3 The characters of $\mathrm{GL}(V)$

All our vector spaces and linear maps are over $\mathbb{C}$. If V is a vector space, let $\mathrm{GL}(V)$ denote the group of bijective linear maps $V \to V$.

A *linear representation* of $\mathrm{GL}(V)$ is a vector space W together with a homomorphism $\mathrm{GL}(V) \to \mathrm{GL}(W)$. The *dimension* of such a representation is $\dim W$. From now on we assume that all representations mentioned are finite-dimensional.

A representation ϕ: $\mathrm{GL}(V) \to \mathrm{GL}(W)$ is called *polynomial,* respectively, *rational* if for one (hence any) choice of bases of V and W, the entries of the matrix $\phi(A)$ are polynomials, respectively, rational functions, in the entries of A. We say that it is *homogeneous of degree* m if $\phi(tA) = t^m \phi(A)$ for all $t \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

Sometimes we identify $\mathrm{GL}(V)$ with $\mathrm{GL}(n, \mathbb{C})$ where $n = \dim V$.

Example 26. Define ϕ: $\mathrm{GL}(2, \mathbb{C}) \to \mathrm{GL}(3, \mathbb{C})$ by

$$\phi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 & 2ab & b^2 \\ ac & ad + bc & bd \\ c^2 & 2cd & d^2 \end{pmatrix}.$$

You can check directly that ϕ is a group homomorphism (but you don't learn much if you do). The entries of the matrix on the right hand side are homogeneous polynomials of degree 2 in a, b, c, d, and so ϕ is a homogeneous polynomial representation of dimension 3 and degree 2.

Examples 27. Here are some simple examples of representations illustrating the terms defined above. In all these examples we take $A \in \mathrm{GL}(n, \mathbb{C})$.

(a). $\phi(A) = A$: the *defining representation.* This is a homogeneous polynomial representation of dimension n and degree 1.

(b). $\phi(A) = (\det A)^m$ where $m \in \mathbb{Z}$. This is a rational homogeneous representation of dimension 1 and degree mn. It is polynomial if and only if $m \geq 0$.

(c). $\phi(A) = |\det A|^{\sqrt{2}}$. Not a rational representation and not homogeneous.

(d). $\phi(A) = A^{-1}$. Not a representation.

(e). $\phi(A) = (\det A)^m (A^{-1})^t$ where t denotes transpose. A homogeneous rational representation of dimension n and degree $mn - 1$. It is polynomial if and only if $m \geq 1$.

(f). $\phi(A) = \left(\begin{smallmatrix} 1 & \log|\det A| \\ 0 & 1 \end{smallmatrix}\right)$. A representation that isn't homogeneous or rational, but it is continuous.

(g). $\phi(A) = (\sigma(a_{ij})$ where to each entry a_{ij} of A we apply a field automorphism σ of $\mathbb{C}$ which is not the identity or complex conjugation (so σ is necessarily discontinous). This representation is not rational and not continuous.

Exercise 28. Let V^* be the dual of a finite-dimensional vector space V and let $\langle \cdot, \cdot \rangle$: $V^* \times V \to \mathbb{C}$ be the pairing. We assume that $\mathrm{GL}(V)$ acts on V from the left.

(a) Prove that there exists a $\mathrm{GL}(V)$-action on V^* from the left, defined by $\langle gx, y\rangle = \langle x, g^{-1}y\rangle$ for all $(x, y) \in V^* \times V$, $g \in \mathrm{GL}(V)$. Prove that it is rational but not polynomial.

(b) Prove that there exists a $\mathrm{GL}(V)$-action on V^* from the right, defined by $\langle xg, y\rangle = \langle x, gy\rangle$. The concept of *polynomial representation* considers actions from the left only, and therefore, the question whether it is polynomial doesn't make sense.

