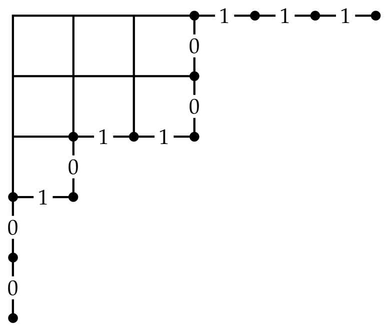
Figure 1. The coding $C_\lambda = \cdots 00101100111 \cdots$ of the partition $\lambda = 331$.



(20) Let $\lambda \vdash n$. Use the hook length formula to prove that $f^\lambda < n$ if and only if $\lambda \in \{(n), (1^n), (n-1,1), (2,1^n), (2,2), (2,2,2), (3,3)\}$.

(21) Let λ be a partition of n of rank r. For $1 \le i \le r$, let $\mu_i = h(i,i;\lambda)$, the hook length of λ at (i,i). Set $\mu = (\mu_1, \mu_2, \ldots, \mu_r)$, so μ is also a partition of n. Show that

$$\chi^\lambda(\mu) = (-1)^t$$

where $t = \sum_{i=1}^r (\lambda_i' - i)$. Prove also that if $\chi^\lambda(\nu) \neq 0$, then $\nu \le \mu$ (dominance order).

(22) Let $\lambda \vdash n$. Prove that λ has precisely n border strips, that is, there are precisely n partitions $\mu \subseteq \lambda$ such that λ/μ is a non-empty border strip.

(23) Let λ be a partition.
  (a) Prove that if λ has an even hook length, then it has a hook of length 2.
  (b) Prove that the number of odd hook lengths of λ minus the number of even hook lengths is a triangular number.

(24) This exercises deals with some basic combinatorial properties of border strips and hooks. Let λ be a partition, and let $p \in \mathbb{P}$. There is a simple bijection between p-hooks (that is, hooks of size p) of λ and borderstrips of λ of size p. Let D_λ denote the diagram of λ with its left-hand edge and upper edge extended to infinity, as shown in figure 1 for $\lambda = 331$. Put a 0 on each vertical edge of the “lower envelope” of D_λ (whose definition should be clear from figure 1) and a 1 on each horizontal edge. If we read these numbers as we move north and east along the lower envelope, then we obtain an infinite binary sequence $C_\lambda = \cdots c_{-1}\, c_0\, c_1 \cdots$. For instance, $C_{331} = \cdots 00101100111 \cdots$. We regard a translate of C_λ as being the same as C_λ. Then the map $\lambda \mapsto C_\lambda$ is a bijection between partitions and infinite binary sequences beginning with infinitely many zeroes and ending with infinitely many ones. The size $|\lambda|$ is the number of pairs (i,j) such that $i < j$ and $(c_i, c_j) = (1,0)$.
  (a) Show that there is a natural bijection between the p-hooks of λ and the integers i such that $(c_i, c_{i+p}) = (1,0)$.
  (b) Show that removing a border strip of size p from λ is equivalent to

choosing i with $(c_i, c_{i+p}) = (1,0)$ and then replacing (c_i, c_{i+p}) with $(0,1)$.

(c) Show that if λ has a hook of size divisible by p, then it has a p-hook.

(d) Start with a partition λ and continually remove border strips of size p until unable to do so. Show that the partition μ that remains is independent of the border strips chosen to be removed. The partition μ is called the *p-core* of λ, and a partition with no p-hooks is called a p-core.

(e) Let μ be a p-core. Let $Y_{p,\mu}$ be the set of all partitions whose p-core is μ. Define $\lambda \leq \nu$ in $Y_{p,\mu}$ if λ can be obtained from ν be removing border strips of size p. Show that $Y_{p,\mu} \cong Y^p$ where $Y = Y_{1,\varnothing}$ is the set of partitions, ordered by containment. Deduce that if $f_\mu(n)$ is the number of partitions of n with p-core μ, then

$$\sum_{n\geq 0} f_\mu(n)\, q^n = q^{|\mu|} \prod_{n\geq 1} (1-q^{pn})^{-p}.$$

(f) Let $n \in \mathbb{P}$. Show that the following three numbers are equal.

(1). The number of p-cores of size n.

(2). The number of solutions $(k_1, \ldots, k_{p-1}) \in \mathbb{N}^{p-1}$ to

$$\left[\sum_{i=1}^{p-1} i\, k_i - p\binom{k_i}{2}\right] - \binom{k_1 + \cdots + k_{p-1}}{2} = n.$$

(3). The coefficient of q^n in $\displaystyle\prod_{n\geq 1} \frac{(1-q^{pn})^p}{(1-q^n)}$.

(g) When $p = 2$, find all partitions satisfying (1) explicitly. What identity results from the equality of (1) and (3)?

(25) Let $H \subset G$ be finite groups. Let V be a (left) H-module and define

$$W := \{p\colon G \to V \mid p(xy) = x \cdot p(y) \text{ for all } x \in H,\ y \in G\}.$$

(a) Prove that $\dim W = [G : H]\ \dim V$.

