MA5P2 The Symmetric Group – Exercises

October 30, 2008

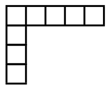
(1) Prove: $\sum_{\lambda \in \mathrm{Par}} q^{|\lambda|} = \prod_{n \geq 1} (1 - q^n)^{-1}$.

(2) Prove Proposition 7.5.3 on page 296.

(3) Prove

$$\sum_{\mu \vdash n} q^{\ell(\mu)-1} m_\mu = \sum_{j=0}^{n-1} (q-1)^j \, s_{n-j,1^j}.$$

Here, $(n-j, 1^j)$ is the *hook* partition, for example

$$(5, 1^3) =$$  .

(4) Prove the following lemma on inverse bumping, used in the proof of Theorem 7.11.5 on page 319. **Lemma.** Let Q be a tableau of shape μ. Let λ be a partition such that $\lambda \subset \mu$ and $|\mu/\lambda| = 1$. Then there exists a unique tableau P of shape λ and a unique $k > 0$ such that $Q = P \leftarrow k$.

(5) Let $d_\lambda(k)$ denote the number of tableaux of shape λ whose entries are taken from $\{1, \ldots, k\}$.

(a) Prove: $k^n = \sum_{\lambda \vdash n} d_\lambda(k) \, f^\lambda$.

(b) Prove: $\binom{n+k\ell-1}{n} = \sum_{\lambda \vdash n} d_\lambda(k) \, d_\lambda(\ell)$.

(6) (a). Lemma 7.11.2 compares the insertion paths $I(P, j)$ and $I(P \leftarrow j, k)$ if $j \leq k$. State and prove a similar result if $j > k$.

(b). Every permutation $w = w_1 \cdots w_n$ gives rise to an *up-down sequence*, a sequence of $n-1$ signs, the i-th one being $+$ if $w_i < w_{i+1}$ and $-$ otherwise. Prove that if $w \overset{\mathrm{RSK}}{\longmapsto} (P, Q)$ then the up-down sequence of w is determined by Q.

(7) Fill in the details of the proof of Corollary 7.12.3 on page 323.

(8) Prove the following claims in the proof of Theorem 7.13.1, all made on page 328.

(a) $|\mu/\lambda| = 1$ in case (L2).

(b) $\lambda = \mu = \nu$ in case (L4).

(c) If a corner p is labelled by λ then $|\lambda|$ is the number of X's to the left and below p.

(9) On the last line of page 324 the book claims "... [in order to prove Theorem 7.13.1] we may assume u and v have no repeated elements". Justify this in full detail.

(10) In the proof of Theorem 7.13.5 we claimed that $T(i,j) = U(i,j)$. Prove this. ($U(i,j)$ is our notation for what the book writes as $((\varnothing \leftarrow i_1) \leftarrow i_2)\cdots i_k)$.

(11) (a). Let $w \in S_n$ be an involution and let $\nu(i,j)$ be the partition at the (i,j) corner of the growth diagram of w (so $\nu(i,j) = \nu(j,i)$). Prove that, for $0 \le i \le n$, the number of columns of $\nu(i,i)$ of odd length is the number of fixed points k of w satisfying $k \le i$.

(b). Suppose that $A \xmapsto{\text{RSK}} (P,Q)$ and that A is symmetric (so $P = Q$). Show that $\operatorname{tr}(A)$ is the number of columns of P of odd length.

(c). Prove

$$\prod_i (1-qx_i)^{-1} \prod_{i<j} (1-x_ix_j)^{-1} = \sum_\lambda q^{c(\lambda)}\, s_\lambda(x) \tag{1}$$

where $c(\lambda)$ denotes the number of odd parts of λ'.

(d). Deduce that

$$\prod_{i<j} (1-x_ix_j)^{-1} = \sum_\mu s_{(2\mu)'}(x)$$

where $2\mu = (2\mu_1, 2\mu_2, \ldots)$.

(e). Fix $k \ge 0$. Evaluate the sum $a(n,k) = \sum_\lambda f^\lambda$ where λ ranges over all partitions of n of k odd parts (and any number of even parts).

(f). What identity results when we apply ω to (1)?