Proposition 29. *Let V, W be finite-dimensional complex vector spaces and write $n = \dim V$, $d = \dim W$. Let $\phi\colon \mathrm{GL}(V) \to \mathrm{GL}(W)$ be a rational representation. Then there exists a d-tuple $(f_1, \ldots, f_d)$ of Laurent monomials in n variables x_1, …, x_n such that for all $A \in \mathrm{GL}(V)$, if A has eigenvalues $\theta_1, \ldots, \theta_n$ then the eigenvalues of $\phi(A)$ are $f_i(\theta_1, \ldots, \theta_n)$ for $1 \le i \le d$. The f_i are unique up to reordering.*

Proof. For $A \in \mathrm{End}(W)$, $\lambda \in \mathbb{C}$, define $\ker(A - \lambda)^d$ to be the *eigenspace* of A at λ. Then W is a direct sum of the eigenspaces of A.

Fix a basis for V and let $T \subset \mathrm{GL}(V)$ denote the subgroup of diagonal elements. Let $W = W_1 \oplus \cdots \oplus W_p$ be such that W_i is T-invariant and nonzero, and p is maximal under these conditions. We claim:

$$\text{For all } t \in T\text{, all eigenvalues of the restriction } \phi(t)|W_i \text{ are equal.} \tag{30}$$

Suppose not. We may suppose it is false for $i = 1$ and some $t \in T$. Let $W_1 = X_1 \oplus \cdots \oplus X_q$ be the decomposition of W_1 into nonzero eigenspaces of $\phi(t)$. So $q \ge 2$. Since T is abelian, X_i is T-invariant. Then

$$W = \left(\bigoplus_{j=1}^{q} X_j\right) \oplus \left(\bigoplus_{i=2}^{p} W_i\right)$$

is a direct decomposition in $p + q - 1 > p$ nonzero T-invariant subspaces. This is a contradiction and proves (30).

Next we prove:

$$\text{For all } t \in T \text{ and all } i\text{, the restriction } \phi(t)|W_i \text{ is scalar.} \tag{31}$$

If not, then there exists $t \in T$ and linear independent $u, v \in W_i$ for some i such that $\phi(t)$ preserves the span of $\{u, v\}$ and its matrix with respect to (u, v) is

$$\lambda \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

for some $\lambda \in \mathbb{C}$. For all $r \in \mathbb{Z}$ then, the matrix with respect to (u, v) of $\phi(t^r)$ is

$$\lambda^r \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}.$$

This contradicts our assumption that ϕ is rational and thus proves (31).

We conclude that there exists a basis of W with respect to which all elements of $\phi(T)$ are diagonal. That is, there are functions $f_i\colon T \to \mathbb{C}^*$ $(1 \le i \le d)$ such that

$$\phi(\mathrm{diag}(\theta_1, \ldots, \theta_n)) = \mathrm{diag}(f_1, \ldots, f_d)$$

for all $\operatorname{diag}(\theta_1, \dots, \theta_n) \in T$. We also know that f_i is given by a rational function. It follows that it is a Laurent monomial in n indeterminates $x_1, \dots, x_n$.

Let B be the set of elements $A \in \operatorname{End}(V)$ such that the eigenvalues of $\phi(A)$ are $f_i(\theta_1, \dots, \theta_n)$ $(1 \le i \le d)$. Then B contains all diagonal invertible elements. Also, B is closed in $\operatorname{End}(V)$, and hence is all of $\operatorname{End}(V)$. □

Definition 32. In the notation of proposition 29, the *character* of the representation ϕ is defined to be $\operatorname{char}\phi = f_1 + \cdots + f_d$. This is a symmetric Laurent polynomial in n variables.

Example 33 (Symmetric powers). Let V be a finite-dimensional vector space. The symmetric group S_k acts on $V^{\otimes k}$ by permuting the factors. The *k-th symmetric power* $\operatorname{Sym}^k V$ is defined to be the subspace of $V^{\otimes k}$ of S_k-invariant vectors.

For $v_1, \dots, v_k \in V$ we write

$$[v_1, \dots, v_k] := \frac{1}{k!} \sum_{w \in S_k} v_{w(1)} \otimes \cdots \otimes v_{w(k)} \in V^{\otimes k}.$$

Then $[v_1, \dots, v_n] \in \operatorname{Sym}^k V$.

