MA4F2 Braid Groups – Sheet 3

Deadline: Wednesday, 2 March 2005, 2:00.

Solutions to Section B are for handing in. Please put your solutions into the MA4F2 Braid Groups box in front of the Undergraduate Office.

(A1) Let $x = s_i$ and $y = s_j$.

(a) Suppose $|i - j| = 1$. Prove $1 < x < xy < xyx$ and $x \vee y = xyx$.

(b) Suppose $|i - j| > 1$. Prove $1 < x < xy$ and $x \vee y = xy$.

(A2) Let G be a group and L the set of subgroups of G. Let $\leq$ be the ordering on L of inclusion. Prove that $(L, \leq)$ is a lattice.

(A3) Prove that the pair (Ω, R) in theorem 11.2 is acyclic and well-founded.

(A4) The generators in our complete rewriting system for B_3^+ can be written $\Omega = \{1, \sigma_1, \sigma_2, \sigma_{12}, \sigma_{21}, \Delta\}$ where $\sigma_i = rs_i$ and $\sigma_{ij} = r(s_i s_j)$. Can you write down all rewriting rules explicitly? There are 25 of them.

(A5) Let $x_1, \ldots, x_k \in \Omega$. Prove that $(x_1, \ldots, x_k)$ is a greedy form if and only if (x_i, x_{i+1}) is a greedy form for all appropriate i.

(A6) (a). Let $M = \{0, 1\}$ be equipped with the multiplication $xy = \max(x, y)$. Prove that M does not embed in a group (that is, there is no injective homomorphism from M to a group).

(b). Consider the monoid

$$S = \langle a, b, c, d, u, v, x, y \mid au = bv,\ cu = dv,\ cx = dy \rangle.$$

Prove that S is not embeddable in a group.

(c)*. Show that S is cancellative, that is, for all $p, q, r \in S$, if $pr = qr$ or $rp = rq$ then $p = q$.

(B1) Let S denote the set of fundamental reflections in Σ_7 and write $s_i < s_j$ if and only if $i < j$. Find the lexicographical smallest minimal expression $(x_1, \ldots, x_k) \in S^k$ for

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 6 & 1 & 5 & 2 \end{pmatrix},$$

that is, with the property that there is no minimal expression of the form $(x_1, \ldots, x_{i-1}, y, \ldots)$ with $y < x_i$.

(B2) Let $a \in \Sigma_n$ and write $x = s_i$ and $y = s_j$. Suppose $a < ax$ and $a < ay$. Prove the following result (which was used in the proof that (Ω, R) is confluent). If $|i - j| = 1$ then $ax < axy < axyx$. If $|i - j| > 1$ then $ax < axy$.

(B3) Compute the greedy form of

$$r\begin{pmatrix}1&2&3&4\\3&1&4&2\end{pmatrix}r\begin{pmatrix}1&2&3&4\\4&2&1&3\end{pmatrix}r\begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix}r\begin{pmatrix}1&2&3&4\\3&1&2&4\end{pmatrix}\Delta^{-17}.$$

(B4) Consider the pair $(\overline{\Omega}, \overline{R})$ of theorem 12.3. You may assume without proving it that it is a well-founded rewriting system for B_n. Prove that it is confluent.

(B5) Prove that the relation $\leq$ on B_n (defined in 12.9) is an ordering.

(B6) We write i for σ_i. Apply the word reversing process to $2^{-1}3^{-1}3^{-1}4^{-1}1223$. Use your result to give an expression for $4332 \vee 1223$.

(B7) Consider the complemented presentation

$$(a, b, c \mid bab = aaa,\ bac = cba,\ abc = cab).$$

(a) Prove that the associated complement f on $A = \{a, b, c\}$ is normed and coherent.

(b) Prove that $\Delta := a \vee b \vee c$ exists and give a representative in A^* for it.

(c) Draw the Dehornoy graph.

(d) Prove that Δ is not a Garside element (without using the following parts of the exercise).

(e) Prove that f is not convergent, by showing that the word reversing process applied to $c^{-1}b^{-1}ac$ does not terminate.

(f) Prove that a Garside element does not exist.

(B8) Consider the complemented presentation

$$(a, b, c \mid acab = bcaa,\ bcaac = cabca,\ cabca = acabc).$$

You may assume that the associated complement f on $A = \{a, b, c\}$ is normed and coherent.

(a) Draw the Dehornoy graph.

(b) Use the Dehornoy graph to prove that $D := a \vee b \vee c$ exists and to compute it.

(c) Prove that D is not a Garside element.

(d) Prove that $\Delta := Da$ is a Garside element.

(e) Draw the generalised Dehornoy graph with vertex set $[1, \Delta] = \{u \in M \mid 1 \leq u \leq \Delta\}$. Hint: Your graph should be closed under "replacing a word $xf(x, y)$ by $yf(y, x)$". This fact was not necessarily used in the examples of the lectures, but it is needed here.

(f) Compute the greedy form (with respect to Δ) for $cbabcaac$.

(C1) Prove that the rewriting system called *word reversing* is a complete rewriting system.

(C2) Prove that Δ^2 generates the centre of B_n.