(12) Let a_k denote the number of tableaux whose entries sum to k. Prove:

$$\sum_{k\ge 0} a_k\, q^k = \prod_{0<i} (1-q^{2i-1})^{-i}(1-q^{2i})^{-i}.$$

(13) Let $\ell(\lambda) \le m$ and $\ell(\lambda') \le n$. Define

$$\bar{\lambda} := (n-\lambda_m, n-\lambda_{m-1}, \ldots, n-\lambda_1). \tag{2}$$

Use the classical definition of the Schur function to prove that

$$(x_1\, x_2 \cdots x_m)^n\, s_\lambda(x_1^{-1}, \ldots, x_m^{-1}) = s_{\bar{\lambda}}(x_1, \ldots, x_m). \tag{3}$$

(14) Prove that

$$\prod_{i=1}^{m} \prod_{j=1}^{n} (x_i + y_j) = \sum_\lambda s_\lambda(x)\, s_{\tilde{\lambda}}(y)$$

summed over all partitions λ with $\ell(\lambda) \le m$ and $\ell(\lambda') \le n$ and where $\tilde{\lambda}$ is defined by (2).

(15) For $m \ge n \ge 0$, let $s(m,n)$ denote the Schur function of the partition $(m,n) = (m,n,0,0,\ldots)$ of at most two parts. Use Pieri's rule to prove $s(m,n) = s_m\, s_n - s_{m+1}\, s_{n-1}$.

(16) For $1 \le m \le n$, let $s\langle 1^{n-m}, m\rangle$ denote the Schur function of the hook partition $\langle 1^{n-m}, m\rangle$ of m boxes in the first row, then $n-m$ rows of one box each. Use Pieri's rule to express $s\langle 1^{n-m}, m\rangle$ as a linear combination of $\{e_i\, h_j \mid i,j \in \mathbb{N}\}$.

(17) Prove corollary 7.15.9: $s_{\lambda/n} = \sum_\nu s_\nu$ where the sum is over all partitions $\nu \subseteq \lambda$ such that λ/ν is a horizontal strip of size n.

(18) (a). Prove that

$$s_\lambda(1, q, q^2, \ldots, q^{m-1}) = q^r \prod_{1 \le i < j \le m} \frac{q^{\lambda_i - \lambda_j + j - i} - 1}{q^{j-i} - 1}$$

where $r = \sum_i (i-1)\lambda_i$.

(b). Prove that

$$s_\lambda(1^m) = \prod_{1 \le i < j \le m} \frac{\lambda_i - \lambda_j + j - i}{j - i}.$$

(19) In this exercise we prove the hook length formula which is a formula for f^λ, the number of standard tableaux of n boxes.

Let $\lambda \vdash n$, $\ell(\lambda) \le k$. Write $\ell_i := \lambda_i + k - i$. Define

$$\Delta(x_1, \ldots, x_k) := \prod_{1 \le i < j \le n} (x_i - x_j), \quad F(\ell_1, \ldots, \ell_k) := \frac{n!\, \Delta(\ell_1, \ldots, \ell_k)}{\ell_1! \cdots \ell_k!}.$$

(a). Prove:

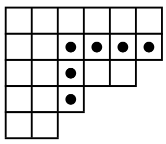
$$\sum_{i=1}^{k} x_i\, \Delta(x_1, \ldots, x_i + t, \ldots, x_k) = \left[x_1 + \cdots + x_k + \tbinom{k}{2} t\right] \Delta(x_1, \ldots, x_k). \tag{4}$$

(b). The numbers $f(\lambda) = f^\lambda$ satisfy a recurrence formula

$$f(\lambda_1, \ldots, \lambda_k) = \sum_{i=1}^{k} f(\lambda_1, \ldots, \lambda_i - 1, \ldots, \lambda_k) \tag{5}$$

where it is understood that $f(\mu_1, \ldots, \mu_k)$ is zero if $(\mu_1, \ldots, \mu_k)$ is not a partition, that is, if $\mu_i < \mu_{i+1}$ or $\mu_i < 0$ for some i. Prove that $f^\lambda = F(\ell_1, \ldots, \ell_k)$, by showing that the right hand side satisfies a recurrence analogous to (5).

(c). By $(i, j) \in \lambda$ we mean that (i, j) is a box in the diagram of λ, that is, $1 \le j \le \lambda_i$. For $(i, j) \in \lambda$, let the *hook length* $h(i, j; \lambda)$ be the number of $(u, v) \in \lambda$ in the same row or column as (i, j) and such that $u + v \ge i + j$:



$h(2, 3; \lambda) = 6.$

Prove the *hook length formula*:

$$f^\lambda = n! \prod_{(i,j) \in \lambda} h(i, j; \lambda)^{-1}. \tag{6}$$