Suppose henceforth that $e_1, \dots, e_n$ is a basis of V. Then

$$B := \left\{ [e_{i_1}, \cdots, e_{i_k}] : 1 \le i_1 \le \cdots \le i_k \le n \right\}$$

is a basis for $\operatorname{Sym}^k V$.

We define

$$[v_1, \dots, v_k][w_1, \dots, w_\ell] := [v_1, \dots, v_k, w_1, \dots, w_\ell]. \tag{34}$$

It is easy to check that this is well-defined. This makes the *symmetric algebra* $\operatorname{Sym} V := \oplus_{k \ge 0} \operatorname{Sym}^k V$ into a graded associative commutative algebra. It is naturally isomorphic to the algebra of polynomials $\mathbb{C}[t_1, \dots, t_n]$ where $t_1, \dots, t_n$ is a basis of the dual of V.

In line with (34), we usually write $v_1 \cdots v_k$ instead of $[v_1, \dots, v_k]$.

Let $A \in \operatorname{End}(V)$ be such that $Ae_i = \theta_i e_i$ for some $\theta_i \in \mathbb{C}$. Then every basis vector $e_{i_1} \cdots e_{i_k} \in B$ is an eigenvector of A with eigenvalue $\theta_{i_1} \cdots \theta_{i_k}$. The trace of A in $\operatorname{Sym}^k V$ is therefore the complete symmetric function $h_k(\theta)$, and the character of $\operatorname{Sym}^k V$ (as $\operatorname{GL}(V)$-module) is $h_k(x_1, \dots, x_n)$.

More common is to define $\operatorname{Sym}^k V$ as a quotient of $V^{\otimes k}$ rather than a subspace. We have chosen the present construction because it fits better into theorem 38 below.

Example 35 (Alternating powers). This is rather analogous to symmetric powers. Let V be a finite-dimensional vector space. Then S_k acts on $V^{\otimes k}$. The *k-th alternating power* of V is the subspace $\wedge^k V \subset V^{\otimes k}$ of vectors v such that $s(v) = \varepsilon_s v$ for all $s \in S_k$. Instead of alternating some say *anti-symmetric* or *skew-symmetric.*

For $v_1, \dots, v_k \in V$ we write

$$\langle v_1, \dots, v_k \rangle := \frac{1}{k!} \sum_{w \in S_k} \varepsilon_w \, v_{w(1)} \otimes \cdots \otimes v_{w(k)} \in V^{\otimes k}.$$

Then $\langle v_1, \dots, v_n \rangle \in \wedge^k V$.

Suppose from now on that $e_1, \dots, e_n$ is a basis of V. Then

$$B := \{\langle e_{i_1}, \dots, e_{i_k} \rangle : 1 \le i_1 < \cdots < i_k \le n\}$$

is a basis for $\wedge^k V$.

We define

$$\langle v_1, \dots, v_k \rangle \wedge \langle w_1, \dots, w_\ell \rangle := \langle v_1, \dots, v_k, w_1, \dots, w_\ell \rangle.$$

It is easy to check that this is well-defined. This makes the *alternating algebra* $\wedge V := \oplus_{k \ge 0} \wedge^k V$ into a graded associative algebra. It is *super-commutative* in the sense that

$$w \wedge v = (-1)^{k\ell}\, v \wedge w \qquad \text{for all } v \in \wedge^k V,\ w \in \wedge^\ell V.$$

We usually write $v_1 \wedge \cdots \wedge v_k$ instead of $\langle v_1, \dots, v_k \rangle$.

Let $A \in \mathrm{End}(V)$ be such that $Ae_i = \theta_i e_i$ with $\theta_i \in \mathbb{C}$. Then every basis vector $e_{i_1} \wedge \cdots \wedge e_{i_k} \in B$ is an eigenvector of A whose eigenvalue is $\theta_{i_1} \cdots \theta_{i_k}$. The trace of A in $\wedge^k V$ is therefore the elementary symmetric function $e_k(\theta)$, and the character of $\wedge^k V$ (as $\mathrm{GL}(V)$-module) is $e_k(x_1, \dots, x_n)$.

