GH := \{gh \mid g \in G,\ h \in H\}$. We say that P is an **internal semi-direct product** of G, H if $H \trianglelefteq P$, $G \cap H = 1$, $P = GH$. We also say that (P, G, H) is an internal semi-direct product.

(b) Prove that an external semi-direct product $G \ltimes H$ is an internal semi-direct product of two subgroups, one isomorphic to G, one to H.

(c) (Not for credit). Prove the following converse. Let (P, G, H) be an internal semi-direct product. Then there exists an action by automorphisms $H \times G \to H$ such that $G \ltimes H \cong P$.

(d) Let G, H be subgroups of a finite group P. Suppose $H \trianglelefteq G$ and $G \cap H = 1$. Prove that (P, G, H) is an internal semi-direct product if and only if $\#P = \#G \cdot \#H$.

(e) Give an example where the group $G \ltimes H$ (internal or external as you prefer) is not isomorphic to $G \times H$ as groups.

(f) (Not for credit). Let G be a group. We define an action by automorphisms $G \times G \to G\colon (x, a) \mapsto x^a := a^{-1} x a$. Prove that $G \ltimes G \cong G \times G$ as groups.