# MA241 Combinatorics – Marking Sheet 2

Deadline: Wednesday, 9 February 2005, 2:00.

For this sheet, (B2)cdef, (B4), (B5), (B6) will be assessed.

(B2) Let $B_3^+$ be the monoid presented by $(S_1, R_1) = (1, 2 \mid 121 = 212)$. (Warning: $\varnothing$ is the identity, 1 is not.) Let $B_3$ be the group presented by the same presentation, regarded as a group presentation. From the general theory of presentations it follows that there is a natural homomorphism $f\colon B_3^+ \to B_3$. One of the things you prove in this exercise is that $f$ is injective.

(a) Prove that $B_3^+$ is presented [1] by

$$(S_2, R_2) := \left(1, 2, \Delta \,\middle|\, \begin{array}{cc} 121 \to \Delta, & 212 \to \Delta, \\ 1\Delta \to \Delta 2, & 2\Delta \to \Delta 1, \end{array}\right)$$

(b) Construct a map $g\colon S_2^* \to \mathbb{Z}_{\geq 0}$ such that $g(x) > g(y)$ if $x \xrightarrow[R]{} y$. You don't need to prove that $g$ has this property. Deduce that $(S_2, R_2)$ is a well-founded rewriting system for $B_3^+$.

(c) Prove that $(S_2, R_2)$ is a complete rewriting system for $B_3^+$.

(d) Prove that you don't get a complete rewriting system for $B_3^+$ if you remove the last rewriting rule $2\Delta \to \Delta 1$.

(e) Compute the $R_2$-minimal form for 22121122121222121.

(f) Prove that

$$(S_3, R_3) := \left(1, 2, \Delta, \delta \,\middle|\, \begin{array}{cc} 121 \to \Delta, & 212 \to \Delta, \\ 1\Delta \to \Delta 2, & 2\Delta \to \Delta 1, \\ 1\delta \to \delta 2, & 2\delta \to \delta 1, \\ \Delta\delta \to \varnothing, & \delta\Delta \to \varnothing \end{array}\right)$$

is a monoid presentation for $B_3$.

(g) From now on you may assume without proving it that $(S_3, R_3)$ is a well-founded rewriting system for $B_3$. Prove that $(S_3, R_3)$ is a complete rewriting system for $B_3$.

(h) Prove that every $R_2$-minimal word in $\{1, 2, \Delta\}^*$ is also $R_3$-minimal. Deduce that $f$ is injective.

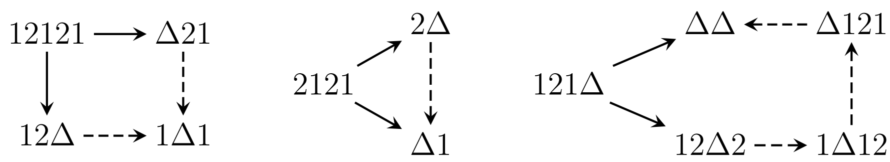
(i) There is a homomorphism (called length) $\ell\colon B_3^+ \to \mathbb{Z}$, $\ell(1) = \ell(2) = 1$. (You don't need to prove this.) Clearly, the rewriting system $(S_2, R_2)$ preserves the length, that is, for all $(x, y) \in R$ one has $\ell(x) = \ell(y)$. Use this to compute the formal power series

$$\sum_{x \in B_3^+} t^{\ell(x)}.$$

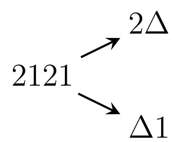
[1] Recall our convention $xRy \Leftrightarrow (x, y) \in R \Leftrightarrow (x = y) \in R \Leftrightarrow (x \to y) \in R$, four notations for the same thing.

Solution.

**[6]** (c). We need to finish the diamonds in the case of any two overlapping rewrite rules $a \to b$ and $a \to c$. Up to interchanging 1 and 2 here they are (solid arrows) together with how to finish them (dashed arrows).




**[3]** (d). Without that rewriting rule the diamond cannot be finished in the following case.




Indeed, the two entries in the right hand column are $r$-minimal words, that is, no rewriting rule can be applied to either.

**[3]** (e).

$$22\underline{121}12\underline{212}1222\underline{121} \longrightarrow \underline{22\Delta}12\Delta1222\Delta$$
$$\longrightarrow \Delta\underline{1112\Delta}1222\Delta \longrightarrow \Delta\Delta\underline{22211222\Delta} \longrightarrow \Delta\Delta\Delta11122111.$$

**[3]** (f). Let $G$ be the monoid thus presented. We first show that $G$ is a group. Clearly, $\Delta$ and $\delta$ have inverses in $G$. Moreover, 1 has an inverse because $\varnothing = \Delta\delta = (121)\delta = 1(21\delta)$ and $\varnothing = \delta\Delta = \delta(121) = (\delta12)1$ (one doesn't need to prove that $\delta12 = 21\delta$ because if an element of any monoid has a left-inverse $a$ and a right-inverse $b$ then $a = b$). Similar for 2. So $G$ is a group.

It remains to show that all relations in $G$ are true in $B_3$. Well, in $B_3$ we have

$$\delta^{-1}1\delta = (\Delta\delta)\delta^{-1}1\delta(\delta^{-1}\Delta) = \Delta1\Delta^{-1} = 2$$

so $1\delta = \delta2$ and likewise $2\delta = 1\delta$.

(B4) Make a planar drawing of the Cayley graph for the presentation $\langle x, y \mid x^5,\ y^2, (xy)^3\rangle$. (It has 60 vertices so you need to draw somewhat small to make it fit on one page.) A proof is not necessary.

**[3]** Solution. To the marker's discretion.

(B5) Prove that there exists $w_0 \in \Sigma_n$ such that $N(w_0) = \text{Ref}$. Give a formula for $w_0(i)$ for all $i \in I_n$. Show that $x \le w_0$ for all $x \in \Sigma_n$.

**[3]** Solution. Define $w_0 \in \Sigma_n$ by $w_0(i) = n + 1 - i$. Then $N(w_0) = \text{Ref}$ as is easily shown using (B6). For any $x \in \Sigma_n$ we have $N(x) \subset \text{Ref} = N(w_0)$ so that $x \le w_0$ by 9.21.

(B6) Let $x \in \Sigma_n$ and $1 \le i < j \le n$. Prove that $(ij) \in N(x^{-1}) \Leftrightarrow xi > xj$.

[4] SOLUTION. Let $a_i \in \mathbb{R}$ $(1 \le i \le n)$ be such that $a = \sum a_i v_i \in C$, that is, $a_i < a_j$ for $i < j$. Let $i < j$. Then

$$\begin{gathered}(ij) \in N(x^{-1}) \Leftrightarrow (ij) \text{ separates } 1 \text{ from } x^{-1} \\ \Leftrightarrow (ij) \text{ separates } a \text{ from } x^{-1}a = \sum_k a_k v_{x^{-1}k} = \sum_k a_{xk} v_k \\ \Leftrightarrow a_{xi} > a_{xj} \Leftrightarrow xi > xj.\end{gathered}$$