Definition 36. Let V be a finite-dimensional vector space. Let V^* be the dual of V and $\langle \cdot, \cdot \rangle \colon V \times V^* \to \mathbb{C}$ the pairing. Let $\phi \colon G \to \mathrm{GL}(V)$ be a linear representation. We define $C(\phi) = C(V)$ to be the vector space of functions $G \to \mathbb{C}$ spanned by those of the form $g \mapsto \langle \phi(g)v, v^* \rangle$ for some $v \in V$, $v^* \in V^*$.

If $\phi \colon G \to \mathrm{GL}(n, \mathbb{C})$ is a representation then $C(\phi)$ is just the space of functions $G \to \mathbb{C}$ spanned by the matrix coefficients.

Theorem 37. *Let G be a group. Let $V_1, \dots, V_k$ be pairwise non-isomorphic irreducible representations of G over an algebraically closed field. Then the $C(V_i)$ are linearly independent.*

Proof. Not on the syllabus. See theorem 27.8 in Curtis and Reiner, Representation theory of finite groups and associative algebras. □

Theorem 38. *The irreducible polynomial representations ϕ^λ of $\mathrm{GL}(V)$ up to isomorphism can be indexed by partitions λ of length at most $n := \dim V$ such that*

$$\operatorname{char} \phi^\lambda = s_\lambda(x_1, \dots, x_n).$$

Proof. Let $k \ge 0$ and put $U = V^{\otimes k}$. The symmetric group S_k acts on U (from the left) by permuting the factors.

Consider the isotypical decomposition of the S_k-module U: $U = \oplus_\lambda U^\lambda$, where λ ranges over some index set.

We can write $U^\lambda = S^\lambda \otimes F^\lambda$ where the S^λ are pairwise non-isomorphic irreducible S_n-modules and $F^\lambda = F^\lambda(V)$ is a trivial S_n-module of as yet unspecified dimension. We can thus assume that the indexing set is precisely the set of partitions of k and that the S^λ are as before (with character χ^λ).

Let

$$B = \{\phi\colon U \to U \mid (gv)\phi = g(v\phi) \text{ for all } g \in S_n,\ v \in U\}.$$

Then U is a right B-module. The S_k-action on U commutes with the B-action on U. We have $B = \oplus_\lambda \operatorname{End}(F^\lambda)$ (use Schur's lemma). Here $\operatorname{End}(F^\lambda)$ is the algebra of $\mathbb{C}$-linear maps from F^λ to itself.

Clearly, B contains the image of the map $\mathrm{GL}(V) \to \operatorname{End}(V)$. Conversely, we shall prove:

- The algebra B is the span of the image of the map $f\colon \operatorname{End}(V) \to \operatorname{End}(U)$. (39)
- It is also the span of the image of the map $f\colon \mathrm{GL}(V) \to \operatorname{End}(U)$. (40)

For any finite-dimensional vector space W, the symmetric power $\operatorname{Sym}^k W$ is spanned by the $w^k := w \otimes \cdots \otimes w$ where w runs through W. Applying this to $W = \operatorname{End}(V)$ proves (39), because $\operatorname{End}(V^{\otimes k}) = (V^*)^{\otimes k} \otimes V^{\otimes k} = W^{\otimes k}$ with compatible S_k-actions.

As $\mathrm{GL}(V)$ is dense in $\operatorname{End}(V)$, (40) is a consequence of (39).

Now F^λ is an irreducible B-module (or zero) and hence, by (40), an irreducible $\mathrm{GL}(V)$-module (or zero).

We have polynomial representations $\phi^\lambda\colon \mathrm{GL}(V) \to \mathrm{GL}(F^\lambda)$.