(b) For $p \in W$, $g \in G$, define $gp = q\colon G \to V$ by $q(x) = p(xg)$. Prove that $qp \in W$.

(c) Prove that the map $G \times W \to W$, $(g, p) \mapsto gp$ makes W into a left G-module.

(d) Define $G \times V \to W$, $(g, v) \mapsto g \boxtimes v$ by $(g \boxtimes v)(x) = xg(v)$ for all $x \in G$. We extend this by linearity to $KG \times V \to W$. Prove that the pair $(W, \boxtimes)$ is a tensor product $KG \otimes_{KH} V$. So W is the induced module $V \uparrow G$.

(e) Prove Frobenius reciprocity.

(26) In this exercise you will prove the *branching rules* for the symmetric group.

(a). Let $\lambda \vdash n$. Prove:

$$\chi^\lambda \uparrow S_{n+1} = \sum_{\lambda \subseteq \mu \vdash n+1} \chi^\mu$$

where we use the usual inclusion $S_n = S_n \times S_1 \subset S_{n+1}$. Hint: Use Pieri's rule and the fact that the Frobenius map is a ring homomorphism.

(b). Let $\mu \vdash n+1$. Prove

$$\chi^{\mu} \downarrow S_{n+1} = \sum_{\mu \supseteq \lambda \vdash n} \chi^{\lambda}.$$

(27) Let $\mathbb{A}_n = \mathbb{A}$ be the alternating S_n-module and let $\chi_{\mathbb{A}}$ be its character. Recall the involution ω of Λ. We define a linear map denoted by the same letter ω: $\mathrm{CF}^n \to \mathrm{CF}^n$ by $\omega(f) = \chi_{\mathbb{A}} f$ (pointwise multiplication). In particular, ω takes a character χ_V to $\chi_{V \otimes \mathbb{A}_n}$.

(a) Prove that $\omega(p_\mu) = \chi_{\mathbb{A}}(\mu)\, p_\mu$ for all $\mu \vdash n$.

(b) Prove that the isomorphism ch: $\mathrm{CF} \to \Lambda$ commutes with the ω maps, that is, $\mathrm{ch}(\omega f) = \omega(\mathrm{ch} f)$ for all $f \in \mathrm{CF}^n$.

(c) Prove that ω: $\mathrm{CF} \to \mathrm{CF}$ is a ring isomorphism.

(d) Prove that $\omega(\chi^{\lambda}) = \chi^{\lambda'}$ for all $\lambda \in \mathrm{Par}$.

(28) Prove that the elements v_T, as T varies over the standard tableaux on λ, form a basis for the Specht module S^{λ}. Hint: prove independence directly; then use the identity $\sum_{\lambda} (f^{\lambda})^2 = n!$.

(29) Prove the claim in the proof of theorem 38 on characters of $\mathrm{GL}(V)$ that $\mathrm{tr}(w, A) = p_{\rho(w)}(\theta)$.

(30) Let V, W be finite-dimensional vector spaces. Prove that there exists an isomorphism of $\mathrm{GL}(V) \times \mathrm{GL}(W)$-modules

$$F^{\lambda}(V \otimes W) \cong \bigoplus_{\mu,\nu} c^{\lambda}_{\mu\nu}\, F^{\mu}(V) \otimes F^{\nu}(W).$$

(31) Let V be a finite-dimensional vector space and $\mu = (\mu_1, \mu_2, \ldots)$ a partition. Prove that there exists an isomorphism of $\mathrm{GL}(V)$-modules

$$\bigotimes_{i \geq 1} \left(\bigwedge^{\mu_i} V \right) \cong \bigoplus_{\lambda} K_{\lambda\mu}\, F^{\lambda'}(V).$$

(32) Let V be an n-dimensional vector space and put $G = \mathrm{GL}(V)$. For $k \in \mathbb{Z}$, let $\mathbb{D}^{\otimes k}$ be the k-th power of the determinant, that is, the 1-dimensional G-module $A \mapsto \det(A)^k$. Note that $\mathbb{D}^{\otimes k} \otimes \mathbb{D}^{\otimes \ell} \cong \mathbb{D}^{\otimes k+\ell}$. Write $\mathbb{D}$ instead of $\mathbb{D}^{\otimes 1}$.

(a) Prove $\mathbb{D} \cong F^{(1^n)}(V)$ (isomorphism of G-modules).

(b) Prove that $F^{\lambda}(V) \otimes \mathbb{D} \cong F^{\mu}(V)$ whenever $\lambda \in \mathrm{Par}$, $\ell(\lambda) \leq n$ and μ is obtained from λ by adding one column of size n.

(c) If $\ell(\lambda) < n$ and $k \in \mathbb{Z}$ then $F^{\ell}(V) \otimes \mathbb{D}^{\otimes k}$ is an irreducible rational G-module. Conversely, every irreducible rational G-module is isomorphic to precisely one of them.