MA241 Combinatorics – Marking Sheet 3

Deadline: Wednesday, 2 March 2005, 2:00.

For this sheet (B1), (B2), (B5), (B7)(cdef) are marked.

(B1) Let S denote the set of fundamental reflections in Σ_7 and write $s_i < s_j$ if and only if $i < j$. Find the lexicographically smallest minimal expression $(x_1, \ldots, x_k) \in S^k$ for

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 6 & 1 & 5 & 2 \end{pmatrix},$$

that is, with the property that there is no minimal expression of the form $(x_1, \ldots, x_{i-1}, y, \ldots)$ with $y < x_i$.

[5] *Solution.* The answer is 213214543 and the computation leading to this result is as follows (it is enough to write the bottom rows of the permutations):

436152	
2	126354
426153	4
1	126345
416253	5
3	125346
316254	4
2	124356
216354	3
1	123456
126354	

(B2) Let $a \in \Sigma_n$ and write $x = s_i$ and $y = s_j$. Suppose $a < ax$ and $a < ay$. Prove the following result (which was used in the proof that (Ω, R) is confluent). If $|i - j| = 1$ then $ax < axy < axyx$. If $|i - j| > 1$ then $ax < axy$.

[5] *Solution.* For all $b \in \Sigma_n$ we have

$$b < bs_i \quad \Leftrightarrow \quad b(i) < b(i+1)$$

because $b < bs_i \Leftrightarrow \ell(b) < \ell(bs_i) \Leftrightarrow b(i) < b(i+1)$.

Let $|i - j| = 1$, say, $j = i + 1$. Then

$$\begin{aligned} ax < axy &\Leftrightarrow ax(i+1) < ax(i+2) \\ &\Leftrightarrow a(i) < a(i+2) \\ &\Leftarrow a(i) < a(i+1) < a(i+2) \\ &\Leftrightarrow a < ax \text{ and } a < ay = \text{true} \end{aligned}$$

and

$$\begin{aligned} axy < axyx &\Leftrightarrow axy(i) < axy(i+1) \\ &\Leftrightarrow ax(i) < ax(i+2) \\ &\Leftrightarrow a(i+1) < a(i+2) \\ &\Leftrightarrow a < ay = \text{true}. \end{aligned}$$

The case of $j = i - 1$ is similar.

Finally, suppose $|i - j| > 1$. Then

$$ax < axy \Leftrightarrow ax(j) < ax(j+1) \Leftrightarrow a(j) < a(j+1) \Leftrightarrow a < ay = \text{true}.$$

(B5) Prove that the relation $\leq$ on B_n (defined in 12.9) is an ordering.

[5] *Solution.* **Transitive.** Let $x \leq xy$ and $xy \leq xyz$ $(x, y, z \in B_n)$. Then $y, z \in B_n^+$. Then $yz \in B_n^+$. Then $x \leq xyz$.

Reflexive. Let $x \in B_n^+$. Since $1 \in B_n^+$ we have $x \leq x \cdot 1 = x$.

Anti-symmetric. We have a homomorphism $\ell \colon B_n \to \mathbb{Z}$ with $\ell(\sigma_i) = 1$ for all i. Since B_n^+ is generated by all σ_i and $\ell(\sigma_i) > 0$ for all i we find that

$$\ell(x) > 0 \text{ for all } x \in B_n^+ - \{1\}.$$

Let $x \leq xy \leq x$ with $x, y \in B_n$ and $y \neq 1$. Then $y, y^{-1} \in B_n^+ - \{1\}$. So $\ell(y) > 0$ and $\ell(y^{-1}) > 0$. Contradiction. We have proved anti-symmetry.

(B7) Consider the complemented presentation

$$(a, b, c \mid bab = aaa,\ bac = cba,\ abc = cab).$$

(a) Prove that the associated complement f on $A = \{a, b, c\}$ is normed and coherent.

(b) Prove that $\Delta := a \vee b \vee c$ exists and give a representative in A^* for it.

(c) Draw the Dehornoy graph.

(d) Prove that Δ is not a Garside element (without using the following parts of the exercise).

(e) Prove that f is not convergent, by showing that the word reversing process applied to $c^{-1}b^{-1}ac$ does not terminate.

(f) Prove that a Garside element does not exist.

Solution.

[3] (c). See attached sheet.

[3] (d). Suppose Δ were a Garside element. Then there exists an automorphism ϕ of (A, f) such that $x\Delta = \Delta\phi(x)$ for all $x \in A$. But there is no nontrivial automorphism. So $a\Delta = \Delta a$. From the Dehornoy graph we immediately see $\Delta = aaaac$. So $a(aaaac) = a\Delta = \Delta a = (aaaac)a$. By the main theorem on Garside elements we have that M is cancellative. So we can cancel a^4 on the left and we get $ac = ca$. This is obviously false in M.

[3] (e). We have

$$\begin{aligned} c^{-1}\,\underline{b^{-1}\,a}\,c &\curvearrowright \underline{c^{-1}\,a}\,b\,a^{-1}\,\underline{a^{-1}\,c} \\ &\curvearrowright a\,b\,c^{-1}\,\underline{b^{-1}\,b}\,\underline{a^{-1}\,b}\,c\,b^{-1}\,a^{-1} \\ &\curvearrowright a\,b\,\underline{c^{-1}\,a}\,a\,b^{-1}\,\underline{a^{-1}\,c}\,b^{-1}\,a^{-1} \\ &\curvearrowright a\,b\,a\,b\,c^{-1}\,b^{-1}\,a\,\underline{b^{-1}\,b}\,c\,b^{-1}\,a^{-1}\,b^{-1}\,a^{-1} \\ &\curvearrowright a\,b\,a\,b\,\underline{c^{-1}\,b^{-1}\,a\,c}\,b^{-1}\,a^{-1}\,b^{-1}\,a^{-1}. \end{aligned}$$

The latter word contains the initial word as subword so the word reversing process does not terminate when applied to the initial word, and is therefore not convergent.

[1] (f). Suppose a Garside element exists. By our main theorem on Garside elements, f is convergent (because we know f to be normed and coherent). Contradiction.