Next we compute the character of ϕ^λ. Let $(w, A) \in S_k \times \mathrm{GL}(V)$, and let $\operatorname{tr}(w, A)$ be the trace of the action on U. As $U = \oplus_\lambda S^\lambda \otimes F^\lambda$ we find

$$\operatorname{tr}(w, A) = \sum_\lambda \chi^\lambda(w)\ \operatorname{tr}(\phi^\lambda(A)).$$

Let $\theta = (\theta_1, \dots, \theta_n)$ be the eigenvalues of A. In an exercise, you will show that $\operatorname{tr}(w, A) = p_{\rho(w)}(\theta)$. It follows that

$$p_{\rho(w)} = \sum_\lambda \chi^\lambda(w)\ \operatorname{char} \phi^\lambda.$$

But by corollary 7.17.4 we know that

$$p_{\rho(w)} = \sum_\lambda \chi^\lambda(w)\, s_\lambda.$$

Since the χ^λ are linearly independent, we have $\operatorname{char} \phi^\lambda = s_\lambda(x_1, \dots, x_n)$.

In particular, the dimension of F^λ is $s_\lambda(1^n)$. So

$$F^\lambda = 0 \iff s_\lambda(1^n) = 0 \iff \ell(\lambda) \le n.$$

Using theorem 37, one easily proves that there are no more irreducible polynomial representations up to isomorphism. □

Exercise 41. Recall that a module is called completely reducible if it is a direct sum of irreducible ones.

(a) Prove that a completely reducible polynomial representation of $\mathrm{GL}(V)$ is determined by its character.

(b) Prove that the following are equivalent:

  (1) All polynomial representations of $\mathrm{GL}(V)$ are completely reducible.

(2) Every polynomial representation of $\mathrm{GL}(V)$ is determined by its character.

(It can be shown that both are true; this is beyond the scope of these notes.)

(c) (Not on the syllabus). Prove that if X, Y are completely irreducible polynomial representations of $\mathrm{GL}(V)$, then so is $X \otimes Y$.

Example 42. Let V, W be vector spaces and write $m = \dim V$, $n = \dim W$, $G = \mathrm{GL}(V) \times \mathrm{GL}(W)$. The character of a rational representation of G is defined in a similar way to those of $\mathrm{GL}(V)$; it is a Laurent polynomial in $x_1, \ldots, x_m$ and $y_1, \ldots, y_n$ and invariant under permutating the x_i among themselves and the y_j among themselves.

We shall show that the G-modules

$$\mathrm{Sym}(V \otimes W), \qquad \bigoplus_\lambda F^\lambda(V) \otimes F^\lambda(W) \tag{43}$$

have equal characters.

Let $g \in G$. Let $\{u_i\}_i$ be the eigenvalues of g in V and $\{v_j\}_j$ those in W. In $V \otimes W$ they are $\{u_i v_j\}_{ij}$. The trace of g in $\mathrm{Sym}^k(V \otimes W)$ is then $h_k(\{u_i v_j\}_{ij})$ by example 33. The trace of g in the symmetric algebra

$$\mathrm{Sym}(V \otimes W) := \bigoplus_{k \geq 0} \mathrm{Sym}^k(V \otimes W)$$

is

$$\sum_{k \geq 0} h_k(\{u_i v_j\}_{ij}) = \prod_{i,j \geq 0} (1 - u_i v_j)^{-1}.$$

On the other hand, the trace of g in the module on the right hand side of (43) is

$$\sum_\lambda s_\lambda(u_1, \ldots, u_m)\, s_\lambda(v_1, \ldots, v_n).$$

From Cauchy's identity (7.44)

$$\prod_{i,j \geq 0} (1 - x_i y_j)^{-1} = \sum_\lambda s_\lambda(x)\, s_\lambda(y)$$

we conclude that the two modules of (43) have the same character.

One may show that they are even isomorphic by an argument similar to exercise 41(c). □

Exercise 44. State and prove a result about $\wedge(V \otimes W)$, the alternating algebra, analogous to the result in example 42 on $\mathrm{Sym}(V \otimes W)$.