MA4F2 Braid Groups – Sheet 4

Deadline: Friday, 11 March 2005, 2:00.

Solutions to Section B are for handing in. Please put your solutions into the MA4F2 Braid Groups box in front of the Undergraduate Office.

(A1) Let G, H be a groups and let

$$\begin{aligned} H \times G &\longrightarrow H \\ (x, g) &\longmapsto x^g \end{aligned}$$

denote a G-action on H on the right. Prove that the binary operation $(a, b)(c, d) := (ac, b^c d)$ defines a group structure on the set $G \times H$.

(A2) Let M be a cancellative monoid. Prove that the following are equivalent.

(1) M is good.

(2) (a) For any $z \in M$ there are only finitely many $(x, y) \in M^2$ such that $xy = z$.

(b) For all $x, y \in M$, if $xy = 1$ then $x = y = 1$.

(B1) Prove the coherence identity

$$(u\backslash v)\backslash(u\backslash w) \equiv^{++} (v\backslash u)\backslash(v\backslash w)$$

for the BKL complemented presentation on B_6 where $(u, v, w) = (a_{14}, a_{25}, a_{36})$.

(B2) Define the following BKL elements in B_4: $a = a_{12}$, $b = a_{23}$, $c = a_{34}$, $d = a_{13}$. Put $A = \{a, b, c, d\}$, let B_4^+ be the submonoid of B_4 generated by A and let $\leq$ be the ordering on B_4^+ defined by $x \leq z \Leftrightarrow \exists y \in B_4^+ \colon xy = z$. Prove that $(B_4^+, \leq)$ is not a lattice. Deduce that B_4^+ is not a Garside monoid, not even if one replaces A with another set of generators.

(B3) Let M be a good monoid. In this exercise you will prove that an element

$$a = \sum_x a_x\, x \in \mathbb{C}[[M]]$$

is invertible if and only if $a_1 \neq 0$.

(a) Prove that there is a homomorphism of rings $f \colon \mathbb{C}[[M]] \to \mathbb{C}$ defined by

$$f\Big(\sum_x a_x\, x\Big) = a_1.$$

Deduce the implication $\Rightarrow$.

(b) Prove $\Leftarrow$. Hint: Why can you assume $a_1 = 1$? Then use the geometric series.

(B4) In the following cases, let M be the monoid given by the presentation. You already know that M is a Garside monoid in cases (a), (b), and you may assume it is in case (c). Each time, $\ell \colon M \to \mathbb{Z}$ will denote the homomorphism such that $\ell(a) = \ell(b) = \ell(c) = 1$. Draw the Dehornoy graph and label each vertex x with $\mu(x)$. Compute

$$\sum_{x \in M} t^{\ell(x)}.$$

Proofs are not necessary.

(a) $(a, b, c \mid abb = bbc,\ bcc = cca,\ caa = aab)$.

(b) $(a, b, c \mid acab = bcaa,\ bcaac = cabca,\ cabca = acabc)$.

(c) $(a, b, c \mid ab = bc = ca)$.

(B5) Prove lemma 17.18: For all $I \subset S$ we have $N(w_I) \cap S = I$. In particular, all w_I are distinct. Hint: The proof of theorem 10.5 shows that $N(w_I)$ is the smallest set A of reflections containing I and such that

$$\big[(ij) \in A \text{ and } (jk) \in A\big] \ \Rightarrow\ (ik) \in A \qquad (1 \le i < j < k \le n).$$

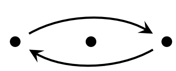
(B6) (a) Let G be any group. Let F_n denote the free group on $p_1, \ldots, p_n$. Prove that there exists a set-theoretic bijection $\phi\colon \operatorname{Hom}(F_n, G) \to G^n$ defined by $\phi(f) = (fp_1, \ldots, fp_n)$.

(b) Let $\theta\colon B_n \times F_n \to F_n$ (pronounciation: θ = theta) denote the action constructed in section 18. By (a) we have a B_n-action on G^n on the left by

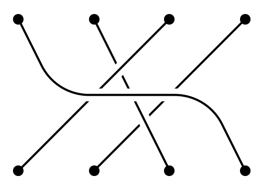
$$\begin{aligned} B_n \times G^n &\longrightarrow G^n \\ (g, \phi f) &\longmapsto g(\phi f) := \phi\big(\theta(g, f)\big). \end{aligned}$$

Find a formula for $\sigma_i(x_1, \ldots, x_n)$.

(B7) Below a braid x is given in horizontal notation and a braid y in vertical. Sketch x in vertical notation and y in horizontal.



x



y

(B8) State and prove the missing part (4) of our proposition on the P_n-action on F_n. That is, if $1 \le i < j < k < m$, express $p_{ik}\, p_{jm}\, p_{ik}^{-1}$ in terms of all $p_{\ell m}$ where $1 \le \ell < m$.

(C1) It's easy to prove that the complemented presentation

$$(a, b, c \mid ab = ccc,\ bc = aaa,\ ca = bbb)$$

is coherent and has a Garside element. A bar of chocolate for the first person to prove that it's normed!

(C2) Let $\ell\colon \Sigma_n \to \mathbb{Z}$ be the length with respect to S, the set of fundamental reflections. Prove:

$$\sum_{x \in \Sigma_n} q^{\ell(x)} = \prod_{k=1}^{n} \frac{1 - q^k}{1 - q}.$$

(C3) Prove the following formula for hZ^{-1} in the case of the BKL positive braid monoid B_m^+, valid for any m:

$$h\Big(\sum_{x \in B_m^+} x\Big)^{-1} = hZ^{-1} = \sum_{k=0}^{m-1} \frac{(m-1+k)!\,(-t)^k}{(m-1-k)!\,k!\,(k+1)!}.$$